Anton Škerlík

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OSCILLATION THEOREMS FOR THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

ANTON ŠKERLÍK

ABSTRACT. The oscillation criterion for the equation

$$\big(r_2(t)\big(r_1(t)y'\big)'\big)' + p(t)y' + q(t)f(y) = 0$$

with nonnegative coefficients p and q is established. This result generalizes some oscillation criteria for third order nonlinear differential equations.

1. Introduction

This paper is concerned with the oscillatory behaviour of solutions of a third order nonlinear differential equation of the form

$$\big(r_2(t)\big(r_1(t)y'\big)'\big)' + p(t)y' + q(t)f(y) = 0\,, \tag{QF}$$

where $f\colon \mathbb{R} \to \mathbb{R} = (-\infty,\infty)$, $r_2, r_1, p, q\colon I \to [0,\infty)$, $I = [a,\infty) \subset \mathbb{R}$ are continuous, $r_2 > 0$, $r_1 > 0$, $q(t)$ not identically zero on any ray of the form $[t^*,\infty)$ for some $t^* \ge a > 0$ and $xf(x) > 0$ for $x \ne 0$.

We restrict our attention to those solutions of equation (QF) which exist on I and satisfy the condition

$$\sup\big\{|y(t)|;\ T \le t < \infty\big\} > 0 \qquad \text{for any} \quad T \in I\,.$$

Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory.

In paper [4] the oscillation theorem for a linear differential equation has been presented

$$y''' + p(t)y' + q(t)y = 0\,. \tag{L}$$

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THEOREM A. (*Theorem* 3.1 *in* [4]) *If* $p \geq 0$, $q \geq 0$, $2q - p' \geq 0$ *and not identically zero in any interval and there exists a number* $m < \frac{1}{2}$ *such that the second order differential equation*

$$z'' + [p(t) + mtq(t)]z = 0$$

is oscillatory, then (L) *has oscillatory solutions. In fact, if* y *is any nonzero solution of* (L) *with*

$$0 \geq F[y(c)] = [2y(t)y''(t) - y'^2(t) + p(t)y^2(t)]_{t=c}$$

for some $c \geq a$, *then* y *is oscillatory.*

The partial generalization of this theorem on third order nonlinear differential equations was presented in [3, 11, 14, 15, 16, 17] and others.

L. Erbe generalized Theorem A on the equation

$$y''' + r(t)y'' + p(t)y' + q(t)y^\alpha = 0, \tag{A}$$

where $\alpha > 0$ is the quotient of odd positive integers and $r\colon I \to [0,\infty)$ is continuous.

THEOREM B. (*Theorem* 4.9 *in* [2]) *Let* $r \geq 0$, $p \geq 0$, $q > 0$ *and* $rp + p' \leq 0$. *Let* y *be a nontrivial solution of* (A) *with* $F[y(c)] \leq 0$ *for some* $c > a$, *where*

$$F[y(t)] = R(t)[2y''(t)y(t) - y'^2(t) + p(t)y^2(t)],$$

$R(t) = \exp\left(\int_c^t r(s)\,\mathrm{d}s\right)$. *Assume further that the equation*

$$(R(t)z')' + R(t)[p(t)z + \lambda^\alpha t^\alpha q(t)z^\alpha] = 0 \tag{B}$$

is oscillatory (*that is, all solutions of* (B) *are oscillatory*) *for some* $0 < \lambda < \frac{1}{2}$. *Then* y *is oscillatory.*

It is therefore natural to ask whether the above results can be extended on more general differential equations than the equations (L) and (A).

Such a extension is possible for equations

$$\big(r_2(t)(r_1(t)y')'\big)' + p(t)y' + q(t)y = 0$$

and

$$\big(r_2(t)(r_1(t)y')'\big)' + p(t)y' + q(t)y^\alpha = 0,$$

where $r_2 \equiv r_1$, since we make use a change of variable to transform these equations into equations of the form (L) or (A), respectively, (see [6], also see [8]). In general, for $r_2 \neq r_1$, such a change of variable does not exist. The purpose of this paper is to answer the question above in the affirmative, also see a similar open question of Philos and Sficas [7, Remark 7]. The methods used patterns after those of Lazer [4], Erbe [2] and Waltman [17].

