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THIN SETS DEFINED BY A SEQUENCE OF CONTINUOUS FUNCTIONS

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ABSTRACT. We investigate thin sets of reals defined from a sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real functions in a similar way to that by which the trigonometric thin sets are defined from the sequence $\{\sin 2\pi nx\}_{n=0}^{\infty}$. We show that all the important classical results on trigonometric thin sets can be proved in a somewhat general case under some natural assumptions. Consequently we may conclude that the basic properties of trigonometric thin sets do not depend on some deep properties of the trigonometric functions.

1. Introduction

In [Bk1], [Bk2], generalizing the classical notion of thin sets of trigonometric series theory, L. Bukovský introduced the abstract notion of a family of thin sets and has established some of its basic classical properties. In this paper we shall investigate thin sets defined from a sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real functions in a similar way to that by which trigonometric thin sets are defined from the sequence $\{\sin 2\pi nx\}_{n=0}^{\infty}$. We shall show that all the important classical results concerning trigonometric thin sets as presented in [Ba], [BKR], [Ka], [Zy] can be proved for example under some natural assumptions in a somewhat general situation. As a consequence we may conclude that the fundamental properties of trigonometric thin sets do not depend on any deep properties of trigonometric functions.

Following S. Kahane in [Ka], we shall work with a compact topological group — the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the operation of addition mod 1. We may identify $\mathbb{T}$ with the interval $[0,1]$ identifying 0 and 1. Any real valued function f

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defined on $\mathbb{T}$ can be identified with a periodic function defined on the whole real line $\mathbb{R}$ with period 1, i.e. $f(x+1) = f(x)$ for every real x. When referring to a set $A \subseteq \mathbb{T}$ we assume that $A \subseteq [0,1]$ and $0 \in A$ if and only if $1 \in A$. Let us recall that a sequence $\{f_n\}_{n=0}^{\infty}$ of real valued functions is said to converge quasinormally to a function f on the set X if there exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of nonnegative reals converging to zero such that

$$(\forall x \in X)(\exists n_0)(\forall n \geq n_0)\big(|f_n(x) - f(x)| \leq \varepsilon_n\big),$$

— compare [Bu2], [CL]. If $\{f_n\}_{n=0}^{\infty}$ converges quasinormally to 0 on X then there exists a sequence $\{n_k\}_{k=0}^{\infty}$ such that $\sum\limits_{k=0}^{\infty} f_{n_k}(x) < \infty$ for every $x \in X$.

A measure μ, defined on a σ-algebra of subsets of $\mathbb{T}$ containing all Borel subsets taking values in the nonnegative reals, is called a positive Borel measure on $\mathbb{T}$. The Lebesgue measure — which is evidently a positive Borel measure — is denoted by λ and the corresponding integral is denoted simply by $\int\limits_A f(x)\,\mathrm{d}x$ instead of $\int\limits_A f(x)\,\mathrm{d}\lambda(x)$.

We shall need a rather elementary result about infinite series:

THEOREM 1. *Let $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence of nonnegative reals such that $\sum\limits_{n=0}^{\infty} a_n = \infty$. Let $s_n = \sum\limits_{k=0}^{n} a_k$. If $\varphi\colon [0,\infty) \to (0,\infty)$ is an increasing unbounded function, then*

$$\sum_{n=0}^{\infty} \frac{1}{\varphi(n)} < \infty$$

if and only if

$$\sum_{n=0}^{\infty} \frac{a_n}{\varphi(s_n)} < \infty .$$

A proof can be found, for example, in [Ba].

The classical Dirichlet–Minkowski theorem (see e.g. [Ba], [BKR]) can be easily generalized as follows:

THEOREM 2. *Let $f\colon \mathbb{T} \to [0,\infty)$ be a continuous function with $f(0) = 0$. Let $\{n_m\}_{m=0}^{\infty}$ be an increasing sequence of natural numbers, and let $\varepsilon > 0$. Let $\delta > 0$ be such that $f(x) \leq \varepsilon$ whenever $|x| \leq \delta$. Then for any $x_1, \dots, x_k \in \mathbb{T}$ there are $m < l < (1/\delta + 1)^k$ such that $f\big((n_l - n_m)x_i\big) \leq \varepsilon$ for $i = 1, \dots, k$.*

P r o o f. Let $x_1, \dots, x_k \in \mathbb{T}$. Let $\varepsilon > 0$, $\delta > 0$ satisfy the condition of the theorem. Take the smallest natural number p such that $1/p \leq \delta$. Thus

$p < 1/\delta + 1$. We divide the k-dimensional cube $[0,1]^k$ into $q = p^k$ small cubes of side $1/p$. Using the pigeon-hole principle, among $q+1$ elements

$$\langle n_i x_1, \dots, n_i x_k \rangle, \qquad i = 0, 1, \dots, q,$$

of the cube $[0,1]^k$, at least two are in the same small cube. Therefore, for some $m < l \le q$ we obtain $|(n_l - n_m)x_i| \le 1/p \le \delta$ for $i = 1, \dots, k$. Then also $f((n_l - n_m)x_i) \le \varepsilon$ for every $i = 1, \dots, k$.

It is easy to see that $m < l \le q = p^k < (1/\delta + 1)^k$. □

Following [Bk2] we define a family $\mathcal{F}$ of subsets of $\mathbb{T}$ to be a *family of thin sets* if the following conditions hold:

(a) $\mathcal{F}$ contains every singleton $\{x\}$, $x \in \mathbb{T}$,
(b) if $A \in \mathcal{F}$, $B \subseteq A$ then also $B \in \mathcal{F}$,
(c) $\mathcal{F}$ does not contain any open interval (a,b), $a < b$, $a, b \in [0,1]$.

