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CHARACTERIZATIONS OF NONEXPANSIVE MULTIPLIERS ON PARTIALLY ORDERED SETS

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ABSTRACT. Having proved some basic characterizations of nonexpansive multipliers on partially ordered sets, we establish some intimate connections between nonexpansive multipliers and interior (quasi-interior) operators.

The results obtained naturally extend and supplement some of the former statements of G. Szász, J. Szendrei, M. Kolibiar, W. H. Cornish and the second author on some particular multipliers on semilattices and partially ordered sets.

Introduction

A function F from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ will be called a *nonexpansive multiplier* if $F(D) \leq D$ and

$$F(D) \wedge E = F(E) \wedge D$$

for all $D, E \in \mathcal{D}$. Moreover, the function F will be called a *quasi-interior operator* if F is nonexpansive, nondecreasing, and quasi-idempotent in the sense that

$$F\big(F(D)\big) = F(D)$$

for all $D \in \mathcal{D}$ with $F(D) \in \mathcal{D}$.

Having established some basic characterizations of nonexpansive multipliers, we show that a function F from a semilattice $\mathcal{D}$ in a poset $\mathcal{A}$ into $\mathcal{A}$ such that $F(E) \wedge D \in F[\mathcal{D}] \cap \mathcal{D}$ for all $D, E \in \mathcal{D}$ with $D \leq E$ is a nonexpansive multiplier if and only if F is a quasi-interior operator or a multiplicative interior operator.

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Moreover, if $\mathcal{A}$ is a semilattice with a least element such that $\mathcal{A}$ is not totally ordered, but directed, then there exists a multiplicative interior operator F on $\mathcal{A}$ such that F is not a multiplier, and even a multiplicative and additive interior operator on a finite totally ordered set need not be a multiplier.

The results obtained naturally extend and supplement some former statements of Szász [7], [8], Szász and Szendrei [9], Kolibiar [4], Cornish [3] and the second author [10] on some particular multipliers on semilattices and posets, respectively. In particular, they show that the "if part" of assertion (ii) of Proposition 2.1 of Cornish [3] is not correct.

1. Partially ordered sets

A nonvoid set $\mathcal{A}$ together with a reflexive, transitive and antisymmetric relation $\leq$ will be called a *poset* [1]. A poset $\mathcal{A}$ can always be thought of as a nonvoid family of sets partially ordered by set inclusion. Namely, each $A \in \mathcal{A}$ can be identified with the set $\{B \in \mathcal{A} : \ B \leq A\}$.

A poset $\mathcal{A}$ will be called *totally ordered* if for each $A, B \in \mathcal{A}$ either $A \leq B$ or $B \leq A$ holds. Moreover, a poset $\mathcal{A}$ will be called *directed* if for each $A, B \in \mathcal{A}$ there exists a $C \in \mathcal{A}$ such that $A \leq C$ and $B \leq C$. Note that thus a totally ordered poset is, in particular, directed.

The infimum (greatest lower bound) and the supremum (least upper bound) of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ will be understood in the usual sense. However, instead of $\inf \mathcal{D}$ and $\sup \mathcal{D}$, we shall also use the lattice theoretic notations meet $\bigwedge \mathcal{D}$ and join $\bigvee \mathcal{D}$, respectively.

Concerning the lattice operations

$$A \wedge B = \inf\{A, B\} \qquad \text{and} \qquad A \vee B = \sup\{A, B\} ,$$

we shall only need the following well-known assertions, which are usually not stressed in the standard books on lattices.

THEOREM 1.1. *If $\mathcal{A}$ is a poset and $A, B, C, D \in \mathcal{A}$, then*

(1) $A \leq B$ *if and only if* $A = A \wedge B$;

(2) $A \leq B$ *and* $C \leq D$ *imply* $A \wedge C \leq B \wedge D$ *whenever* $A \wedge C$ *and* $B \wedge D$ *exist.*

COROLLARY 1.2. *If $\mathcal{A}$ is a poset and $A, B, C \in \mathcal{A}$, then*

(1) $A = A \wedge A$;

(2) $A = A \wedge (A \vee B)$ *whenever* $A \vee B$ *exists;*

(3) $A \leq B$ *implies* $A \wedge C \leq B \wedge C$ *whenever* $A \wedge C$ *and* $B \wedge C$ *exist.*

THEOREM 1.3. *If $\mathcal{A}$ is a poset and $A, B, C \in \mathcal{A}$, then*

(1) $A \wedge B = B \wedge A$ *whenever either* $B \wedge A$ *or* $A \wedge B$ *exist;*
(2) $A \wedge (B \wedge C) = (A \wedge B) \wedge C$ *whenever* $A \wedge B$ *and* $B \wedge C$ *and moreover either* $(A \wedge B) \wedge C$ *or* $A \wedge (B \wedge C)$ *exist.*

Remark 1.4. Hence, by using the dual $\mathcal{A}(\geq)$ of the poset $\mathcal{A}(\leq)$, one can easily get some analogous statements for the operation $\vee$.

