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Mathematica Slovaca, Vol. 32 (1982), No. 4, 321--325

Persistent URL: http://dml.cz/dmlcz/129764

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COMPACT PARTIALLY ORDERED SETS AND COMPACTIFICATION OF PARTIALLY ORDERED SETS

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We call a partially ordered set P compact if and only if every subset S of P has a nonzero lower bound in P (i.e. a lower bound which is not the least element of P), provided every finite subset of S has a nonzero lower bound in P. A compact extension Q of a partially ordered set P which preserves all the existing infima and suprema of subsets of P, except perhaps the zero infima (if P has a zero) of certain infinite subsets of P is called a compactification of P. Every partially ordered set without zero has a compactification. For partially ordered sets with zero it is not the case. We give one necessary and one sufficient condition for the existence of a compactification of a partially ordered set.

In what follows we refer for the sake of simplicity to a partially ordered set simply as a poset. The least element of a poset P, if it exists, is called the zero of P and is denoted by 0.

Definition 1. *A poset P is called compact if and only if for every subset S of PS has a nonzero lower bound if every finite subset of S has a nonzero lower bound.*

Let S be a subset of a set P. We say that S has the finite lower bound property if and only if every finite subset of S has a nonzero lower bound. Thus, Definition 1 can be rephrased as follows:

Definition 2. *A poset is called compact if and only if every subset of it which has the finite lower bound property has a nonzero lower bound.*

Let P be a poset without zero. We denote by $P\cup\{0\}$ the poset with the zero element 0 which is obtained by adjoining 0 to P in the most obvious way (we assume that 0 is not used as a symbol for an element of P). But then from the above Definitions it follows:

Lemma 1. *Let P be a poset without zero. Then P is compact if and only if the poset $P\cup\{0\}$ is compact.*

The significance of Lemma 1 is revealed by Lemma 2 which shows that compactness is formulated rather conveniently in posets with a zero element.

Lemma 2. *Let P be a poset with zero 0. Then P is compact if and only if for every subset G of P in the case of 0 being the infimum of G, 0 is already the infimum of a finite subset of G.*

Proof. By Definition 2, P is compact if and only if every subset of it which has no nonzero lower bound, fails to have the finite lower bound property. Since a subset of P has no nonzero lower bound if and only if 0 is its infimum, the statement is evident.

Let $(P, \leqslant)$, $(Q, \leqslant)$ be posets and $P \subseteq Q$. We say that $(Q, \leqslant)$ is an extension of $(P, \leqslant)$ if and only if the order relation between the elements of P in $(Q, \leqslant)$ is the same as that of P in $(P, \leqslant)$. Clearly, an extension $(Q, \leqslant)$ of $(P, \leqslant)$ need not preserve the zero or the infima or the suprema of the subsets of P; however, if it does preserve them, then we say that the extension $(Q, \leqslant)$ is zero-, or infima-, or suprema- (depending on the case) preserving. For instance, if $(P, \leqslant)$ has no zero element, then the extension $(P \cup \{0\}, \leqslant)$ is both infima- and suprema-preserving.

Let $(P, \leqslant)$ be a poset. We shall be interested in the existence of a poset $(Q, \leqslant)$ with the following properties:

(1) $(Q, \leqslant)$ is an extension of $(P, \leqslant)$ such that the zero of $(P, \leqslant)$ (if it exists) is also the zero of $(Q, \leqslant)$.

(2) $(Q, \leqslant)$ preserves all the existing infima and suprema of the subsets of P, except the zero infima of those infinite subsets of P which have the finite lower bound property, each of which, however, acquires a nonzero infimum in $(Q, \leqslant)$.

(3) $(Q, \leqslant)$ is compact.

Definition 3. *An extension $(Q, \leqslant)$ of $(P, \leqslant)$ satisfying (1), (2) and (3) is called a compactification of $(P, \leqslant)$.*

The following theorem is evident.

Theorem 1. *Let $(P, \leqslant)$ be a poset without zero. Then the ordinal sum any two element chain and $(P, \leqslant)$ is a compactification of $(P, \leqslant)$.*

In what follows we shall suppose that $(P, \leqslant)$ is a poset with zero 0.

If A is a subset of P having the finite lower bound property, then by Zorn's Lemma there exists a subset of P maximal with respect to the finite lower bound property and containing A.