2. Basic lemma

For the sake of brevity, we denote

$$L_0y(t) = y(t), \quad L_iy(t) = r_i(t)\big(L_{i-1}y(t)\big)', \qquad i = 1,2, \qquad (1)$$
$$L_3y(t) = \big(L_2y(t)\big)' \qquad \text{for} \quad t \in I.$$

So the equation (QF) can be written as

$$L_3y + p(t)y' + q(t)f(y) = 0.$$

Remark 1. If y is solution of (QF), then $z = -y$ is a solution of the equation

$$L_3z + p(t)z' + q(t)f^*(z) = 0.$$

where $f^*(z) = -f(-z)$ and $zf^*(z) > 0$ for $z \neq 0$.

DEFINITION 1. *Let y be a solution of (QF). We say that the solution y has property V_2 on $[T,\infty)$, $T \geq a$ if and only if*

$$L_0y(t)L_ky(t) > 0, \quad k = 0,1,2; \qquad L_0y(t)L_3y(t) \leq 0 \qquad (2)$$

for every $t \in [T,\infty)$.

Define the functions

$$R_2(t,T) = \int_T^t \frac{\mathrm{d}s}{r_2(s)}, \qquad R_{12}(t,T) = \int_T^t \frac{R_2(s,T)}{r_1(s)}\,\mathrm{d}s, \qquad (3)$$

$a \leq T \leq t < \infty$.

We assume that

$$R_2(t,a) \to \infty \qquad \text{as} \quad t \to \infty. \qquad (4)$$

LEMMA 1. *Let the assumption (4) hold and y be a non-oscillatory solution of (QF) such that $y(t)L_1y(t) \geq 0$ for every $t \geq T \geq a$. Then y has property V_2 for all large t.*

Proof. Suppose without loss of generality that $y(t) > 0$, $L_1y(t) \geq 0$, $t \geq T$ (see Remark 1). From the equation (QF) we see that $L_3y(t) \leq 0$, $t \geq T$, and L_3y not identically zero on any ray on the form $[t^*,\infty)$ for some $t^* \geq T$. So either y has property V_2 for large t or there exists a point $t_0 \geq T$ such that $L_2y(t_0) = A < 0$. Hence $\big(L_1y(t)\big)' \leq A/r_2(t)$, $t \geq t_0$, (see (1)) and by integration of this inequality we obtain $L_1y(t) < 0$ for large t, a contradiction.

Lemma 2. *Let* (4) *hold. Suppose that* r_2/r_1 , $p \in C^1(I,\mathbb{R})$ *and*

$$p'(t) \le 0\,, \quad [r_2/r_1(t)]' \ge 0 \qquad \text{for} \quad t \ge a\,. \tag{5}$$

Let y *be a solution of* (QF) *and assume further that there exists* $t_0 \in I$ *such that* $F\big[y(t_0)\big] \le 0$, *where*

$$F\big[y(t)\big] = 2y(t)L_2y(t) - \frac{r_2(t)}{r_1(t)}\big(L_1y(t)\big)^2 + p(t)y^2(t)\big)\,. \tag{6}$$

Then either y *is oscillatory or* y *has property* V_2 *for every large* t .

P r o o f . Let y be a nonoscillatory solution of (QF) satisfying the condition $F\big[y(t_0)\big] \le 0$ for some $t_0 \ge a$. Suppose without loss of generality that $y(t) > 0$ for every $t \ge T > t_0$. Then a calculation shows that

$$\Big[F\big[y(t)\big]\Big]' = -\big[r_2(t)/r_1(t)\big]'\big(L_1y(t)\big)^2 + p'(t)y^2(t) - 2q(t)f\big(y(t)\big)y(t) \le 0\,, \qquad t \ge T\,.$$