However, in the sequel, we shall mainly need the operation $\wedge$. Therefore, in connection with posets, we shall assume here some particular terminology.

For instance, a nonvoid subset $\mathcal{B}$ of a poset $\mathcal{A}$ will be called a semilattice in $\mathcal{A}$ if $D \wedge E$ exists in $\mathcal{A}$ and belongs to $\mathcal{B}$ for all $D, E \in \mathcal{B}$.

2. Nonexpansive and nondecreasing functions

According to [10], we shall assume here the following notion.

DEFINITION 2.1. A function F from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ will be called *nonexpansive* if $F(D) \leq D$ for all $D \in \mathcal{D}$.

Because of Theorem 1.1(1), we evidently have the following proposition.

PROPOSITION 2.2. *If F is a function from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$, then the following assertions are equivalent:*

(1) F *is nonexpansive.*
(2) $F(D) = F(D) \wedge D$ *for all* $D \in \mathcal{D}$.
(3) $F(D) = F(D) \wedge E$ *for all* $D \in \mathcal{D}$ *and* $E \in \mathcal{A}$ *with* $D \leq E$.

Moreover, in the sequel, we shall also need the following obvious definition.

DEFINITION 2.3. A function F from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into another poset $\mathcal{B}$ will be called *nondecreasing* if $F(D) \leq F(E)$ for all $D, E \in \mathcal{D}$ with $D \leq E$.

Again, by Theorem 1.1(1), it is clear that we also have the following proposition.

PROPOSITION 2.4. *If F is a function from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into another poset $\mathcal{B}$, then the following assertions are equivalent:*

(1) F *is nondecreasing;*
(2) $F(D) = F(D) \wedge F(E)$ *for all* $D, E \in \mathcal{D}$ *with* $D \leq E$.

Now, by using the above propositions, we can also easily establish the following theorem.

THEOREM 2.5. *If F is a function from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$, then the following assertions are equivalent:*

(1) *F is nonexpansive and nondecreasing;*
(2) *$F(D) \wedge E = F(D) \wedge F(E)$ for all $D, E \in \mathcal{D}$ with $D \leq E$.*

Hint. If assertion (2) holds and $D, E \in \mathcal{D}$ such that $D \leq E$, then by using Corollary 1.2 we can see that

$$F(D) = F(D) \wedge F(D) = F(D) \wedge D \leq F(D) \wedge E = F(D) \wedge F(E) \leq F(D).$$

Therefore, we also have

$$F(D) = F(D) \wedge E \qquad \text{and} \qquad F(D) = F(D) \wedge F(E).$$

Thus, by Propositions 2.2 and 2.4, assertion (1) also holds. □

In addition to the above theorem, it is also worth proving the following theorem.

THEOREM 2.6. *If F is a function from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ such that $F(E) \wedge D$ exists for all $D, E \in \mathcal{D}$ with $D \leq E$, then the following assertions are equivalent:*

(1) *F is nonexpansive and nondecreasing;*
(2) *$F(D) \leq F(E) \wedge D$ for all $D, E \in \mathcal{D}$ with $D \leq E$.*

Proof. If assertion (1) holds and $D, E \in \mathcal{D}$ such that $D \leq E$, then $F(D) \leq D$ and $F(D) \leq F(E)$, and hence $F(D) \leq F(E) \wedge D$. That is, assertion (2) also holds.

While, if assertion (2) holds and $D, E \in \mathcal{D}$ such that $D \leq E$, then we evidently have $F(D) \leq F(E) \wedge D \leq D$ and $F(D) \leq F(E) \wedge D \leq F(E)$. Therefore, assertion (1) also holds. □

Remark 2.7. Note that the extra condition on the domain and the range of F should actually be included in assertion (1).