A family $\mathcal{G} \subseteq \mathcal{F}$ is called a *basis* for $\mathcal{F}$ if every $A \in \mathcal{F}$ is a subset of some $B \in \mathcal{G}$. If every set from $\mathcal{G}$ is an F_σ-set, Borel set etc., we speak of respectively, an F_σ-basis, Borel basis etc.

A family of thin sets $\mathcal{F}$ is said to be *trigonometric like* if for every $A \in \mathcal{F}$ the arithmetic difference

$$A - A = \{x \in \mathbb{T};\ x = y - z \text{ for some } y, z \in A\}$$

also belongs to $\mathcal{F}$.

As an easy consequence of a theorem of Steinhaus, in [Bk2], the author shows that:

THEOREM 3. *Every member of a trigonometric like family of thin sets with a Borel basis is meager (= the first Baire category) and has Lebesgue measure zero.*

All classical families of trigonometric thin sets $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{A}$, and $w\mathcal{D}$, as defined e.g. in [BKR] (see also Section 2 below), are trigonometric like families of thin sets with Borel bases in the sense of the above definition.

2. Thin sets defined by a sequence of functions

From now on, let $\mathfrak{f} = \{f_n\}_{n=0}^{\infty}$ be a sequence of continuous functions defined on $\mathbb{T}$ taking values in the nonnegative reals

$$f_n \colon \mathbb{T} \to [0, \infty) \qquad \text{for} \quad n = 0, 1, \dots .$$

We shall follow classical definitions of trigonometric thin sets as presented e.g. in [Ba], [BKR], [El], [Ka], [Zy] and we define: a set $A \subseteq \mathbb{T}$ is called an $\mathfrak{f}$-*Dirichlet* *set* (briefly a $D^{\mathfrak{f}}$*-set*), a *pseudo* $\mathfrak{f}$*-Dirichlet set* (briefly a $pD^{\mathfrak{f}}$*-set*), an $A^{\mathfrak{f}}$*-set* if there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that the sequence $\{f_{n_k}(x)\}_{k=0}^{\infty}$ converges uniformly, quasinormally and pointwise to 0 on the set A, respectively. Further, a set $A \subseteq \mathbb{T}$ is called an $N_0^{\mathfrak{f}}$*-set* ($B_0^{\mathfrak{f}}$*-set*) if there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ (and a positive real c) such that the series $\sum_{k=0}^{\infty} f_{n_k}(x)$ converges $\left(\sum_{k=0}^{\infty} f_{n_k}(x) < c\right)$ for every $x \in A$. A set $A \subseteq \mathbb{T}$ is called an $N^{\mathfrak{f}}$*-set* ($B^{\mathfrak{f}}$*-set*) if there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of nonnegative reals (and a positive real c) such that $\sum_{n=0}^{\infty} a_n = \infty$ and the series $\sum_{n=0}^{\infty} a_n f_n(x)$ converges $\left(\sum_{n=0}^{\infty} a_n f_n(x) < c\right)$ for every $x \in A$. Finally, a set $A \subseteq \mathbb{T}$ is called a *weak* $\mathfrak{f}$*-Dirichlet set* (briefly a $wD^{\mathfrak{f}}$*-set*) if there exists a Borel set B, $A \subseteq B \subseteq \mathbb{T}$, such that for every positive Borel measure μ on $\mathbb{T}$ there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} \int_B f_{n_k}(x)\, \mathrm{d}\mu(x) = 0\,. \tag{1}$$

The corresponding families will be denoted by $\mathcal{D}^{\mathfrak{f}}$, $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{A}^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, $\mathcal{B}_0^{\mathfrak{f}}$, $\mathcal{N}^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$, and $w\mathcal{D}^{\mathfrak{f}}$, respectively.

If $f_n(x) = |\sin \pi n x|$ then we obtain the classical families of trigonometric thin sets $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, and $w\mathcal{D}$, respectively.

In the definitions of an $\mathrm{N}^{\mathfrak{f}}$-set and a $\mathrm{B}^{\mathfrak{f}}$-set we may assume that the sequence $\{a_n\}_{n=0}^{\infty}$ is bounded. In fact we may assume that $a_0 \geq 1$ and replace every a_n by $a_n/(a_0 + \cdots + a_n)$. Evidently

$$\sum_{n=0}^{\infty} \frac{a_n}{a_0 + \cdots + a_n} = \infty$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{a_0 + \cdots + a_n} f_n(x) \leq \sum_{n=0}^{\infty} a_n f_n(x)$$

for every $x \in \mathbb{T}$.

Let us note that in the definition of a weak $\mathfrak{f}$-Dirichlet set we followed [HMP] rather than [BKR] or [Ka]. However, our results can easily be modified for the definition corresponding to that of [BKR] and [Ka].

One can easily see that each of the families $\mathcal{D}^{\mathfrak{f}}$, $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{A}^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, $\mathcal{B}_0^{\mathfrak{f}}$, $\mathcal{N}^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$, $w\mathcal{D}^{\mathfrak{f}}$ contains every singleton if and only if the following condition holds:

(α) 0 is an accumulation point of the sequence $\{f_n(x)\}_{n=0}^{\infty}$ for every $x \in \mathbb{T}$.

