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PARTIALLY DIRECTED MOORE GRAPHS

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1. Introduction

Undirected T-graphs (sometimes called strongly geodetic graphs) have been studied in several papers [3, 7, 13]. Plesník and Znám [14] have dealt with analogous problems for directed graphs. In the present paper we generalize this concept for partially directed graphs. We show that any T-graph is either an undirected tree or a homogeneous graph with a finite diameter, i.e., a tied graph. The most interesting T-graphs — called Moore graphs — are finite tied graphs. We get a class much richer than the class of undirected Moore graphs (cf. [1, 2, 8, 10, 12, 13]). For example, there exist infinitely many (partially directed) Moore graphs of diameter two. However, given the number of vertices of a Moore graph G of diameter two the following invariants are uniquely determined: the valency of G, the directed and the undirected valencies of G, the degree of G and the spectrum of G (that is, the eigenvalues of the adjacency matrix of G and their multiplicities). Therefore our main attention is devoted to finding conditions for the existence of a Moore graph of diameter two with a given number of vertices. Some results are concerned with the Moore graphs of other diameters, with the T-graphs or a generalization of them, namely the P-graphs. Several open problems are stated.

Most results of the present paper were reported at international conferences held in Kalamazoo [4] and Keszthely [5] in 1976.

2. Notation and terminology

We shall give here only definitions of the notions that are not clear from the context.

Graphs considered in this paper are *partially directed,* i.e., they may contain directed edges as well as undirected ones. A graph is called *directed [undirected]* if all its edges are directed [undirected, respectively]. A graph is called *mixed* if it contains at least one directed edge and at least one undirected edge. An edge is said to be a *link* [a *loop*] if its end vertices are different [equal] (cf. [15]). Two edges with the same end vertices are called *parallel.* If they are both undirected or both directed with the same orientation, they are called *multiple.* Two directed parallel links that are not multiple form a *pair of oppositely directed links.* A graph is said to be *complete* if it has no loops and if any two different vertices are joined either by exactly one undirected link or by just a pair of oppositely directed links. The undirected complete graph with n vertices will be denoted by K_n. A graph is said to be *finite [infinite]* if it has a finite [an infinite] number of elements (vertices and edges). By a *factor* of a graph G we mean a subgraph of G containing all the vertices of G. Graphs G and H are said to be *isomorphic* if there exists a bijection from the vertex set of G onto the vertex set of H preserving the number (generally, the cardinal number) of both directed (in each orientation) and undirected edges joining any two vertices.

Let v be a vertex of a graph G. Denote by E_1 the (cardinal) number of undirected links incident with v, by E_2 [E_3] the number of links directed from [to] v and by E_4 [E_5] the number of directed [undirected] loops incident with v. Then the following cardinal numbers can be defined:

$\deg_G v = E_1 + E_2 + E_3 + E_4 + 2E_5$ (the *degree* of v in G),
$\mathrm{dd}_G v = E_2 + E_3 + E_4$ (the *directed degree* of v in G),
$\mathrm{ud}_G v = E_1 + 2E_5$ (the *undirected degree* of v in G),
$\mathrm{od}_G v = E_1 + E_2 + E_4 + E_5$ (the *outdegree* of v in G),
$\mathrm{dod}_G v = E_2 + E_4$ (the *directed outdegree* of v in G),
$\mathrm{id}_G v = E_1 + E_3 + E_4 + E_5$ (the *indegree* of v in G),
$\mathrm{did}_G v = E_3 + E_4$ (the *directed indegree* of v in G),
$\mathrm{val}_G v = E_1 + \max\{E_2, E_3\} + E_4 + E_5$ (the *valency* of v in G),
$\mathrm{dv}_G v = \max\{E_2, E_3\} + E_4$ (the *directed valency* of v in G),
$\mathrm{uv}_G v = E_1 + E_5$ (the *undirected valency* of v in G).

The *degree [valency, directed valency, undirected valency]* of a graph G is defined as the supremum of the set of the degrees [valencies, directed valencies, undirected valencies, respectively] of all the vertices of G.

Let d be a cardinal number. A graph G is said to be *regular of degree d* if $\deg_G v = d$ for every vertex v of G. A graph G is said to be *homogeneous of valency d* if $\mathrm{od}_G v = \mathrm{id}_G v = d$ for every vertex v of G. In the case of G being an undirected loopless graph these two concepts coincide.