Lemma 3. *If M is a subset of P maximal with respect to the finite lower bound property, then* $\inf M$ *exists. If* $\inf M = p \neq 0$, *then p is an atom in $(P, \leqslant)$ and $M = \{x \in P\colon x \geqslant p\}$.*

Proof. Let $M (\subseteq P)$ be maximal with respect to the finite lower bound property. If 0 is the unique lower bound of M, then $0 = \inf M$. Suppose that M has a nonzero lower bound p. Then $M \cup \{p\}$ has evidently the finite lower bound property and using the maximality of M we obtain $p \in M$. Hence $p = \inf M$. Assume that there exists p_1 such that $0 < p_1 < p$. Then p_1 is also a nonzero lower bound of M, hence

$p_1 = \inf M$. We have a contradiction. Evidently $M \subseteq \{x \in P: x \geqslant p\}$. Assume that $q \in P$ and $q \geqslant p$. Then $M \cup \{q\}$ has the finite lower bound property. Consequently by the maximality of M we have $q \in M$.

Denote by $\mathcal{M}$ the system of all subsets of P maximal with respect to the finite lower bound property with zero infima.

Theorem 2. *If* $(Q, \leqslant)$ *is a compactification of* $(P, \leqslant)$, *then* $\{\inf_Q M: M \in \mathcal{M}\}$ *is an antichain. Further if* $M \in \mathcal{M}$ *and* $x \in P - (M \cup \{0\})$, *then* $\inf_Q M$ *and* x *are incomparable.*

Proof. If $M \in \mathcal{M}$, then M has the finite lower bound property and $\inf_P M = 0$, hence M must be infinite. By (2) we see that $\inf_Q M \in Q - P$ exists. Let $M_1, M_2 \in \mathcal{M}$, $M_1 \neq M_2$ and suppose e.g. $\inf_Q M_1 \leqslant \inf_Q M_2$. Pick $m \in M_1 - M_2$. The maximality of M_2 ensures the existence of a finite subset K of M_2 with $\inf_P(K \cup \{m\}) = 0$. Then (2) implies that $\inf_Q(K \cup \{m\}) = 0$. On the other hand $\inf_Q M_1 < m$ and $\inf_Q M_1 \leqslant \inf_Q M_2 < k$ for every $k \in K$, hence $\inf_Q M_1$ is a nonzero lower bound of $K \cup \{m\}$ in $(Q, \leqslant)$.

Further let $M \in \mathcal{M}$ and $x \in P - (M \cup \{0\})$. Suppose that $x < \inf_Q M$. Then x is a nonzero lower bound of M in $(P, \leqslant)$ and we have a contradiction. Assume that $\inf_Q M < x$. Since $x \notin M$, there exists a finite subset L of M such that $\inf_P(L \cup \{x\}) = 0$. Then we have $\inf_Q(L \cup \{x\}) = 0$, a contradiction. Therefore $\inf_Q M$ and x are incomparable.

Consider the following conditions:

(a) *If* $N \subseteq M$ *for some* $M \in \mathcal{M}$ *and* N *has in* P *a nonzero infimum, then the latter belongs to* M.

(b) *If* $M_1, M_2 \in \mathcal{M}$ *and* $M_1 \neq M_2$, *then* $\inf_P(M_1 \cap M_2) \neq 0$ *(i.e.* $M_1 \cap M_2$ *has in* P *a nonzero lower bound).*

Theorem 3. *If* $(P, \leqslant)$ *has a compactification, then* $(P, \leqslant)$ *satisfies condition* (a).

Proof. Suppose that $(Q, \leqslant)$ is a compactification of $(P, \leqslant)$. Let $M \in \mathcal{M}$, $N \subseteq M$ and $\inf_P N = p \neq 0$. By (2) we have $\inf_Q N = p$ and since $\inf_Q M \leqslant \inf_Q N = p$, in view of Theorem 2, p must belong to M.

Let $\mathcal{M} = \{M_h: h \in H\}$. Denote by Q the disjoint join of P and H and define a relation $\leqslant$ in Q as follows: for every $x, y \in P$ let $x \leqslant y$ in Q if and only if $x \leqslant y$ in P; for every $x, y \in H$ let $x \leqslant y$ if and only if $x = y$; for every $x \in P$ and $y \in H$ let $x \leqslant y$ if and only if $x = 0$ and $y \leqslant x$ if and only if $x \in M_y$. It is easy to verify that $(Q, \leqslant)$ is a poset.

Theorem 4. *Let* $(P, \leqslant)$ *satisfy condition* (a). *The poset* $(Q, \leqslant)$ *defined above is a compactification of* $(P, \leqslant)$ *if and only if* $(P, \leqslant)$ *satisfies* (b).