We shall also need the following three properties of the sequence $\mathfrak{f}$:

(β) For every open interval $(c,d) \subseteq \mathbb{T}$ there exist $K > 0$ and $n_0 \in \mathbb{N}$ such that $\int\limits_{(c,d)} f_n(x)\,\mathrm{d}x \geq K$ for every $n \geq n_0$.

(γ) The sequence $\mathfrak{f}$ is uniformly bounded, i.e. there exists a real d such that $f_n(x) \leq d$ for all $x \in \mathbb{T}$ and $n \in \mathbb{N}$.

(δ) $f_n(x-y) \leq f_n(x) + f_n(y)$ for every $n \in \mathbb{N}$ and for every $x, y \in \mathbb{T}$.

One can easily see that the sequence $f_n(x) = |\sin 2\pi n x|$, $n = 0, 1, \dots$ satisfies conditions (α), (β), (γ), and (δ).

We begin with an extension of the classical results on trigonometric thin sets for the families of thin sets introduced above.

THEOREM 4.

(i) *The following inclusions hold (an arrow "$\rightarrow$" means the inclusion "$\subseteq$"):*

$$\begin{array}{ccccc} & & \mathcal{A}^{\mathfrak{f}} & & \\ & & \uparrow & & \\ p\mathcal{D}^{\mathfrak{f}} & \longrightarrow & \mathcal{N}_0^{\mathfrak{f}} & \longrightarrow & \mathcal{N}^{\mathfrak{f}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D}^{\mathfrak{f}} & \longrightarrow & \mathcal{B}_0^{\mathfrak{f}} & \longrightarrow & \mathcal{B}^{\mathfrak{f}} \end{array}$$

(ii) *If condition (γ) holds true then also $\mathcal{A}^{\mathfrak{f}} \subseteq w\mathcal{D}^{\mathfrak{f}}$ and $\mathcal{N}^{\mathfrak{f}} \subseteq w\mathcal{D}^{\mathfrak{f}}$.*

(iii) *Assume that conditions (α) and (β) hold. Then every family $\mathcal{D}^{\mathfrak{f}}$, $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{B}_0^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$, $\mathcal{N}^{\mathfrak{f}}$, $\mathcal{A}^{\mathfrak{f}}$, $w\mathcal{D}^{\mathfrak{f}}$ is a family of thin sets.*

(iv) *If (α), (β), and (δ) hold then the families $\mathcal{D}^{\mathfrak{f}}$, $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{B}_0^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$, $\mathcal{N}^{\mathfrak{f}}$, and $\mathcal{A}^{\mathfrak{f}}$ are trigonometric like.*

Proof.

(i) Since a uniformly convergent sequence is also quasinormally convergent, we have $\mathcal{D}^{\mathfrak{f}} \subseteq p\mathcal{D}^{\mathfrak{f}}$. From a sequence converging quasinormally to zero, we may easily choose a subsequence with converging series; therefore $p\mathcal{D}^{\mathfrak{f}} \subseteq \mathcal{N}_0^{\mathfrak{f}}$. Similarly, from a sequence uniformly converging to zero, we can easily choose a subsequence with bounded sum; therefore $\mathcal{D}^{\mathfrak{f}} \subseteq \mathcal{B}_0^{\mathfrak{f}}$. If the sequence $\{n_k\}_{k=0}^{\infty}$ indicates that A is an $\mathrm{N}_0^{\mathfrak{f}}$-set ($\mathrm{B}_0^{\mathfrak{f}}$-set) just take $a_{n_k} = 1$ and all other $a_n = 0$ and we obtain the inclusion $\mathcal{N}_0^{\mathfrak{f}} \subseteq \mathcal{N}^{\mathfrak{f}}$ ($\mathcal{B}_0^{\mathfrak{f}} \subseteq \mathcal{B}^{\mathfrak{f}}$). Since a bounded series of positive functions is convergent we have $\mathcal{B}_0^{\mathfrak{f}} \subseteq \mathcal{N}_0^{\mathfrak{f}}$ and $\mathcal{B}^{\mathfrak{f}} \subseteq \mathcal{N}^{\mathfrak{f}}$. The inclusion $\mathcal{N}_0^{\mathfrak{f}} \subseteq \mathcal{A}^{\mathfrak{f}}$ is trivial.

(ii) Let $A \in \mathcal{A}^{\mathfrak{f}}$, $\lim\limits_{k\to\infty} f_{n_k}(x) = 0$ for every $x \in A$. We denote

$$B = \left\{ x \in \mathbb{T};\ \lim_{k\to\infty} f_{n_k}(x) = 0 \right\}.$$

Then B is a Borel (in fact an $F_{\sigma\delta}$) set and $A \subseteq B$. Using condition (γ), for any Borel measure on $\mathbb{T}$ we can use the Lebesgue dominated convergence theorem to obtain (1).