By a *semitrail* from u to v in a graph G we mean a finite sequence

$$S = [v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n],$$

where n is a non-negative integer (called the length of S); $v_0 = u, v_1, v_2, \ldots, v_{n-1}, v_n = v$ are vertices of G; $e_1, e_2, \ldots, e_n$ are mutually different edges of G and (provided that $n \geqslant 1$) for $i \in \{1, 2, ..., n\}$ v_{i-1} and v_i are the end vertices of e_i. If, moreover, every e_i is either undirected, or directed from v_{i-1} to v_i, then S is said to be a *trail.* A semitrail [trail] whose vertices are mutually different, is called a *semipath [path].* A semitrail [trail] from u to v is said to be a *semicircuit [circuit]* if it has a positive length and if its vertices are mutually different with the exception of $u = v$. The undirected graph consisting of the vertices and the edges of a path with n vertices will be denoted by P_n. The graph consisting of the vertices and the edges of a circuit will be called a *cycle.* A directed [undirected] cycle with n edges will be denoted by $Z_n[C_n]$.

A graph G is said to be a *tree* if for each ordered pair $[u, v]$ of vertices of G there exists in G exactly one semitrail from u to v.

Let u and v be vertices of a graph G. Denote by $\varrho_G(u, v)$ the smallest length of a path from u to v. If there is no such path, we put $\varrho_G(u, v) = \infty$. A graph G is said to be *connected [strongly connected]* if for every ordered pair $[u, v]$ of vertices of G there exists a semipath [path] from u to v. The *diameter* of G is defined as $k_G = \sup \varrho_G(u, v)$, where the supremum is taken through all the ordered pairs $[u, v]$ of vertices of G. If G is not strongly connected, then $k_G = \infty$.

A vertex v of a graph G is said to be a *cutpoint* of G if there exist two different edges e and f of G such that every semitrail containing e and f contains v between e and f. A maximal connected subgraph H of G containing no cutpoint of H is called a *block* of G.

Let p be a positive integer and G be a finite graph with p vertices $v_1, v_2, ..., v_p$. By the *adjacency matrix* of G we mean the $p \times p$ matrix $A = [a_{i,j}]$, where $a_{i,j}$ is the number of edges of G joining v_i and v_j that are either undirected or directed from v_i to v_j $(i, j \in \{1, 2, ..., p\})$.

3. T-graphs

A (partially directed) graph G is said to be a *T-graph* if for each ordered pair $[u, v]$ of vertices of G there exists in G exactly one trail from u to v of a length not greater than the diameter of G. (In [3, 7, 13, 14] T-graphs were called strongly geodetic graphs.) In [7] the following result has been proved:

Lemma 1. *Every undirected T-graph is either a tree or a regular graph with a finite diameter.*

A directed version of this result has been proved in [14]. We shall generalize it for partially directed graphs.

Theorem 1. *Every T-graph is either an undirected tree or a homogeneous graph with a finite diameter.*

Proof. Let G be a T-graph of diameter k. Distinguish three cases:

I. The diameter k is infinite. Let there exist in G an edge e directed from a vertex u to a vertex v. If $v = u$, then there are in G two trails from u to u, namely $[u, e, u]$ and $[u]$. If $v \neq u$, then there is in G a trail from v to u,

$$[v, e_1, v_1, e_2, v_2, \dots, u]$$

so that there are again two trails from u to u, namely

$$[u, e, v, e_1, v_1, e_2, v_2, \dots, u]$$

and $[u]$. As these possibilities lead to a contradiction, we may suppose that all edges of G are undirected. But then the assertion follows from Lemma 1.

II. The diameter $k = 0$ or $k = 1$. Then every T-graph is homogeneous and the assertion holds.

III. The diameter k of G is finite, $k \geqslant 2$. According to Lemma 1 we may suppose that in G there exists a directed edge. We shall prove that every directed edge of G belongs to a directed cycle with $k+1$ edges. Let e be an edge directed from u to v (evidently $v \neq u$). Then there exists a trail from v to u

$$[v, e_1, v_1, \dots, u]$$

of a length $\leqslant k$ so that

$$C = [u, e, v, e_1, v_1, \dots, u]$$

is a circuit of a length $\leqslant k+1$. Obviously C cannot contain an undirected edge as then its end vertices would be connected by two trails of lengths $\leqslant k$. As G is a T-graph, G cannot have a directed circuit of a length $\leqslant k$. Hence C has the length $k+1$. Thus e lies in a directed cycle consisting of elements of C and this cycle has $k+1$ edges.