So there exists a point $t_0^* \ge T$ such that $F\big[y(t)\big] < 0$ for every $t \ge t_0^*$ and $\lim\limits_{t\to\infty} F\big[y(t)\big] = F_0 < 0$ exists (finite or infinite). From (6) we obtain

$$\begin{aligned} 2r_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{L_1y(t)}{y(t)}\right] &= \frac{2}{y^2(t)}\left[y(t)L_2y(t) - \frac{r_2(t)}{r_1(t)}\big(L_1y(t)\big)^2\right] \\ &\le \left[2y(t)L_2(t) - \frac{r_2(t)}{r_1(t)}\big(L_1y(t)\big)^2\right]y^{-2}(t) < -p(t) \le 0\,, \end{aligned} \tag{7}$$

$t \ge t_0^*$. Hence the function L_1y/y is decreasing on $[t_0^*,\infty)$. This means that either $L_1y(t) > 0$, $t \ge t_0^*$, and by Lemma 1 y has property V_2 or there exists some $T_1 \ge t_0^*$ such that $L_1y(t) < 0$ for $t \ge T_1$. We shall prove that the case $y(t) > 0$, $L_1y(t) < 0$, $t \ge T_1$, is contradictory to assumptions of Lemma 2.

Let $y(t) > 0$, $L_1y(t) < 0$, $t \ge T_1$. From (4) and (5) we have $r_2(t) \ge Ar_1(t)$, $t \ge a$, where $A = r_2(a)/r_1(a) > 0$ and so

$$\begin{gathered} \lim_{t\to\infty} R_1(t,a) = \lim_{t\to\infty}\int_a^t \frac{\mathrm{d}s}{r_1(s)} = \infty\,, \\ r_2(t)y'(t) \le AL_1y(t) < 0\,, \qquad t \ge T_1\,. \end{gathered} \tag{8}$$

We consider the function L_2y. The case $L_2y(t) \le 0$ cannot hold for all large t, say $t \ge T_2 \ge T_1$, since by integration of inequality $y'(t) \le L_1y(T_2)/r_1(t)$, $t \ge T_2$ we obtain from (8) $y(t) < 0$ for all large t, a contradiction.

Let $y(t) > 0$, $L_1y(t) < 0$, $L_2y(t) \ge 0$ for all large t, say $t \ge T_3 \ge T_1$. We assert that $\lim_{t\to\infty} L_1y(t) = \limsup_{t\to\infty} r_2(t)y'(t) = 0$. (If $\limsup_{t\to\infty} r_2(t)y'(t) < 0$, i.e. there exist numbers $B < 0$ and $T_4 \ge T_3$ such that $r_2(t)y'(t) \le B$, $t \ge T_4$, then integrating the inequality $y'(t) \le B/r_2(t)$, $t \ge T_4$, yields a contradiction for all large t). Otherwise, a calculation shows that

$$0 > F_0 = \lim_{t\to\infty} F\big[y(t)\big] = \limsup_{t\to\infty}\big[2y(t)L_2y(t) - r_2(t)y'(t)L_1y(t) + p(t)y^2(t)\big]$$
$$= \limsup_{t\to\infty}\big[2y(t)L_2(t) + p(t)y^2(t)\big] \ge 0,$$

a contradiction.

Finally, $y(t) > 0$, $L_1y(t) < 0$, $t \ge T_1$ and L_2y changes the sign for arbitrarily large t. Denote

$$G(t) = \big(r_2(t)y'(t)\big)' = \left(\frac{r_2(t)}{r_1(t)}\right)' L_1y(t) + \frac{L_2y(t)}{r_1(t)}, \qquad t \ge T_1.$$

The function G cannot be nonpositive on $[T_2, \infty)$ for some $T_2 \ge T_1$, since there is $r_2(t)y'(t) \le r_2(T_2)y'(T_2) < 0$, $t \ge T_2$, and from (4) we obtain a contradiction with positivity of y for all large t. Let G change the sign. Hence there exists an unboundary sequence of zeros of the function G. Choose a sequence (t_n), $n = 1, 2, \dots$ from the set of zeros of G (i.e. $G(t_n) = 0$) such that $a_1 \le a_2 \le \dots \le a_n \le \dots$, where $a_n = r_2(t_n)y'(t_n)$, $n = 1, 2, \dots$ are nondecreasing relative maxima of r_2y'. It is clear that $\lim_{n\to\infty} a_n = 0$. (If $\lim_{n\to\infty} a_n = a_0 < 0$, i.e. $r_2(t)y'(t) \le a_0$ for all $t \ge t_1$, then we obtain again a contradiction.) From (8) it follows that $\lim_{n\to\infty} L_1y(t_n) = 0$. We see that $L_2y(t_n) \ge 0$ since $G(t_n) = \Big[\big(r_2(t)/r_1(t)\big)'L_1y(t) + L_2y(t)/r_1(t)\Big]_{t=t_n} = 0$. So a calculation shows that