Now, let $A \in \mathcal{N}^{\mathfrak{f}}$, $\sum\limits_{n=0}^{\infty} a_n f_n(x) < \infty$ for every $x \in A$ and $\sum\limits_{n=0}^{\infty} a_n = \infty$. As above, we put

$$B = \left\{ x \in \mathbb{T};\ \sum_{n=0}^{\infty} a_n f_n(x) < \infty \right\}.$$

Then B is also a Borel (in fact an F_σ) set. For every $x \in B$ we have

$$\lim_{n\to\infty} \frac{\sum\limits_{k=0}^{n} a_k f_k(x)}{\sum\limits_{k=0}^{n} a_k} = 0.$$

Since by (γ)

$$\frac{\sum\limits_{k=0}^{n} a_k f_k(x)}{\sum\limits_{k=0}^{n} a_k} \le \frac{\sum\limits_{k=0}^{n} a_k d}{\sum\limits_{k=0}^{n} a_k} = d,$$

for an arbitrary Borel measure μ on $\mathbb{T}$ we can again use the Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to\infty} \frac{\sum\limits_{k=0}^{n} a_k \int\limits_B f_k(x)\ \mathrm{d}\mu(x)}{\sum\limits_{k=0}^{n} a_k} = 0$$

and consequently

$$\liminf_{n\to\infty} \int\limits_B f_n(x)\ \mathrm{d}\mu(x) = 0.$$

Thus, for some increasing sequence $\{n_k\}_{k=0}^{\infty}$ we have (1).

(iii) Condition (a) follows from (α). Condition (b) follows directly from the definition. By part (i) it suffices to show that $\mathcal{N}^{\mathfrak{f}}$, $\mathcal{A}^{\mathfrak{f}}$, and $w\mathcal{D}^{\mathfrak{f}}$ satisfy condition (c).

Assume that $(a,b) \in \mathcal{N}^{\mathfrak{f}}$, $0 \le a < b \le 1$. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative reals such that $\sum\limits_{n=0}^{\infty} a_n = \infty$ and the series $\sum\limits_{n=0}^{\infty} a_n f_n(x)$ converges for every $x \in (a,b)$. For $m \in \mathbb{N}$ we put

$$F_m = \left\{ x \in (a,b);\ \sum_{n=0}^{\infty} a_n f_n(x) \le m \right\}. \tag{2}$$

Every set F_m is closed in (a,b) and $\bigcup_{m=0}^{\infty} F_m = (a,b)$. By the Baire category theorem, at least one of the sets F_m has a nonempty interior, i.e. there exists $m \in \mathbb{N}$ and an open interval $(c,d) \subseteq F_m$. Using (2) and condition (β), by integrating we obtain

$$m \cdot (d-c) \geq \int\limits_{(c,d)} \sum_{n=0}^{\infty} a_n f_n(x)\ \mathrm{d}x \geq \sum_{n=n_0}^{\infty} a_n \int\limits_{(c,d)} f_n(x)\ \mathrm{d}x \geq K \cdot \sum_{n=n_0}^{\infty} a_n\,,$$

contradicting $\sum_{n=0}^{\infty} a_n = \infty$.

Assume that $(a,b) \in \mathcal{A}^{\mathfrak{f}}$, i.e. there exists a sequence $\{n_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} f_{n_k}(x) = 0$ for every $x \in (a,b)$. Then, by the Baire category theorem, for some $m \in \mathbb{N}$, the closed set

$$\{x \in \mathbb{T};\ (\forall k \geq m)(f_{n_k}(x) \leq 1)\}$$

has nonempty interior, and hence for some open interval $(c,d) \subseteq (a,b)$ by the Lebesgue dominated convergence theorem we obtain

$$\lim_{k\to\infty} \int\limits_{(c,d)} f_{n_k}(x)\ \mathrm{d}x = 0\,,$$

which contradicts condition (β).

For any $0 \leq c < d \leq 1$, $(c,d) \notin w\mathcal{D}^{\mathfrak{f}}$, since by (β)

$$\liminf_{n\to\infty} \int\limits_{(c,d)} f_n(x)\ \mathrm{d}x \geq K > 0\,.$$

(iv) The assertion follows immediately from the corresponding definitions. □

Now we present two simple examples showing that the thin sets defined above are different from classical trigonometric thin sets.

EXAMPLE 1. We construct a rather trivial example of a countable set $A \subseteq \mathbb{T}$ which is not Dirichlet but for which there exists a sequence of function $\mathfrak{f}$ satisfying conditions (α), (β) and (γ) such that $A \in \mathcal{D}^{\mathfrak{f}}$.

For $0 \leq a < b \leq 1$ we set

$$g_n^{a,b}(x) = \begin{cases} \left|\sin \frac{n\pi}{b-a}(x-a)\right| & \text{if } x \in [a,b]\,, \\ 0 & \text{if } x \in \mathbb{T} \setminus [a,b]\,. \end{cases}$$

Then

$$\int_0^1 g_n^{a,b}(x)\,\mathrm{d}x = \int_a^b g_n^{a,b}(x)\,\mathrm{d}x = \frac{2(b-a)}{\pi}\,.$$

Now, let $a_0 = 1$ and $a_n = 1/2n$ for $n > 0$. Since $\sin n\pi a_n = \sin \pi/2 = 1$ for arbitrary $n > 0$, the set $A = \{a_n;\ n \in \mathbb{N}\}$ is not a Dirichlet set.

We define

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \le x \le a_n\,, \\ g_n^{a_k,a_{k-1}}(x) & \text{for } x \in [a_k, a_{k-1}]\,,\ 1 \le k \le n\,. \end{cases}$$

It is easy to check that the sequence $\mathfrak{f} = \{f_n\}_{n=0}^{\infty}$ satisfies conditions (α), (β) and (γ). Since $f_n(x) = 0$ for any $x \in A$, the set A is $\mathfrak{f}$-Dirichlet.

EXAMPLE 2. Another example was given by J. Arbault [Ar], who has proved that the set

$$A = \Big\{x \in \mathbb{T};\ \sum_{n=0}^{\infty} \big(\sin 2^{2^n}\pi x\big)^2 < \infty\Big\}$$

is not an N_0-set. If we set $f_n(x) = \big(\sin 2^{2^n}\pi x\big)^2$ then A is an $\mathrm{N}_0^{\mathfrak{f}}$-set, where $\mathfrak{f} = \{f_n(x)\}_{n=0}^{\infty}$.