Proof. Suppose that $(Q, \leqslant)$ defined above is a compactification of $(P, \leqslant)$. Let $h_1, h_2 \in H$ and $h_1 \neq h_2$. Assume $\inf_P(M_{h_1} \cap M_{h_2}) = 0$. Then, as the set $M_{h_1} \cap M_{h_2}$ has the finite lower bound property, it must be infinite. By (2) there exists $q \in Q$ and

$q \neq 0$ with $\inf_Q(M_{h_1} \cap M_{h_2}) = q$. Evidently $q \notin P$. Since h_1, h_2 are lower bounds of $M_{h_1} \cap M_{h_2}$, we must have h_1, $h_2 \leq q$. We have a contradiction.

Now suppose that $(P, \leq)$ satisfies (b). Evidently $(Q, \leq)$ is a zero- and suprema-preserving extension of $(P, \leq)$. Let now $\inf_P A = p \neq 0$ for some $A \subseteq P$. Suppose that $h(\in H)$ is a lower bound of A. Then $A \subseteq M_h$ and by (a) we have $p \in M_h$. Hence $h \leq p$. Therefore $p = \inf_Q A$. Further let $\inf_P A = 0$ for some $A \subseteq P$. If 0 is the unique lower bound of A in $(Q, \leq)$, then $\inf_Q A = 0$. Suppose that there exists a lower bound $h(\in H)$ of A in $(Q, \leq)$. Then $A \subseteq M_h$ which implies that A has the finite lower bound property and it is infinite. We show that $h = \inf_Q A$. Let $h_1 \in H$ be a lower bound of A different from h. Then $A \subseteq M_{h_1}$. From the relation $A \subseteq M_h \cap M_{h_1}$ it follows that $\inf_P(M_h \cap M_{h_1}) = 0$, which contradicts (b).

It remains to prove that $(Q, \leq)$ is compact. Let A be a subset of Q having the finite lower bound property. If $A \subseteq P$, then there exists a subset M of P maximal with respect to the finite lower bound property containing A. If $\inf_P M = p \neq 0$, then p is a nonzero lower bound of A. If $\inf_P M = 0$, then $M = M_h$ for some $h \in H$ and h is a nonzero lower bound of A. Now suppose that $A \cap H \neq \emptyset$. Then evidently A contains just one element of the set H. Let $A \cap H = \{h\}$. For every $x \in A$ and $x \neq h$ the set $\{x, h\}$ has a nonzero lower bound, hence we have $h < x$. Thus h is a lower bound of A.

Remark. In view of Theorem 2 every compactification of $(P, \leq)$ is an extension of the one mentioned above if P is a poset satisfying (a) and (b).

We recall that a poset is called lower semilattice [1] if and only if every two elements of it have an infimum. But then, based on Theorem 4, we have:

Corollary 1. *Let $(P, \leq)$ be a lower semilattice satisfying* (a) *and* (b). *Then the compactification $(Q, \leq)$ of $(P, \leq)$ mentioned in Theorem 4 is also a lower semilattice.*

Proof. By (2) the compactification $(Q, \leq)$ preserves all the infima of two-element subsets of $(P, \leq)$. It remains to show that every two elements of Q, where at least one of them belongs to H, have an infimum in $(Q, \leq)$. But this is obvious, since $\inf\{h_1, h_2\} = \inf\{x, h_1\} = 0$ for every h_1, $h_2 \in H$, $h_1 \neq h_2$ and $x \in P$ such that $x \ngeq h_1$.

Let us recall that a poset is called complete lattice if and only if every subset of it has an infimum (or equivalently, a supremum). Based on Theorem 4, we have:

Corollary 2. *Let $(P, \leq)$ be a complete lattice satisfying* (a) *and* (b). *Then the compactification $(Q, \leq)$ of $(P, \leq)$ mentioned in Theorem 4 is also a complete lattice.*

Proof. Let S be a subset of Q. If $S \subseteq P$, then from (2) it follows that $\inf_Q S$ exists. If $S \cap H$ contains more than one element, then $\inf_Q S = 0$ since the elements of H are pairwise incomparable and 0 is the only element of Q which is less than every one of them. Finally if $S \cap H = \{h\}$, then either $\inf_Q S = h$ if h is a lower bound of S, or $\inf_Q S = 0$ otherwise.

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Received March 19, 1979

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КОМПАКТНЫЕ ЧАСТИЧНО УПОРЯДОЧЕННЫЕ МНОЖЕСТВА И КОМПАКТИФИКАЦИЯ ЧАСТИЧНО УПОРЯДОЧЕННЫХ МНОЖЕСТВ

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Резюме

В работе определяется понятие компактного частично упорядоченного множества и компактификации. Исследуется вопрос существования компактификации частично упорядоченного множества (теоремы 1, 3, 4).