$$\begin{aligned} 0 > F_0 &= \lim_{t\to\infty} F\big[y(t)\big] = \lim_{n\to\infty} F\big[y(t_n)\big] \\ &= \lim_{n\to\infty}\big[2y(t_n)L_2y(t_n) - r_2(t_n)y'(t_n)L_1y(t_n) + p(t_n)y^2(t_n)\big] \\ &= \lim_{n\to\infty}\big[2y(t_n)L_2y(t_n) + p(t_n)y^2(t_n)\big] \ge 0, \end{aligned}$$

a contradiction, too. This completes the proof of Lemma 2.

DEFINITION 2. *The equation* (QF) *is called weak superlinear if the function* f *has the property:*

for any $u \neq 0$ *there exists a number* $m > 0$ *such that* $uf(u) \geq mu^2$.

Let us note that any linear equation is weak superlinear with $m = 1$.

Remark 2. If the equation (QF) is weak superlinear (linear, $m = 1$), then the condition $p' \leq 0$ in (5) of the Lemma 2 may be replaced by a weaker condition $2mq(t) - p'(t) \geq 0$, $t \in I$ and $2mq(t) - p'(t)$ not identically zero on any ray of the form $[t^*, \infty]$ for some $t^* \geq a > 0$.

Remark 3. From Lemma 2 it follows that any nonoscillatory solution y of (QF) with $F[y(t_0)] \leq 0$ for some $t_0 \in I$ is unbounded.

Example 1. Consider the differential equation

$$\begin{gathered}\left(t^{\frac{1}{2}}(t^{-\frac{1}{2}}y')'\right)' + (1/36t^2)y' + (5/108)t^{-(5\alpha+4)/3}|y|^\alpha \operatorname{sgn} y = 0\,, \\ \alpha > 0 \quad \text{and} \quad t > 0\,.\end{gathered} \tag{9}$$

The conditions of Lemma 2 are satisfied and the equation has the unbounded nonoscillatory solution $y(t) = t^{5/3}$.

3. Conditions of nonexistence of property V_2

We assume that the function f satisfies conditions:

f is nondecreasing, (10)

there exists a constant $C > 0$ such that

$$|f(uv)| \geq Cf(u)|f(v)| \qquad \text{for} \quad u \geq 0\,, \quad v \in \mathbb{R}\,. \tag{11}$$

Suppose further that

$$R_{12}(t, a) \to \infty \qquad \text{as} \quad t \to \infty\,. \tag{12}$$

Let conditions (4), (10)–(12) hold and y be a nonoscillatory solution of (QF), say $y(t) > 0$, with property V_2 for $t \geq c \geq a$. By the third Kiguradze lemma (see Lemma 2 in [13])

$$y(t) \geq \frac{R_{12}(t,c)}{R_2(t,c)} L_1 y(t) \qquad \text{for every} \quad t \geq t_0 > c$$

holds. Thus, for every $\lambda \in (0,1)$ there exists a number T, $T = t_\lambda \geq t_0$ such that

$$\frac{R_{12}(t,c)}{R_2(t,c)} \geq \lambda \frac{R_{12}(t,a)}{R_2(t,a)}, \qquad t \geq T,$$

since $\lim\limits_{t\to\infty} \dfrac{R_{12}(t,c)}{R_2(t,c)} \dfrac{R_2(t,a)}{R_{12}(t,a)} = 1$. Using conditions (10) and (11) we obtain