3. Borel bases and subgroups

As in part (ii) of the proof of Theorem 4, we can easily prove the existence of a Borel basis for every family considered. In fact we can prove more (compare [Bu1]):

THEOREM 5.

(i) *The families $\mathcal{D}^{\mathfrak{f}}$, $\mathcal{B}_0^{\mathfrak{f}}$, and $\mathcal{B}^{\mathfrak{f}}$ have closed bases.*

(ii) *The families $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, and $\mathcal{N}^{\mathfrak{f}}$ have F_σ bases. If the sequence $\mathfrak{f}$ satisfies condition (δ) then there exist such bases which in addition consist of subgroups of $\mathbb{T}$.*

(iii) *The family $\mathcal{A}^{\mathfrak{f}}$ has an $F_{\sigma\delta}$ basis. If the sequence $\mathfrak{f}$ satisfies condition (δ) then there exists such a basis which in addition consists of subgroups of $\mathbb{T}$.*

(iv) *The family $w\mathcal{D}^{\mathfrak{f}}$ has a Borel basis.*

Proof.

(i) If $A \in \mathcal{D}^{\mathfrak{f}}$ and f_{n_k}, $k = 0, 1, \dots$, converges uniformly to zero on A then there are positive reals ε_k converging to zero such that $f_{n_k}(x) \le \varepsilon_k$ for every k and every $x \in A$. Evidently A is a subset of the closed set

$$\{x \in \mathbb{T};\ (\forall k)\big(f_{n_k}(x) \le \varepsilon_k\big)\}\,.$$

The proof is similar in the case of the families $\mathcal{B}_0^{\mathfrak{f}}$ and $\mathcal{B}^{\mathfrak{f}}$.

(ii) Let $A \in p\mathcal{D}^{\mathfrak{f}}$, the sequence of positive reals $\{\varepsilon_k\}_{k=0}^{\infty}$ converging to zero and let $\{n_k\}_{k=0}^{\infty}$ be such that for every $x \in A$ there exists a k_0 such that $f_{n_k}(x) \le \varepsilon_k$ whenever $k \ge k_0$. Let

$$B_i = \{x \in \mathbb{T};\ (\forall k \ge i)(f_{n_k}(x) \le \varepsilon_k)\}.$$

Then every B_i is a closed set and A is a subset of the pseudo $\mathfrak{f}$-Dirichlet set $\bigcup_{i=0}^{\infty} B_i$.

Now, assume that $\mathfrak{f}$ satisfies condition (δ). By induction we define:

$$C_0 = \bigcup_{i=0}^{\infty} B_i, \qquad C_{n+1} = C_n - C_n.$$

We define the continuous mapping $h\colon \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ by $h(x,y) = x - y$. Then

$$C_1 = \bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} h(B_i \times B_j)$$

is an F_σ set. Proceeding by induction we obtain that every C_n is an F_σ set. It is easy to see that $C = \bigcup_{i=0}^{\infty} C_i$ is an F_σ subgroup of $\mathbb{T}$ containing A as a subset. We have to show that C is a pseudo $\mathfrak{f}$-Dirichlet set.

We shall use the following simple fact proved e.g. in [Bu2], [CL]: if a sequence $\{g_n\}_{n=0}^{\infty}$ converges quasinormally to a function g on X_k, $k = 0, 1, \dots$, then it does so on the union $\bigcup_{k=0}^{\infty} X_k$.

Thus we have to show that the sequence $\{f_{n_k}\}_{k=0}^{\infty}$ converges quasinormally to zero on every C_i, $i = 0, 1, \dots$. By definition, it does so on C_0. By induction, assume that $f_{n_k}(x)$, $k = 0, 1, \dots$, converges quasinormally to zero on C_i. Thus, there exists a sequence of positive reals β_i, $i = 0, 1, \dots$, converging to zero such that for every $x \in C_i$ there exists a j such that $f_{n_k}(x) \le \beta_k$ for every $k \ge j$. By condition (δ), for $x = y - z \in C_{i+1}$, $y, z \in C_i$, we obtain

$$f_{n_k}(x) \le f_{n_k}(y) + f_{n_k}(z) \le 2\beta_k$$

for sufficiently large k. Thus, $f_{n_k}(x)$, $k = 0, 1, \dots$, converges quasinormally to zero on C_{i+1} and we are finished.

The case of $\mathcal{N}_0^{\mathfrak{f}}$ and $\mathcal{N}^{\mathfrak{f}}$ is simple. If A is such that $\sum_{k=0}^{\infty} f_{n_k}(x) < \infty$ for $x \in A$ then the set

$$B = \left\{x \in \mathbb{T};\ \sum_{k=0}^{\infty} f_{n_k}(x) < \infty\right\}$$

is an F_σ set of $\mathbb{T}$ containing A as a subset. Moreover, if condition (δ) holds true then B is a subgroup of $\mathbb{T}$.