Now we shall prove that if x is any vertex of G, then $\mathrm{id}_G\, x = \mathrm{od}_G\, x$. Since undirected edges have no influence upon the validity of this equality, we may restrict ourselves to directed edges. (In fact, we shall prove that $\mathrm{did}_G\, x = \mathrm{dod}_G\, x$.) Every edge directed from x, as we have proved, lies in a directed cycle with $k+1$ edges. As G is a T-graph, no edge can be contained in two such cycles so that they are edge-disjoint. In each of the cycles there is an edge directed to x so that $\mathrm{od}_G\, x \leqslant \mathrm{id}_G\, x$. Analogously the inequality $\mathrm{id}_G\, x \leqslant \mathrm{od}_G\, x$ can be proved. Hence $\mathrm{id}_G\, x = \mathrm{od}_G\, x$.

Let y and z be vertices of G such that $\varrho_G(y, z) = k$. We shall prove that $\mathrm{od}_G\, y = \mathrm{id}_G\, z$. Let

$$[y, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, z]$$

be a path of the length k from y to z. Let $y_1, y_2, \dots, y_s$ be all the vertices of G different from v_1 such that there exists an edge directed from y to one of them or an undirected edge joining y to one of them. Evidently for $i \in \{1, 2, \dots, s\}$ we have

$\varrho_G(y_i, z) = k$ and no two of the corresponding paths of length k have a vertex different from z in common. Therefore $\mathrm{od}_G\, y \leqslant \mathrm{id}_G\, z$. Analogously, considering the edges directed to z, the inequality $\mathrm{od}_G\, y \geqslant \mathrm{id}_G\, z$ can be obtained. Consequently, $\mathrm{od}_G\, y = \mathrm{id}_G\, z$.

Let e be an edge of G directed from a vertex U to a vertex V. Then evidently $\varrho_G(V, U) = k$ so that according to what has been just proved $\mathrm{id}_G\, V = \mathrm{od}_G\, V = \mathrm{id}_G\, U = \mathrm{od}_G\, U$. The edge e lies in a circuit $C(e)$ of the length $k+1$ with all edges directed. Let the vertices of $C(e)$ be $v_0 = U$, $v_1 = V$, $v_2, \ldots, v_k$ such that there exists an edge directed from v_{i-1} to v_i $(i = 1, 2, \ldots, k)$ and an edge directed from v_k to v_0. Using the preceding results we get

$$\mathrm{id}_G\, v_i = \mathrm{od}_G\, v_i = \mathrm{id}_G\, V$$

for each $i \in \{0, 1, \ldots, k\}$.

Let w be a vertex of G not contained in $C(e)$. Let v_j be the vertex of $C(e)$ such that

$$\varrho_G(w, v_j) \leqslant \varrho_G(w, v_i)$$

for all $i = 0, 1, 2, \ldots, k$. Put $\varrho_G(w, v_j) = J$, $1 \leqslant J \leqslant k$. Then the vertex v_{j+k-J} has the property that

$$\varrho_G(w, v_{j+k-J}) = k,$$

where the index $j + k - J$ is to be taken modulo $k+1$. From the preceding parts of the proof it follows that

$$\mathrm{id}_G\, w = \mathrm{od}_G\, w = \mathrm{id}_G\, V.$$

Thus G is a homogeneous graph. Q.E.D.

A homogeneous T-graph of valency d and with a finite diameter k will be called a tied graph of type (d, k). Thus from Theorem 1 it follows:

Corollary 1. *A graph G is a T-graph if and only if G is either an undirected tree or a tied graph.*

Evidently, we have:

Lemma 2. *A tied graph of type (d, k) is finite if and only if its valency d is finite.*

Concerning infinite tied graphs, the following result is known:

Proposition 1. *For every infinite cardinal number d and finite cardinal number k there exists an undirected [directed] tied graph of type (d, k).*

The undirected version of this result has been proved in [3, 7, 11], the directed one in [14]. Using the method of these papers it is easy to prove that the following version of Proposition 1 is true:

Proposition 2. *For every infinite cardinal number d and a finite cardinal number k there exists a mixed tied graph of type (d, k).*

In view of these results we shall be concerned now only with finite tied graphs. A finite tied graph of type (d, k) will be called also a Moore graph of type (d, k). (For undirected Moore graphs see [1, 2, 8, 10, 12, 13].)