$$f(y(t)) \geq f\left[\lambda \frac{R_{12}(t,a)}{R_2(t,a)} L_1 y(t)\right] \geq C f(\lambda) f\left[\frac{R_{12}(t,a)}{R_2(t,a)}\right] f(L_1 y(t))$$

for some $C > 0$ and every $t \geq T$. Substituting $f(y)$ by the estimate above we obtain

$$L_3 y(t) + \frac{p(t)}{r_1(t)} L_1 y(t) + C f(\lambda) f\left[\frac{R_{12}(t,a)}{R_2(t,a)}\right] q(t) f(L_1 y(t)) \leq 0$$

and so

$$L_1 y(t) \left\{ L_3 y(t) + \frac{p(t)}{r_1(t)} L_1 y(t) + C f(\lambda) f\left[\frac{R_{12}(t,a)}{R_2(t,a)}\right] q(t) f(L_1 y(t)) \right\} \leq 0 \quad \text{for every} \quad t \geq T. \tag{13}$$

It is clear that the inequality (13) for a negative solution y of (QF) with property V_2 holds, too.

Let conditions (4) and (12) hold and the equation (QF) be weak superlinear. Let y be a nonoscillatory solution of (QF). Similarly as above we derive the inequality

$$L_1 y(t) \left\{ L_3 y(t) + \left[\frac{p(t)}{r_1(t)} + m\lambda \frac{R_{12}(t,a)}{R_2(t,a)} q(t)\right] L_1 y(t) \right\} \leq 0 \tag{13'}$$

for every $t \geq T > a$.

THEOREM 1. *Let conditions* (4) *and* (10)–(12) *hold and assume that the equation*

$$\left(r_2(t) z'\right)' + \frac{p(t)}{r_1(t)} z + C f(\lambda) f\left[\frac{R_{12}(t,a)}{R_2(t,a)}\right] q(t) f(z) = 0 \tag{14}$$

is oscillatory (that is, all solutions of (14) *are oscillatory) for some* $0 < \lambda < 1$ *and* $C > 0$. *Then no nonoscillatory solution* y *of* (QF) *has property* V_2 *for all large* t.

Proof. Let y be a nonoscillatory solution of (QF) with property V_2 for all large t. Thus inequality (13) holds for all large t. By Theorem 1 in [10] the equation (14) is oscillatory if and only if the inequality

$$z\left\{\left(r_2(t)z'\right)' + \frac{p(t)}{r_1(t)}z + Cf(\lambda)f\left[\frac{R_{12}(t,a)}{R_2(t,a)}\right]q(t)f(z)\right\} \le 0 \tag{15}$$

is oscillatory, too. This is a contradiction, since $z = L_1 y$ is a nonoscillatory solution of (15) for large t.

Remark 4. Under the hypotheses of Theorem 1 it is clear by the generalized Sturm comparison theorem (see Theorem 2 in [10]) that any criterion which guarantees that

$$\left(r_2(t)z'\right)' + \frac{p(t)}{r_1(t)}z = 0 \tag{16}$$

or

$$\left(r_2(t)z'\right)' + Cf(\lambda)f\left[\frac{R_{12}(t,a)}{R_2(t,a)}\right]q(t)f(z) = 0 \tag{17}$$

for some $0 < \lambda < 1$ and $C > 0$ is oscillatory, also guarantees that (14) is oscillatory.

Oscillation criteria for (16) may be found in [1], [5], and [12], for example.

Example 2. The equation (9) from Example 1 has the solution $y(t) = t^{5/3}$ with property V_2. Both, the equation

$$(t^{1/2}z')' + (1/36)t^{-3/2}z = 0 \tag{16'}$$

and the equation

$$(t^{1/2}z')' + \frac{5}{108}\left(\frac{\lambda}{6}\right)^{\alpha}\left[3t^{1/2} - a^{1/2}t - at^{1/2} - a^{3/2}\right]^{\alpha} t^{-(5\alpha+4)/3}|z|^{\alpha}\operatorname{sgn} z = 0\,, \qquad t \ge a\,, \tag{17'}$$