3. Quasi-interior operators

DEFINITION 3.1. A function F from a subset $\mathcal{D}$ of a set $\mathcal{A}$ into $\mathcal{A}$ will be called *quasi-idempotent* if

$$F\big(F(D)\big) = F(D)$$

for all $D \in \mathcal{D}$ with $F(D) \in \mathcal{D}$.

Remark 3.2. Now, a quasi-idempotent function F from a subset $\mathcal{D}$ of a set $\mathcal{A}$ into $\mathcal{A}$ may be called idempotent if $F[\mathcal{D}] \subset \mathcal{D}$.

Simple applications of the corresponding definitions immediately yield the following proposition.

PROPOSITION 3.3. *If F is a function from a subset $\mathcal{D}$ of a set $\mathcal{A}$ into $\mathcal{A}$, then the following assertions are equivalent:*

(1) *F is quasi-idempotent;*
(2) *$F[\mathcal{D}] \cap \mathcal{D}$ is the family of all fixed points of F.*

P r o o f. If D is a fixed point of F, then $D \in \mathcal{D}$ such that $D = F(D)$. Therefore, $D \in F[\mathcal{D}] \cap \mathcal{D}$.

While, if $E \in F[\mathcal{D}] \cap \mathcal{D}$, then $E \in \mathcal{D}$ and moreover there exists a $D \in \mathcal{D}$ such that $E = F(D)$. Hence, if assertion (1) holds, it follows that $F(E) = F\big(F(D)\big) = F(D) = E$. That is, E is fixed point of F. Therefore, assertion (2) also holds.

On the other hand, if $D \in \mathcal{D}$ such that $F(D) \in \mathcal{D}$, then we also have $F(D) \in F[\mathcal{D}] \cap \mathcal{D}$. Hence, if assertion (2) holds, it follows that $F\big(F(D)\big) = F(D)$. Therefore, assertion (1) also holds. □

Remark 3.4. From Proposition 3.3, we can at once see that a function F from a subset $\mathcal{D}$ of a set $\mathcal{A}$ into $\mathcal{A}$ is idempotent if and only if $F[\mathcal{D}]$ is the family of all fixed points of F.

Analogously to [1; p. 111], we may also have the following definition.

DEFINITION 3.5. A function F from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ will be called a *quasi-interior operator* if it is nonexpansive, nondecreasing and quasi-idempotent.

Remark 3.6. Now, a quasi-interior operator from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ may be called an interior operator if $F[\mathcal{D}] \subset \mathcal{D}$.

Note that combining Theorems 2.5 and 2.6 with Remark 3.4 and Proposition 3.3, we can at once get some useful characterizations of interior and quasi-interior operators.

Moreover, by using Theorem 2.6, Remark 3.4 and Proposition 3.3, we can also easily prove the following theorem.

THEOREM 3.7. *If F is a function from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ such that $F(E) \wedge D \in F[\mathcal{D}] \cap \mathcal{D}$ for all $D, E \in \mathcal{D}$ with $D \leq E$, then the following assertions are equivalent:*

(1) *F is an interior operator;*
(2) *F is a quasi-interior operator;*
(3) *$F(D) = F(E) \wedge D$ for all $D, E \in \mathcal{D}$ with $D \leq E$, and $F[\mathcal{D}]$ is the family of all fixed points of F;*
(4) *$F(D) = F(E) \wedge D$ for all $D, E \in \mathcal{D}$ with $D \leq E$, and $F[\mathcal{D}] \cap \mathcal{D}$ is the family of all fixed points of F.*

P r o o f . From Theorem 2.6, Remark 3.4 and Proposition 3.3, we can at once see that the following assertions are equivalent:

(a) F is an interior (resp. a quasi-interior) operator;
(b) $F(D) \leq F(E) \wedge D$ for all $D, E \in \mathcal{D}$ with $D \leq E$, and $F[\mathcal{D}]$ (resp. $F[\mathcal{D}] \cap \mathcal{D}$) is the family of all fixed points of F.

On the other hand, if F is a quasi-interior operator, then because of the nonexpansivity of F and the assumption of the theorem, we have

$$F(D) = F(D) \wedge D \in F[\mathcal{D}] \cap \mathcal{D} \subset \mathcal{D}$$

for all $D \in \mathcal{D}$. Therefore, $F[\mathcal{D}] \subset \mathcal{D}$, and thus F is actually an interior operator and $F[\mathcal{D}] = F[\mathcal{D}] \cap \mathcal{D}$.