4. Moore graphs of diameter two

Theorem 2. *Let G be a Moore graph of diameter two. Then there exist non-negative integers z and r satisfying the following conditions:*

A. $\operatorname{dod}_G v = \operatorname{did}_G v = z$ *for every vertex* v *of* G.

B. $\operatorname{uv}_G v = r$ *for every vertex* v *of* G.

C. *The number of vertices of* G *is*

$$p = (z+r)^2 + z + 1 \geqslant 3 \tag{1}$$

and exactly one of the following cases occurs:

I. $z=1,\ r=0,\ p=3$.

II. $z=0,\ r=2,\ p=5$.

III. *There is an odd positive integer* c *such that*

$$c \mid (4z-3)(4z+5) \tag{2}$$

and

$$r = \tfrac{1}{4}(c^2+3). \tag{3}$$

Proof. G is a homogeneous graph. Therefore there exists a non-negative integer d such that

$$\operatorname{val}_G v = \operatorname{od}_G v = \operatorname{id}_G v = d$$

for every vertex v of G. Thus

$$\operatorname{dv}_G v = \operatorname{dod}_G v = \operatorname{did}_G v$$

for every vertex v of G. Put

$$z(v) = \operatorname{dv}_G v,$$
$$r(v) = d - z(v).$$

Using trails of lengths 0, or 1, or 2 beginning with v we can arrive at 1, or d, or $r(v)(d-1) + z(v)d$ vertices of G, respectively. Since G is a T-graph of diameter two, we have

$$p = 1 + d + r(v)(d-1) + z(v)d$$

so that

$$p = 1 + d + d^2 - r(v). \tag{4}$$

As p and d do not depend on v, the same is valid for

$$r(v)=d^2+d+1-p$$

and

$$z(v)=d-r(v).$$

Put $r(v)=r$, $z(v)=z$. Thus A and B hold. Moreover, substituting $r(v)=r$ and

(5) $$d=z+r$$

in (4) we obtain (1) (evidently $p\geqslant 3$ as G has diameter 2).

Let A be the adjacency matrix of G. Then

$$A^2+A-(r-1)I=J,$$

where I is the identity matrix of order p and J is the matrix of order p each entry of which is 1. The eigenvalues of J are

$$\lambda_1=\lambda_2=\ldots=\lambda_{p-1}=0, \quad \lambda_p=p.$$

Thus A has the eigenvalues $\mu_1, \mu_2, \ldots, \mu_p$ such that $\mu_1, \mu_2, \ldots, \mu_{p-1}$ satisfy the equation

(6) $$x^2+x-(r-1)=0$$

and μ_p satisfies the equation

$$x^2+x-(r-1)=p.$$

As G is a homogeneous graph of valency $d=z+r$, A has an eigenvalue d. From (1) it follows that $d=\mu_p$. The roots of (6) are

(7) $$s=\frac{-1+\sqrt{4r-3}}{2}, \quad t=\frac{-1-\sqrt{4r-3}}{2}.$$

(For $r=0$ we put $\sqrt{4r-3}=\mathrm{i}\sqrt{3}$.) Thus A has at most 3 different eigenvalues, namely d, s and t. The eigenvalue d is, according to the preceding considerations, simple (cf. [16]). Denote the multiplicity of s or t in A by a or b, respectively. Thus

(8) $$a+b+1=p.$$

As the trace of A is zero, the sum of the eigenvalues of A is

(9) $$as+bt+d=0.$$

Substituting (7) into (9) gives

(10) $$a+b+\sqrt{4r-3}(b-a)=2d.$$

Distinguish two cases:

Case 1. $b=a$. Then from (10) and (8) it follows that $2d=a+b=p-1$ so that

$$p=2d+1. \tag{11}$$

Since $p \geqslant 3$, we have

$$d \geqslant 1. \tag{12}$$

Using (11), (1) and (5) we get

$$2d+1=p=(z+r)^2+z+1=d^2+d-r+1$$

so that

$$r=d^2-d. \tag{13}$$

However, $r \leqslant d$. Hence $d^2-d \leqslant d$ and (12) implies

$$1 \leqslant d \leqslant 2. \tag{14}$$

According to (14), (13), (5) and (1) we have only two possibilities:

I. $d=1$, $r=0$, $z=1$, $p=3$.

II. $d=2$, $r=2$, $z=0$, $p=5$.

Case 2. $b \neq a$. Put

$$c=\sqrt{4r-3}. \tag{15}$$

Then (3) holds. From (10) it follows that $r \neq 0$, hence $r \geqslant 1$ and $c \geqslant 0$. By (10) c is rational. As $4r-3$ is an integer, c is an integer, too. From (10), (8) and (15) it follows that

$$p-1-2d+c(p-2a-1)=0. \tag{16}$$

If in (16) we successively substitute

$$\begin{aligned} p&=d^2+d-(r-1),\\ d&=r+z,\\ r&=\tfrac{1}{4}(c^2+3), \end{aligned}$$

we get after simple calculations

$$c^5+c^4+c^3(8z+6)+c^2(8z-2)+c(16z^2+40z-32a+9)+ \\ +(16z^2+8z-15)=0.$$

Since

$$16z^2+8z-15=(4z-3)(4z+5)$$

and c is an integer, we get (2). Consequently, c is an odd integer. Q.E.D.