The proof of (iii) is almost identical with the previous one. (iv) follows from the definition. □

In contrast to part (ii) and (iii) we have:

COROLLARY 6. *Assume that $\mathcal{D}^{\mathfrak{f}}$ ($\mathcal{B}_0^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$) satisfies condition* (c). *Then every subgroup of $\mathbb{T}$ belonging to the family $\mathcal{D}^{\mathfrak{f}}$ ($\mathcal{B}_0^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$) is finite.*

P r o o f. Since every infinite subgroup of $\mathbb{T}$ is a dense subset of $\mathbb{T}$, the corollary follows from (i). □

4. Well distributed sequences

One can easily see that for a finite set $A \subseteq \mathbb{T}$ the four equivalent conditions $A \in \mathcal{D}^{\mathfrak{f}}$, $A \in \mathcal{A}^{\mathfrak{f}}$, $A \in \mathcal{N}^{\mathfrak{f}}$, $A \in w\mathcal{D}^{\mathfrak{f}}$ are also equivalent to the following: for every $\varepsilon > 0$ there exists arbitrary large n such that $f_n(x) < \varepsilon$ for $x \in A$. This suggests the definition that a sequence $\mathfrak{f} = \{f_n\}_{n=0}^{\infty}$ is *well distributed* if the following condition holds true

(ε) for every finite number of points $x_1, \dots, x_k \in \mathbb{T}$ and for every $\varepsilon > 0$ there exists arbitrary large n such that $f_n(x_i) < \varepsilon$ for $i = 1, \dots, k$.

Condition (ε) holds true for the sinus sequence by the classical Dirichlet-Minkowski theorem on Diophantine approximation — see e.g. [Ba], [BKR].

THEOREM 7. *If the sequence $\mathfrak{f}$ is well distributed then every countable subset of $\mathbb{T}$ is a $p\mathcal{D}^{\mathfrak{f}}$-set.*

P r o o f. Let $\{x_n ;\ n \in \mathbb{N}\}$ be an enumeration of the countable subset A of $\mathbb{T}$. Using (ε), by induction, one can easily find an $n_{k+1} > n_k$ such that $f_{n_{k+1}}(x_i) < \frac{1}{k+1}$ for every $i = 0, 1, \dots, k+1$. It is then easy to see that the sequence $\{f_{n_k}\}_{k=0}^{\infty}$ converges quasinormally to zero on the set A. □

COROLLARY 8. *If the sequence $\mathfrak{f}$ is well distributed then the families $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, $\mathcal{N}^{\mathfrak{f}}$, and $\mathcal{A}^{\mathfrak{f}}$ contain all countable subsets of $\mathbb{T}$. Moreover, if $\mathfrak{f}$ satisfies condition (γ) then also the family $w\mathcal{D}^{\mathfrak{f}}$ contains every countable subset of $\mathbb{T}$.*

Since the families $\mathcal{D}^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$, and $\mathcal{B}_0^{\mathfrak{f}}$ each contain along with every set also its closure, if any of them is a family of thin sets it cannot contain every countable set, in fact it cannot contain the set $\mathbb{Q} \cap \mathbb{T}$.

One can easily check that Theorem 10 of [Bu1] can be generalized in a similar way — a definition of the cardinal $\mathfrak{p}$ may be found e.g. in [Bu1], [BKR]:

THEOREM 9. *Let $E_s \subseteq \mathbb{T}$ be a $D^{\mathfrak{f}}$-set for every $s \in S$, $|S| < \mathfrak{p}$. If for every finite $T \subseteq S$ the union $\bigcup_{s\in T} E_s$ is a $D^{\mathfrak{f}}$-set then the union $\bigcup_{s\in S} E_s$ is a $pD^{\mathfrak{f}}$-set.*

S k e t c h o f t h e p r o o f . For every finite $T \subseteq S$ and for every natural number m we denote

$$B(T,m) = \Big\{\langle k,n\rangle \in \mathbb{N}\times\mathbb{N};\ \ k,n \geq m \wedge \Big(\forall x \in \bigcup_{s\in T} E_s\Big)\Big(f_n(x) < \tfrac{1}{k+1}\Big)\Big\}.$$

As in [Bu1] we may show that

$$\mathcal{F} = \{B(T,m);\ T \subseteq S \text{ finite},\ \ m \in \mathbb{N}\}$$

is a family of infinite subsets of $\mathbb{N}\times\mathbb{N}$ with the finite intersection property and of cardinality smaller than $\mathfrak{p}$. By the definition of the cardinal $\mathfrak{p}$ there exists an infinite $C \subseteq \mathbb{N}\times\mathbb{N}$ such that $C \setminus B(T,m)$ is finite for every finite $T \subseteq S$ and every m. In the same way as in [Bu1] one can construct two increasing sequences $\{n_i\}_{i=0}^{\infty}$ and $\{k_i\}_{i=0}^{\infty}$ such that

$$\Big(\forall x \in \bigcup_{s\in S} E_s\Big)(\exists i_0)(\forall i \geq i_0)\Big(f_{n_i}(x) < \frac{1}{k_i+1}\Big).$$

Thus the union $\bigcup_{s\in S} E_s$ is a pseudo $\mathfrak{f}$-Dirichlet set. □

COROLLARY 10. *If the sequence $\mathfrak{f}$ is well distributed then the families $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, $\mathcal{N}^{\mathfrak{f}}$, and $\mathcal{A}^{\mathfrak{f}}$ contain every subset of $\mathbb{T}$ of cardinality $< \mathfrak{p}$. If $\mathfrak{f}$ satisfies condition (γ) then the family $w\mathcal{D}^{\mathfrak{f}}$ also contains every subset of $\mathbb{T}$ of cardinality $< \mathfrak{p}$.*

5. Condition (α) is strictly weaker than (ε)

We give a rather complicated example showing that condition (α) is weaker than (ε). In fact the example provides a counterexample for some other implications.