are not oscillatory. The equation (16') is nonoscillatory since its general solution is $z(t) = t^{1/4}\left(C_1 t^{\sqrt{5}/12} + C_2 t^{-\sqrt{5}/12}\right)$. The equation $(17')$ is not oscillatory (that is, there exists at least one nonoscillatory solution; see §4 in [18]) by the generalized Atkinson theorem (Theorem 4.1 in [18], $\alpha > 1$) or generalized Belohorec theorem (Theorem 4.7 in [18], $0 < \alpha < 1$), respectively, since

$$\int^{\infty} 2\left(t^{1/2} - a^{1/2}\right)\left[3t^{3/2} - a^{1/2}t - at^{1/2} - a^{3/2}\right]^{\alpha} t^{-(5\alpha+4)/3}\,\mathrm{d}t < \infty \qquad \text{for} \quad \alpha > 1$$

or

$$\int^{\infty} 2\left(t^{\frac12} - a^{\frac12}\right)^{\alpha}\left[3t^{3/2} - a^{1/2}t - at^{1/2} - a^{3/2}\right]^{\alpha} t^{-(5\alpha+4)/3}\,\mathrm{d}t < \infty \qquad \text{for} \quad 0 < \alpha < 1.$$

DEFINITION 3. *The equation* (QF) *is called superlinear if the function* f *for every* $\varepsilon > 0$ *satisfies*

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{du}{f(u)} < \infty , \tag{18}$$

and (QF) *is called sublinear if* f *satisfies*

$$\int_{0}^{\pm\varepsilon} \frac{du}{f(u)} < \infty \qquad \text{for every} \quad \varepsilon > 0 . \tag{19}$$

Let us give examples of the functions which satisfy the conditions (10), (11), and (18) or (19).

Example 3. The functions f_1 and $f_2 \colon \mathbb{R} \to \mathbb{R}$, where $f_1(u) = |u|^\alpha \operatorname{sgn} u$, $\alpha > 0$ and $f_2(u) = \dfrac{|u|^{2\alpha} \operatorname{sgn} u}{1 + |u|^\alpha}$, $\alpha > 0$ are continuous on $\mathbb{R}$, satisfy $uf(u) > 0$ for $u \neq 0$ and conditions (10), (11). Further, the function f_1 satisfies (18) for $\alpha > 1$ and (19) for $0 < \alpha < 1$. The function f_2 satisfies (18) for $\alpha > 1$.

COROLLARY 1. *Let conditions* (4) *and* (10)–(12) *hold and assume that*

$$\int^{\infty} f\big(R_2(t,a)\big) f\left[\frac{R_{12}(t,a)}{R(t,a)}\right] q(t)\, dt = \infty , \tag{20}$$

if (QF) *is sublinear*
or

$$\int^{\infty} R_2(t,a) f\left[\frac{R_{12}(t,a)}{R_2(t,a)}\right] q(t)\, dt = \infty , \tag{21}$$

if (QF) *is superlinear.*

Then no nonoscillatory solution y *of* (QF) *has property* V_2 *for all large* t.

Proof. Condition (20) is sufficient for oscillation of all solutions of (17) in the sublinear case (that is, f satisfies (19)), see Theorem 1.8 in [9]. Likewise, condition (21) is sufficient for oscillation of (17) in the superlinear case (see Theorem 4 in [10]). Therefore, the Corollary 1 follows by Remark 4 and Theorem 1.

THEOREM 2. *Let conditions* (4), (10)–(12) *hold and the equation* (QF) *be sublinear. If*

$$\int^{\infty} f\big(R_{12}(t,a)\big)q(t)\,\mathrm{d}t = \infty \tag{22}$$

holds, then no nonoscillatory solution y *of* (QF) *has property* V_2 *for all large* t.