Moreover, if F is a quasi-interior operator, then by Proposition 3.3, the assumption of the theorem and the nondecreasingness of F it is clear that

$$F(E) \wedge D = F\big(F(E) \wedge D\big) \leq F(D)$$

for all $D, E \in \mathcal{D}$ with $D \leq E$. Therefore, by Theorem 2.6, we actually have $F(D) = F(E) \wedge D$ for all $D, E \in \mathcal{D}$ with $D \leq E$. And hence it is clear that not only assertions (a) and (b), but also assertions (1) through (4) are equivalent. □

In the sequel, we shall also need the following definition.

DEFINITION 3.8. A function F from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into a poset $\mathcal{B}$ will be called *quasi-multiplicative* if

$$F(D \wedge E) = F(D) \wedge F(E)$$

for all $D, E \in \mathcal{D}$ such that $D \wedge E$ exists in $\mathcal{A}$ and belongs to $\mathcal{D}$.

Remark 3.9. Now, a quasi-multiplicative function from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into a poset $\mathcal{B}$ may be called multiplicative if $\mathcal{D}$ is a semilattice in $\mathcal{A}$.

By Proposition 2.4, it is clear that a quasi-multiplicative function is, in particular, nondecreasing. On the other hand, by defining a function F on the poset of all subsets of $\{1, 2, 3\}$ such that

$$F(A) = \begin{cases} A & \text{if } A \in \big\{\{1,2\}, \{2,3\}, \{1,2,3\}\big\}, \\ \emptyset, & \text{otherwise}, \end{cases}$$

we can easily see that even an interior operator need not be quasi-multiplicative.

4. Partial meet multipliers

Analogously to [10], we may also have the following definition.

DEFINITION 4.1. A function F from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ will be called a *multiplier* if

$$F(D) \wedge E = F(E) \wedge D$$

for all $D, E \in \mathcal{D}$.

EXAMPLE 4.2. If $\mathcal{D}$ is a subset of a poset $\mathcal{A}$ such that $D \wedge E$ exists in $\mathcal{A}$ for all $D, E \in \mathcal{D}$, then the identity function $\Delta_{\mathcal{D}} = \{(D, D) : D \in \mathcal{D}\}$ of $\mathcal{D}$ is a nonexpansive multiplier.

EXAMPLE 4.3. If A is an element and $\mathcal{D}$ is a semilattice in a poset $\mathcal{A}$ such that $A \wedge D$ exists for all $D \in \mathcal{D}$, then the function F defined by

$$F(D) = A \wedge D$$

for all $D \in \mathcal{D}$ is a nonexpansive multiplier.

Namely, by Theorem 1.3, we have

$$(A \wedge D) \wedge E = A \wedge (D \wedge E) = A \wedge (E \wedge D) = (A \wedge E) \wedge D,$$

and hence $F(D) \wedge E = F(E) \wedge D$ for all $D, E \in \mathcal{D}$.

Remark 4.4. In the sequel, we shall see that a nonexpansive multiplier from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$ is in particular a quasi-multiplicative quasi-interior operator.

Therefore, it is also of some interest to point out that even a quasi-idempotent multiplicative multiplier from an ideal $\mathcal{D}$ of a distributive lattice $\mathcal{A}$ into $\mathcal{A}$ need not be nonexpansive.

EXAMPLE 4.5. Let $\mathcal{A}$ be a distributive lattice [1; p. 12] with a least element O and a greatest element X such that $X \neq O$. Fix $A \in \mathcal{A} \setminus \{O\}$, and define

$$\mathcal{D} = \{D \in \mathcal{A} : A \wedge D = O\}$$

and

$$F(D) = A \vee D \qquad (D \in \mathcal{D}).$$

Then, it can be easily seen that $\mathcal{D}$ is an ideal of $\mathcal{A}$, and F is a multiplicative multiplier with $F[\mathcal{D}] \cap \mathcal{D} = \emptyset$ such that $D < F(D)$ for all $D \in \mathcal{D}$.

Note that if $D \in \mathcal{D}$, then we have

$$A \wedge F(D) = A \wedge (A \vee D) = (A \wedge A) \vee (A \wedge D) = A \vee O = A \neq O,$$

and hence $F(D) \notin \mathcal{D}$. Therefore, $F[\mathcal{D}] \cap \mathcal{D} = \emptyset$, and thus in particular F is quasi-idempotent.