We shall use the following simple fact: if A, B are disjoint closed subsets of $\mathbb{T}$ then there exists a continuous function $f\colon \mathbb{T} \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. If $A \subseteq \mathbb{T}$, $x \in \mathbb{T}$ we denote the distance of the point x from the set A by

$$d(x,A) = \inf\{|x-y|;\ y \in A\}.$$

Let $\{r_n;\ n \in \mathbb{N}\}$ be a one-to-one enumeration of the set $\mathbb{T}\cap\mathbb{Q}$. Let $H = \bigcap_{n\ 0}^{\infty} H_n$ be a G_δ subset of $\mathbb{T}$, $H_n \supseteq H_{n+1}$ being open, such that $\mathbb{T}\cap\mathbb{Q} \subseteq H$

and $\lambda(H) = 0$. The existence of such a set is evident. For an arbitrary natural number n, let $g_n\colon \mathbb{T} \to [0,1]$ be a continuous function such that $g_n(r_i) = 0$ for $i = 0, \dots, n$ and $g_n(x) = 1$ for $x \notin H_n$. Put

$$G = \Bigl\{x \in \mathbb{T};\ \liminf_{n\to\infty} g_n(x) = 0\Bigr\}. \tag{3}$$

One can easily see that G is a G_δ subset of H, and $\mathbb{T}\cap\mathbb{Q} \subseteq G$. Let $G_n \supseteq G_{n+1}$ be open sets such that $G = \bigcap_{n=0}^{\infty} G_n$. We denote $F_n = \mathbb{T} \setminus G_n$ and $F = \bigcup_{n=0}^{\infty} F_n$. Then F is a meager F_σ subset of $\mathbb{T}$, $\lambda(F) = 1$, and hence the closure of F is the whole circle $\mathbb{T}$.

Let $\{a_n;\ n \in \mathbb{N}\} \subseteq \mathbb{T}$ be a countable dense set disjoint from F. We denote

$$\delta_{n,m} = d(a_n, F_m), \qquad d_m = \min\{\delta_{0,m}, \dots, \delta_{m,m}\}.$$

One can easily see that for every n, $\{\delta_{n,m}\}_{m=0}^{\infty}$ is a nonincreasing sequence of positive reals and $\lim_{m\to\infty} \delta_{n,m} = 0$, $d_m > 0$ and $d_m \geq d_{m+1}$ for every m. Moreover, for every $m \geq n$ we have $d_m \leq \delta_{n,m}$ and $\lim_{m\to\infty} d_m = 0$.

For arbitrary n, m we denote by $h_{n,m}$ a continuous function from $\mathbb{T}$ into $[0,1]$ such that $h_{n,m}(x) = 0$ if $x \in F_n$ and $h_{n,m}(x) = 1$ if $d(x, F_n) \geq d_{n+m}$. One can easily see that for arbitrary n we have

$$\lim_{m\to\infty} h_{n,m}(x) = \begin{cases} 0 & \text{if } x \in F_n, \\ 1 & \text{if } x \notin F_n. \end{cases}$$

Let $\mathfrak{f} = \{f_n\}_{n=0}^{\infty}$ be a one-to-one enumeration of the set of functions

$$\{g_n;\ n \in \mathbb{N}\} \cup \{h_{m,n};\ n, m \in \mathbb{N}\}.$$

We put

$$L = \bigl\{n \in \mathbb{N};\ (\exists k)(f_n = g_k)\bigr\}, \qquad K_m = \bigl\{n \in \mathbb{N};\ (\exists k)(f_n = h_{m,k})\bigr\}.$$

The sets L, K_m, $m = 0, 1, \dots$, are pairwise disjoint and $L \cup \bigcup_{m=0}^{\infty} K_m = \mathbb{N}$. We may assume that the enumeration $\mathfrak{f}$ is such that there are increasing functions $\pi\colon \mathbb{N} \xrightarrow{\text{onto}} L$ and $\pi_m\colon \mathbb{N} \xrightarrow{\text{onto}} K_m$, $m = 0, 1, \dots$, such that $f_{\pi(n)} = g_n$ and $f_{\pi_m(n)} = h_{m,n}$ for every $n, m \in \mathbb{N}$.

THEOREM 11. *Let $\mathfrak{f}$ be the sequence of functions defined above. Then*

(i) *$\mathfrak{f}$ satisfies conditions (α) and (γ).*

(ii) *In each open interval $(a,b) \subseteq [0,1]$, there are $u, v \in (a,b)$, $u \neq v$, such that $\{u, v\} \notin w\mathcal{D}^{\mathfrak{f}}$.*

(iii) *$\mathfrak{f}$ is not well distributed, i.e. $\mathfrak{f}$ does not satisfy the condition (ε).*

(iv) *Every family $\mathcal{D}^{\mathfrak{f}}$, $p\mathcal{D}^{\mathfrak{f}}$, $\mathcal{N}_0^{\mathfrak{f}}$, $\mathcal{B}_0^{\mathfrak{f}}$, $\mathcal{N}^{\mathfrak{f}}$, $\mathcal{B}^{\mathfrak{f}}$, $w\mathcal{D}^{\mathfrak{f}}$ is a family of thin sets.*

(v) *$\mathfrak{f}$ does not satisfy condition (β).*

Proof.