P r o o f. Let y be a positive solution of (QF) with property V_2 on $[c,\infty)$, $c \geq a$. By the third generalized Kiguradze lemma (see Lemma 2 in [13])

$$y(t) \geq R_{12}(t,c)L_2 y(t) \qquad \text{for every} \quad t \geq t_0 > c$$

holds. Thus, for every $\lambda \in (0,1)$ there exists a number $T = t_\lambda$, $T \geq t_0$ such that

$$R_{12}(t,c) \geq \lambda R_{12}(t,a)\,, \qquad t \geq T\,,$$

since $\lim\limits_{t\to\infty} R_{12}(t,c)\big(R_{12}(t,a)\big)^{-1} = 1$. Using conditions (10) and (11) we obtain

$$f\big(y(t)\big) \geq f\big[\lambda R_{12}(t,a)L_2 y(t)\big] \geq Cf(\lambda)f\big(R_{12}(t,a)\big)f\big(L_2 y(t)\big)$$

for some $C > 0$ and every $t \geq T$. Dividing (QF) by $f\big(L_2 y(t)\big)$ and integrating from T to $t \geq T$, we get

$$\int_T^t \frac{L_3 y(s)}{f\big(L_2 y(s)\big)}\,\mathrm{d}s \leq -Cf(\lambda)\int_T^t f\big(R_{12}(s,a)\big)q(s)\,\mathrm{d}s\,.$$

Since equation (QF) is sublinear, we have

$$\int_T^t \frac{L_3 y(s)}{f\big(L_2 y(s)\big)}\,\mathrm{d}s = -\int_{L_2 y(t)}^{L_2 y(T)} \frac{\mathrm{d}u}{f(u)} \geq -\int_0^{L_2 y(T)} \frac{\mathrm{d}u}{f(u)} > -\infty\,,$$

contradicting the condition (22). This completes the proof of the theorem.

R e m a r k 5. The condition (22) is weaker than the condition (20) because $f(R_{12}) = f\left(R_2 \dfrac{R_{12}}{R_2}\right) \geq Cf(R_2)f\left(\dfrac{R_{12}}{R_2}\right)$.

THEOREM 3. *Let conditions* (4) *and* (12) *hold and the equation* (QF) *be weak superlinear. If the equation*

$$\left(r_2(t)z'\right)' + \left[\frac{p(t)}{r_1(t)} + m\lambda \frac{R_{12}(t,a)}{R_2(t,a)} q(t)\right] z = 0 \tag{23}$$

for some $m > 0$, $0 < \lambda < 1$ *is oscillatory, then no nonoscillatory solution* y *of* (QF) *has property* V_2 *for all large* t.

The proof is similar to that of Theorem 1 (see (13′)) and hence is omitted.

Remark 6. Let conditions (4) and (12) hold. By the generalized Kneser theorem (Theorem 2.3 in [5]) or by the criterion Moore-Ráb (see Theorem 11 or Theorem 12 in [1] with $u = (R_2)^\delta$), respectively, the equation (23) is oscillatory if the condition

$$\liminf_{t\to\infty} r_2(t) R_2^2(t,a) \left[\frac{p(t)}{r_1(t)} + m\lambda \frac{R_{12}(t,a)}{R_2(t,a)} q(t)\right] > \frac{1}{4}$$

or

$$\int^{\infty} \left(R_2(t,a)\right)^\delta \left[\frac{p(t)}{r_1(t)} + m\lambda \frac{R_{12}(t,a)}{R_2(t,a)} q(t)\right] \mathrm{d}t = \infty\,, \qquad 0 \le \delta < 1$$

holds.

THEOREM 4. *Let the function* f *satisfy the condition*

$$\liminf_{|u|\to\infty} |f(u)| > 0\,. \tag{24}$$

If

$$\int^{\infty} q(t)\,\mathrm{d}t = \infty\,, \tag{25}$$

then no nonoscillatory solution y *of* (QF) *has property* V_2 *for large* t.

Proof. Let y be a positive solution of (QF) with property V_2 on $[T,\infty)$, $T \ge a$. Since $yL_1y > 0$ on $[T,\infty)$, $\lim_{t\to\infty} y(t)$ exists. If $\lim_{t\to\infty} y(t) = \infty$, then from (24) and (25) we obtain

$$\int^{\infty} q(t) f\big(y(t)\big)\,\mathrm{d}t = \infty\,. \tag{26}$$

If $\lim_{t\to\infty} y(t) = K < \infty$, then from (25) and the continuity f (26) holds, too. Integrating the inequality $L_3y + q(t)f(y) \le 0$ from T to $t \ge T$ and using (26) we get $L_2y(t) < 0$ for all sufficiently large t, a contradiction. This completes the proof of the theorem.