Moreover, we evidently have $D \leq A \vee D = F(D)$, and if $D = F(D) = A \vee D$ were true, then $A \leq D$, and thus $A = A \wedge D = O$ would also be true. Therefore, $D < F(D)$. Thus, in particular, F is not nonexpansive.

Remark 4.6. In connection with the above example, it is also worth noticing that if $D, E \in \mathcal{D}$, then

$$F(D \vee E) = A \vee (D \vee E) = (A \vee D) \vee E = F(D) \vee E,$$

and thus $F(D) \vee E = F(D \vee E) = F(E \vee D) = F(E) \vee D$. Therefore, F is not only a meet multiplier, but also a join multiplier.

Note that F as a join multiplier can be extended to all of $\mathcal{A}$, but F as a meet multiplier cannot be extended to a larger domain. Namely, if $E, B \in \mathcal{A}$ such that $F(D) \wedge E = B \wedge D$ for all $D \in \mathcal{D}$, then

$$A \wedge E = (A \vee O) \wedge E = F(O) \wedge E = B \wedge O = O.$$

Therefore, $E \in \mathcal{D}$, and thus F has the required maximality property.

5. Basic properties of nonexpansive multipliers

The importance of nonexpansive multipliers lies mainly in the following theorems whose origins go back to Szász [7; Satz 1] and Szász and Szendrei [9; Satz 1].

THEOREM 5.1. *If F is a function from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$, then each of the following assertions implies the subsequent one:*

(1) *F is a nonexpansive multiplier;*
(2) *$F(D) = F(E) \wedge D$ for all $D, E \in \mathcal{D}$ with $D \leq E$;*
(3) *$F(D \wedge E) = F(D) \wedge E$ for all $D, E \in \mathcal{D}$ with $D \wedge E \in \mathcal{D}$.*

Proof. If assertion (1) holds and $D, E \in \mathcal{D}$ such that $D \leq E$, then by Proposition 2.2 and the multiplier property of F we have

$$F(D) = F(D) \wedge E = F(E) \wedge D.$$

Therefore, assertion (2) also holds.

While, if assertion (2) holds and $D, E \in \mathcal{D}$ such that $D \wedge E \in \mathcal{D}$, then since $D \wedge E \leq D$, it is clear that

$$F(D \wedge E) = F(D) \wedge (D \wedge E) = \big(F(D) \wedge D\big) \wedge E = F(D) \wedge E.$$

Therefore, assertion (3) also holds. □

Now, as an immediate consequence of Theorem 5.1, we can also state:

COROLLARY 5.2. *If F is a function from a semilattice $\mathcal{D}$ in a poset $\mathcal{A}$ into $\mathcal{A}$, then the following assertions are equivalent:*

(1) *F is a nonexpansive multiplier;*
(2) *$F(D) = F(E) \wedge D$ for all $D, E \in \mathcal{D}$ with $D \leq E$;*
(3) *$F(D \wedge E) = F(D) \wedge E$ for all $D, E \in \mathcal{D}$.*

Proof. For this, by Theorem 5.1, it is enough to note only that if assertion (3) holds and $D, E \in \mathcal{D}$, then

$$F(D) = F(D \wedge D) = F(D) \wedge D$$

and

$$F(D) \wedge E = F(D \wedge E) = F(E \wedge D) = F(E) \wedge D.$$

Therefore, assertion (1) also holds. □

Moreover, as a more satisfactory characterization of nonexpansive multipliers, we can also prove:

THEOREM 5.3. *If F is a function from a subset $\mathcal{D}$ in a poset $\mathcal{A}$ into $\mathcal{A}$, then the following assertions are equivalent:*

(1) *F is a nonexpansive multiplier;*
(2) *$F(D) \wedge E = F(D) \wedge F(E)$ for all $D, E \in \mathcal{D}$.*

Proof. If assertion (1) holds and $D, E \in \mathcal{D}$, then by Corollary 1.2, Theorem 1.3 and Proposition 2.2 it is clear that

$$\begin{aligned} F(D) \wedge E &= \big(F(D) \wedge F(D)\big) \wedge E = F(D) \wedge \big(F(D) \wedge E\big) \\ &= F(D) \wedge \big(F(E) \wedge D\big) = F(D) \wedge \big(D \wedge F(E)\big) = \big(F(D) \wedge D\big) \wedge F(E) \\ &= F(D) \wedge F(E). \end{aligned}$$

Therefore, assertion (2) also holds.