Remark. If we put in Theorem 2 $z=0$ or $r=0$, respectively, we easily obtain the following two well-known results.

Corollary 2 [10]. *Any undirected Moore graph of diameter two has* 5, 10, 50 *or* 3250 *vertices.*

Corollary 3 [14]. *A directed graph is a Moore graph of diameter two if and only if it is isomorphic to* Z_3.

Theorem 3. *Let there exist a Moore graph G of diameter two with p vertices. Then the following invariants* d, z, r, D, s, t, a, b *of G are uniquely determined by* p:

(17) $d=[\sqrt{p}]$ *(the valency of G),*

(18) $z=p-d^2-1$ *(the directed valency of G),*

(19) $r=d-z$ *(the undirected valency of G),*

(20) $D=2z+r$ *(the degree of G),*

(21) $s=\dfrac{-1+c}{2}$,

(22) $t=\dfrac{-1-c}{2}$ *(the eigenvalues of the adjacency matrix of G different from d),*

(23) $a=\dfrac{\frac{1}{2}(c+1)(p-1)-d}{c}$ *(the multiplicity of s),*

(24) $b=p-1-a$ *(the multiplicity of t),*

where

(25) $c=\sqrt{4r-3}$.

(*We put* $\sqrt{-3}=\mathrm{i}\sqrt{3}$.)

Proof. Denote the valency, directed valency, undirected valency and the degree of G by d, z, r and D, respectively. Evidently, (19) and (20) hold. According to (1) we have

$$p=d^2+z+1.$$

Therefore we get (18) and the inequalities

$$d^2<d^2+z+1\leq d^2+d+1<d^2+2d+1=(d+1)^2$$

imply

$$d<\sqrt{p}<d+1.$$

Hence we obtain (17). Define c by (25). Then (21) and (22) follow from (7). Solving the system of the two equations (8) and (10) we get (23) and (24). Q.E.D.

Theorem 4. *Let there exist a Moore graph of diameter two with p vertices. Then either $p \in \{3, 5\}$ or p is an even integer of the form*

$$p = d(d+1) - s(s+1) \geq 6, \tag{26}$$

where d and s are integers, $d \geq 2$ (d is the valency of the graph and the greatest eigenvalue of its adjacency matrix), $s \geq 0$ (s is the second greatest eigenvalue),

$$s(s+1) < d \tag{27}$$

and

$$2s+1 \mid (2d+1)(2d-3). \tag{28}$$

Proof. We use the same notation as in the preceding theorems. Thus p, d, z and r are integers, $p \geq 3$, $0 \leq r \leq d$, $0 \leq z \leq d$, $d = z + r$. According to (1) we have

$$p = d^2 + d - (r-1).$$

From (3) and (21) it follows that

$$r - 1 = \tfrac{1}{4}(c^2 + 3) - 1 = s(s+1) \tag{29}$$

and we get (26).

According to Theorem 2 either $p \in \{3, 5\}$ or c is an odd positive integer and by (21) s is a non-negative integer. But then from (26) we get that p is an even integer. Evidently, $p \neq 2$. If $p = 4$, then (17) and (18) give $d = 2$, $z = -1$, a contradiction. Therefore $p \geq 6$. Now (17) implies $d \geq 2$. From (29) we get

$$s(s+1) = r - 1 < r \leq d$$

and (27) holds. It remains to prove (28). We shall use (2). From (21) we get

$$c = 2s + 1.$$

As $z = d - r = d - 1 - s(s+1)$, we have

$$(4z-3)(4z+5) = (4z+1)^2 - 16 = (4d - 3 - 4s(s+1))^2 - 16.$$

However,

$$4s(s+1) = (2s+1)^2 - 1 \equiv -1 \pmod{2s+1}.$$

Thus

$$\begin{aligned}(4z-3)(4z+5) &\equiv (4d-3+1)^2 - 16 = \\ &= 4(2d+1)(2d-3) \pmod{2s+1}.\end{aligned}$$

Therefore (2) is equivalent to

$$2s+1 \mid 4(2d+1)(2d-3).$$

Since $2s+1$ is relatively prime to 4, we get (28). Q.E.D.