4. Main result

The last theorem is an oscillation criterion for (QF). It generalizes not only Theorem A and Theorem B but some partial generalizations of Theorem A for third order nonlinear differential equations, too (see [11, 14, 15, 16, 17]). See also Corollary 3.4 in [3].

We recall that

$$R_2(t,T) = \int_T^t \frac{\mathrm{d}s}{r_2(s)}, \qquad t \ge T \ge a,$$

$$L_1 y(t) = r_1(t) y'(t), \qquad L_2 y(t) = r_2(t)\big[L_1 y(t)\big]', \qquad (\text{see } (1)),$$

and

$$F\big[y(t)\big] = 2y(t)L_2 y(t) - \frac{r_2(t)}{r_1(t)}\big[L_1 y(t)\big]^2 + p(t)y^2(t).$$

Assume further that r_2/r_1, $p \in C^1(I,\mathbb{R})$.

THEOREM 5. *Let $p \ge 0$, $q \ge 0$, $[r_2/r_1]' \ge 0$, $p' \le 0$ on I, $R_2(t,a) \to \infty$ as $t \to \infty$. In addition assume that the hypotheses of any theorem 1–4 are fulfilled. Let y be a solution of* (QF) *which exists on the interval $[T,\infty)$, $T \ge a$. Then y is oscillatory if and only if there exist a point $t_0 \ge T$ such that $F\big[y(t_0)\big] \le 0$.*

Proof. If $F\big[y(t)\big] > 0$ for all $t \ge T$, it is clear that y cannot have any zeros for $t \ge T$. Hence y is nonoscillatory.

Now suppose that $F\big[y(t_0)\big] \le 0$ for some $t_0 \ge T$. By the Lemma 2 either y is oscillatory or y is nonoscillatory with the property V_2 for all large t (see (2)). On the other hand applying some of Theorems 1–4 we get that a nonoscillatory solution y has not property V_2. Consequently y is oscillatory. This completes the proof of theorem.

Remark 7. Any solution y of (QF) which has a zero (that is, $y(t^*) = 0$ for some $t^* \ge T$) satisfies $F\big[y(t^*)\big] \le 0$. So by Theorem 5 any solution which has a zero is oscillatory.

Remark 8. The assertion of Theorem 5 can be written as: Then y is nonoscillatory if and only if $F\big[y(t)\big] > 0$ for all $t \in [T,\infty)$.

Remark 9. Let us recall that if the equation (QF) is weak superlinear, (see Definition 2), then the condition $p' \le 0$ of Theorem 5 may be replaced with a weaker condition $2mq(t) - p'(t) \ge 0$, $t \in I$ and $2mq(t) - p'(t)$ not identically zero any ray of the form $[t^*,\infty]$ for some $t^* \ge a > 0$, (see proof of Lemma 2).

E x a m p l e 4. Consider the weak superlinear equation

$$\big(t(ty')'\big)' + (t^2-1)y' + \frac{3t}{2+\sin 2t}(y+y^3) = 0\,, \qquad t \ge a > 0\,. \tag{26}$$

All the conditions of Theorem 5 (see Theorem 3 and Remark 9, $m = 1$) are satisfied since the equation

$$(tz')' + \left[\frac{t^2-1}{t} + \frac{\lambda}{2}\big(\ln\frac{t}{a}\big)\frac{3t}{2+\sin 2t}\right]z = 0\,, \qquad \text{some} \quad 0<\lambda<1$$

is oscillatory (see Remark 6). Hence any solution of (26) with $F\big[y(t_0)\big] \le 0$ (e.g. if $y(t_0) = 0$, then $F\big[y(t_0)\big] \le 0$) is oscillatory. An example of such solution is $y(t) = \sin t + \cos t$.

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Department of Mathematics
Faculty of Mechanical Engineering
Technical University
Švermova 9
041 87 Košice