While, if the assertion (2) holds and $D, E \in \mathcal{D}$, then we evidently have

$$F(D) = F(D) \wedge F(D) = F(D) \wedge D$$

and

$$F(D) \wedge E = F(D) \wedge F(E) = F(E) \wedge F(D) = F(E) \wedge D.$$

Therefore, assertion (1) also holds. □

Now, as an immediate consequence of Theorems 5.1 and 5.3, we can also state:

COROLLARY 5.4. *If F is a nonexpansive multiplier from a subset $\mathcal{D}$ of a poset $\mathcal{A}$ into $\mathcal{A}$, then F is a quasi-multiplicative quasi-interior operator.*

Proof. From Theorem 5.1 it is clear that F is nondecreasing. Moreover, combining Theorems 5.1 and 5.3, we can at once see that F is quasi-multiplicative.

On the other hand, if $D \in \mathcal{D}$ such that $F(D) \in \mathcal{D}$, then by Proposition 2.2 and Theorem 5.1 it is clear that

$$F\big(F(D)\big) = F\big(F(D) \wedge D\big) = F\big(D \wedge F(D)\big) = F(D) \wedge F(D) = F(D)\,.$$

Therefore, F is also quasi-idempotent. □

6. The relationship with interior operators

Now, analogously to Szász [7; Satz 2], we can also naturally establish the following theorem.

THEOREM 6.1. *If F is a function from a semilattice $\mathcal{D}$ in a poset $\mathcal{A}$ into $\mathcal{A}$ such that $F(E) \wedge D \in F[\mathcal{D}] \cap \mathcal{D}$ for all $D, E \in \mathcal{D}$ with $D \le E$, then the following assertions are equivalent:*

(1) *F is a nonexpansive multiplier;*
(2) *F is a multiplicative interior operator;*
(3) *F is a quasi-interior operator.*

Proof. If assertion (1) holds, then by Corollary 5.4, F is a quasi-multiplicative quasi-interior operator. Hence, since $\mathcal{D}$ is now a semilattice, it is clear that F is multiplicative. Moreover, by using Theorem 3.7, we can also see that F is an interior operator. Therefore, assertion (2) also holds.

Now, since the implication (2) $\Longrightarrow$ (3) is automatic, to complete the proof we need only note that if assertion (3) holds, then by Theorem 3.7 and Corollary 5.2, assertion (1) also holds. □

In addition to Theorem 6.1, it is also worth proving the following theorem which has been suggested by Cornish [3; p. 343].

THEOREM 6.2. *If $\mathcal{A}$ is a semilattice with a least element O such that $\mathcal{A}$ is not totally ordered, but directed, then there exists a multiplicative interior operator F from $\mathcal{A}$ into $\mathcal{A}$ such that F is not a multiplier.*

Proof. Since $\mathcal{A}$ is not totally ordered, but directed, there exist $A, B, C \in \mathcal{A}$ such that A and B are incomparable, but $A < C$ and $B < C$. For each $D \in \mathcal{A}$, define

$$F(D) = O \quad \text{if } C \not\le D \qquad \text{and} \qquad F(D) = C \quad \text{if } C \le D\,.$$

Then, it can be easily seen that F is a multiplicative interior operator. Moreover, it is clear that

$$F(A) \wedge C = O \wedge C = O \neq A = C \wedge A = F(C) \wedge A,$$

and thus F is not a multiplier. □

Now, as an immediate consequence of Theorems 6.2, we can also state:

COROLLARY 6.3. *A directed semilattice $\mathcal{A}$ with a least element is totally ordered if each multiplicative interior operator on $\mathcal{A}$ is a multiplier.*

The following example shows that, in contrast to Cornish [3; Proposition 2.1(ii)], even a multiplicative and additive interior operator on a finite totally ordered set need not be a multiplier.

EXAMPLE 6.4. Consider the set $\mathcal{A} = \{1, 2, 3\}$ to be equipped with its natural order and define a function F on $\mathcal{A}$ such that

$$F(1) = 1, \qquad F(2) = 1, \qquad F(3) = 3.$$

Then F is a multiplicative and additive interior operator such that F is not a multiplier. Namely, for instance,

$$F(2) \wedge 3 = 1 \wedge 3 = 1, \qquad \text{but} \qquad F(3) \wedge 2 = 3 \wedge 2 = 2.$$

Remark 6.5. This example has been constructed with the help of Zoltán Boros, and is closely related to the examples of Szász [7; p. 168] and Kolibiar [4; p. 455].

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