Remark. If $s=0$, the conditions (27) and (28) always hold. Checking (28) can be simplified in the case when $2s+1$ is a prime power (i.e., if $s \in \{1, 2, 3, 5, 6, 8, 9, 11, 14, \ldots,\}$):

Proposition 3. *Let d and s be integers and let $2s+1$ be a power of a prime. Then (28) holds if and only if*

$$d \equiv s \quad \text{or} \quad s+2 \pmod{2s+1}.$$

Proof. (28) can be written in the form

$$(2d+1)(2d-3) \equiv 0 \pmod{2s+1}.$$

Let $2s+1$ be a power of a prime p. Evidently, $p \neq 2$. As

$$(2d+1)-(2d-3)=4$$

is not divisible by p, we have either

$$2d+1 \equiv 0 \pmod{2s+1},$$

or

$$2d-3 \equiv 0 \pmod{2s+1}.$$

The former congruence has the unique solution $d \equiv s$, the latter $d \equiv s+2 \pmod{2s+1}$. Q.E.D.

We need the following result proved in [10] and closely related to Corollary 2:

Lemma 3. *Undirected Moore graphs of diameter two with p vertices for $p \in \{5, 10, 50\}$ exist and are unique up to isomorphism.*

The Moore graph of diameter two with 5 vertices is C_5 (pentagon), with 10 vertices is the Petersen graph, with 50 is the Hoffman-Singleton graph (see [10, 2]).

Theorem 5. *A Moore graph of diameter two exists for every $p \in \{3, 5, 10, 18, 50\}$ and every $p=d(d+1)$, where d is an integer, $d \geqslant 2$.*

Proof. For $p=3$ it is sufficient to take the directed cycle Z_3 (cf. Corollary 3). The cases $p \in \{5, 10, 50\}$ are included in Lemma 3.

Let $p=18$. We construct a Moore graph M of diameter two with 18 vertices as follows: The vertices of M are the residue classes 1, 2, 3, ..., 18 modulo 18. An arbitrary vertex x is incident with one edge directed from x to a vertex

$$\operatorname{dir} x = \begin{cases} x+1, & \text{if} \quad x \equiv 1 \quad \text{or} \quad 2 \pmod 6; \\ x-2, & \text{if} \quad x \equiv 3 \pmod 6; \\ x+2, & \text{if} \quad x \equiv 4 \pmod 6; \\ x-1, & \text{if} \quad x \equiv 5 \quad \text{or} \quad 6 \pmod 6) \end{cases}$$

and with three undirected edges joining x with the vertices

$$x+3, \quad x-3, \quad \operatorname{dir} x+9.$$

The graph M (decomposed into two factors) is given in Fig. 1. (Another representation of M is given in [4].)

Finally, let $p = d(d+1)$, $d \geqslant 2$. We shall construct a Moore graph $B(d)$ of type $(d, 2)$. (For $d=3$ see Fig. 2.) The vertices of $B(d)$ are ordered pairs $[x, y]$ of

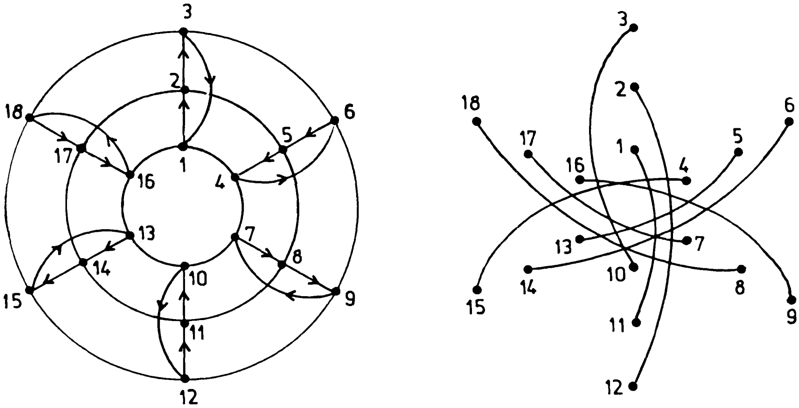

Fig. 1. A Moore graph M with 18 vertices

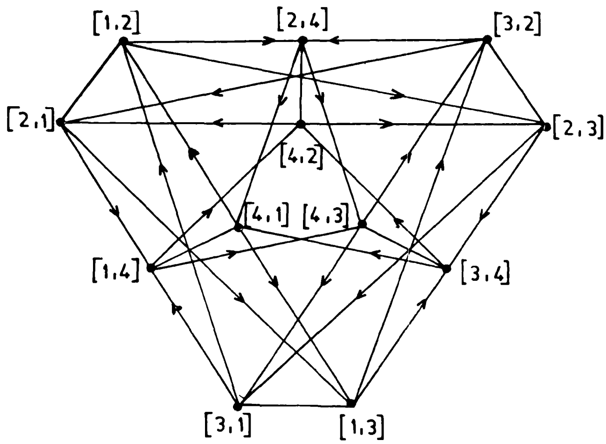

Fig. 2. A Moore graph $B(3)$ with 12 vertices

integers such that $1 \leqslant x \leqslant d+1$, $1 \leqslant y \leqslant d+1$, $x \neq y$. An arbitrary vertex $[x, y]$ is joined with a vertex $[X, Y]$ by an undirected edge if and only if $X = y$, $Y = x$ and by an edge directed from $[x, y]$ to $[X, Y]$ if and only if $X = y$, $Y \neq x$. (The graph $B(d)$ is isomorphic to the graph $B^+(d, 1)$ from [4].)

It is easy to check that all these graphs are Moore graphs of diameter two. Q.E.D.

Table 1

p	d	z	r	D	c	s	t	a	b	example
3	1	1	0	2	$i\sqrt{3}$	ε	$\bar{\varepsilon}$	1	1	Z_3
5	2	0	2	2	$\sqrt{5}$	α	β	2	2	C_5
6	2	1	1	3	1	0	–1	3	2	$B(2)$
10	3	0	3	3	3	1	–2	5	4	Petersen graph
12	3	2	1	5	1	0	–1	8	3	$B(3)$
18	4	1	3	5	3	1	–2	10	7	M
20	4	3	1	7	1	0	–1	15	4	$B(4)$
30	5	4	1	9	1	0	–1	24	5	$B(5)$
40	6	3	3	9	3	1	–2	24	15	unknown
42	6	5	1	11	1	0	–1	35	6	$B(6)$
50	7	0	7	7	5	2	–3	28	21	Hoffman-Singleton graph
54	7	4	3	11	3	1	–2	33	20	unknown
56	7	6	1	13	1	0	–1	48	7	$B(7)$
72	8	7	1	15	1	0	–1	63	8	$B(8)$
84	9	2	7	11	5	2	–3	48	35	unknown
88	9	6	3	15	3	1	–2	55	32	unknown
90	9	8	1	17	1	0	–1	80	9	$B(9)$

A survey of "small" Moore graphs ($p \leqslant 100$) is given in Table 1, where

$$\varepsilon = \frac{-1 + i\sqrt{3}}{2}, \qquad \bar{\varepsilon} = \frac{-1 - i\sqrt{3}}{2},$$

$$\alpha = \frac{-1 + \sqrt{5}}{2}, \qquad \beta = \frac{-1 - \sqrt{5}}{2}.$$

Problem 1. *Are there two non-isomorphic Moore graphs of diameter two with the same number of vertices?*

Problem 2. *Are there Moore graphs of diameter two with* 40, 54, 84, *or* 88 *vertices?*

5. Moore graphs of diameter $k \neq 2$

The cases $k = 0$ or 1 are not interesting as we obviously have:

Lemma 4. *Every finite graph of diameter* 0 *is a Moore graph.*

Lemma 5. *A finite graph G of diameter* 1 *is a Moore graph if and only if G is loopless and simple (so that G is a complete graph).*

For diameters $k \geqslant 3$ the following results are known:

Lemma 6. [1, 8]. *An undirected graph of a finite diameter $k \geqslant 3$ is a Moore graph if and only if it is isomorphic to C_{2k+1}.*

Lemma 7 [14]. *A directed graph of a finite diameter $k \geqslant 3$ is a Moore graph if and only if it is isomorphic to Z_{k+1}.*

From Theorem 1, Corollary 3, Proposition 1, and Lemmas 2, 4, 5 and 7 it immediately follows:

Corollary 4 [14]. *A directed graph G is a T-graph if and only if one of the following cases occurs:*

(i) *G consists of a single vertex and an arbitrary number of loops.*
(ii) *G is a complete directed graph.*
(iii) *G is a directed cycle.*
(iv) *G is an infinite directed tied graph.*

Problem 3. *Is there a mixed Moore graph of diameter $k \geqslant 3$?*

6. P-graphs

A (partially directed) graph G is said to be a *P-graph* if for each ordered pair $[u, v]$ of vertices of G there exists in G exactly one path from u to v of a length not greater than the diameter of G.

A graph G is said to be a quasitree if for each ordered pair $[u, v]$ of vertices of G there exists exactly one path from u to v.

Theorem 6. *A graph G is a quasitree if and only if G is connected and every block of G is isomorphic to K_2, C_1, or a directed cycle.*

Proof. Let G be a connected graph whose every block is K_2, C_1, or a directed cycle. Then evidently G is a quasitree.

Conversely, let G be a quasitree. If we replace every undirected loop of G by a directed loop and every undirected link of G by a pair of oppositely directed links, we get a directed graph G^* that is again a quasitree. Therefore every edge of G^* is contained in exactly one (directed) circuit.

Evidently, it now suffices to prove that every block of G^* is a (directed) cycle. Let B be a block of G^* with n vertices. If $n = 1$, then B is isomorphic to Z_1. Therefore, let $n \geqslant 2$. Then B has no loops. Let C be a cycle contained in B. Evidently, every edge joining two vertices of C lies in C. Suppose that there exists a vertex v of B that does not lie in C. As B is a block, there exists a semipath $[u_0, e_1, u_1, e_2, \ldots, u_r = v, \ldots, u_{s-1}, e_s, u_s]$, where $s \geqslant 2$, $1 \leqslant r \leqslant s-1$, u_0 and u_s are in C, but $u_1, u_2, \ldots, u_{s-1}$ are not in C. Without loss of generality we may suppose that e_1 is directed from u_0 to u_1. Each e_i, $i \in \{2, 3, \ldots, t\}$ lies in a cycle of B. Therefore in B

there exists a path $[v_1 = u_1, f_1, v_2, \ldots, v_{t-1}, f_t, v_t = u_s]$. But then in B there are two paths from u_0 to u_s, the first is inside C and the second one is

$$[u_0, e_1, u_1 = v_1, f_1, v_2, \ldots, v_{t-1}, f_t, v_t = u_s].$$

However, this is a contradiction to the definition of a quasitree. Q.E.D.

Graphs G and H are said to be similar if deleting all the loops and replacing every undirected link by a pair of oppositely directed links in both G and H yields two isomorphic (directed) graphs. We obviously have:

Proposition 4. 1. *Every graph similar to a T-graph is a P-graph.*
2. *Every quasitree is a P-graph.*

Problem 4. *Is there a P-graph that is neither a quasitree nor a graph similar to a T-graph?*

Evidently, we have:

Lemma 8. *Let G be a loopless undirected graph. Then G is a P-graph if and only if G is a T-graph.*

From Lemmas 2—6, 8 and Corollaries 1—2 we immediately have:

Corollary 5 (cf. [7], Theorem 3). *A looples undirected graph G is a P-graph (or, equivalently, a T-graph) if and only if one of the following cases occurs:*
(i) *G is a tree.*
(ii) *G is a complete graph.*
(iii) *G is a cycle with an odd number of edges.*
(iv) *G is the Petersen graph.*
(v) *G is the Hoffman-Singleton graph.*
(vi) *G is a Moore graph of type* (57, 2).
(vii) *G is an infinite tied graph.*

Remark 1. The existence and uniqueness of graphs (vi) from Corollary 5 is an open problem (see [10, 13]).

Remark 2. The condition "loopless" in Corollary 5 is not an essential restriction as a graph G is a P-graph if and only if deleting all loops from G yields a P-graph. On the other hand, a T-graph with a non-zero diameter cannot have loops.

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ЧАСТИЧНО ОРИЕНТИРОВАННЫЕ ГРАФЫ МУРА

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Резюме

(Частично ориентированный) граф G называется T-графом [P-графом] если для всякой упорядоченной пары $[u, v]$ его вершин существует в G точно одна цепь [точно один путь, соответственно] длины непревышающей диаметр графа G. Граф G называется однородным валентности d, если внешняя и внутренняя степень всякой вершины равны d. Показано, что всякий T-граф является или неориентированным деревом, или однородным графом конечного диаметра – завязанным графом. Конечные завязанные графы называются графами Мура. В работе изучаются свойства P-графов, T-графов и графов Мура. Главные результаты касаются вопросов существования графов Мура диаметра два – кроме других результатов – построен бесконечный класс таких графов.