(i) All the functions have their values in the interval $[0,1]$ thus (γ) holds. If $x \in G$ then by (3), 0 is an accumulation point of the sequence $\{g_n(x)\}_{n=0}^{\infty}$ and therefore, 0 is also an accumulation point of the sequence $\{f_n(x)\}_{n=0}^{\infty}$. If $x \notin G$ then $x \in F_n$ for some n. Then $h_{n,m}(x) = 0$ for every m and consequently, 0 is an accumulation point of the sequence $\{f_n(x)\}_{n=0}^{\infty}$.

(ii) Let $(a,b) \subseteq [0,1]$ be an open interval. Then there exists a natural number n such that $a_n \in (a,b)$. Since $\lambda(G) = 0$, there is an $u \in F \cap (a,b)$. Let $v = a_n$.

Now, to get a contradiction, suppose that $\{u,v\} \in w\mathcal{D}^{\mathfrak{f}}$ and therefore also that $\{u,v\} \in \mathcal{D}^{\mathfrak{f}}$. Thus there is an infinite set $M \subseteq \mathbb{N}$ such that

$$\lim_{k\in M} f_k(u) = \lim_{k\in M} f_k(v) = 0\,.$$

Since $u \notin G$ by (3), the intersection $M \cap L$ must be finite. Thus we may assume that $M \subseteq \bigcup_{m=0}^{\infty} K_m$. Then either $M \cap K_m$ is infinite for some m or $M \cap K_m$ is nonempty for infinitely many m's.

Since $v \notin F$, if $M \cap K_m$ is infinite then

$$\limsup_{k\in M} f_k(v) \geq \lim_{k\in M\cap K_m} f_k(v) = \lim_{k\in M\cap K_m} h_{m,\pi_m^{-1}(k)}(v) = 1\,,$$

— a contradiction.

On the other hand, if $M \cap K_m$ is nonempty for infinitely many m's, then there are sequences $\{m_i\}_{i=0}^{\infty}$, $\{l_i\}_{i=0}^{\infty}$ such that $\pi_{m_i}(l_i) \in M \cap K_{m_i}$ for every $i \in \mathbb{N}$. Thus $f_{\pi_{m_i}(l_i)} = h_{m_i,l_i}$. We may assume that $\{m_i\}_{i=0}^{\infty}$ is increasing. If $m_i \geq n$ then

$$d(v, F_{m_i}) = \delta_{n,m_i} \geq d_{m_i} \geq d_{m_i+l_i}$$

and for all i such that $m_i \geq n$ we obtain

$$f_{\pi_{m_i}(l_i)}(v) = h_{m_i,l_i}(v) = 1\,,$$

— again a contradiction.

(iii) The assertion is a simple consequence of (ii).

(iv) This is an immediate consequence of (i) and (ii).

(v) Since $\lambda(G) = 0$ we have $\lim_{n\to\infty} \lambda(F_n) = 1$. Thus, for any m we obtain

$$\lim_{n\to\infty} \int_{\mathbb{T}} h_{n,m}(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \lambda(\mathbb{T} \setminus F_n) = 0\,.$$

□

6. The case $f_n(x) = f(nx)$

We shall investigate the case when the sequence $\mathfrak{f} = \{f_n\}_{n=0}^{\infty}$ is generated by a single continuous non-zero function $f\colon \mathbb{T} \to [0,\infty)$ by the equation

$$f_n(x) = f(nx)\,, \qquad \text{for} \quad x \in \mathbb{T}, \;\; n \in \mathbb{N}.$$

Of course, we must at least assume that 0 is an accumulation point of the sequence $\{f(n0)\}_{n=0}^{\infty}$, i.e. $f(0) = 0$. However, it turns out that this condition implies everything necessary. To simplify our notation, in this case the corresponding family $\mathcal{F}^{\mathfrak{f}}$ will be denoted simply by $\mathcal{F}_f$. We start with main result:

THEOREM 12. *Let $f_n(x) = f(nx)$ for $n \in \mathbb{N}$, $x \in \mathbb{T}$, $f\colon \mathbb{T} \to [0,\infty)$ being a continuous function satisfying $f(0) = 0$, $f(x) > 0$ for some $x \in \mathbb{T}$. Then the sequence $\mathfrak{f} = \{f_n\}_{n=0}^{\infty}$ satisfies conditions (β), (γ), and (ε).*

P r o o f. Denote $K = \int\limits_{\mathbb{T}} f(x)\,\mathrm{d}x > 0$. For a given open interval (a,b), let n_0 be such that $4/n_0 \le b-a$. For arbitrary $n \ge n_0$, let k be the smallest natural umber such that $k/n > a$ and let l be the greatest number such that $l/n < b$. Evidently

$$\frac{l-k}{n} \ge \frac{b-a}{2}$$

and therefore

$$\begin{aligned}\int\limits_{(a,b)} f(nx)\,\mathrm{d}x &\ge \int\limits_{(\frac{k}{n},\frac{l}{n})} f(nx)\,\mathrm{d}x \\ &= \int\limits_{(k,l)} \frac{1}{n} f(x)\,\mathrm{d}x \ge \frac{l-k}{n} \int\limits_{(0,1)} f(x)\,\mathrm{d}x \ge \frac{1}{2}\cdot(b-a)K\,.\end{aligned}$$

Thus, the sequence $\{f_n\}_{n=0}^{\infty}$ satisfies condition (β).

Since f is continuous on a compact set, f is bounded by a real d and condition (γ) is fulfilled.

Now, we show that condition (ε) is fulfilled. Assume that $x_1,\dots,x_k \in \mathbb{T}$, $\varepsilon > 0$, and $j \in \mathbb{N}$. Since f is continuous there exists a $\delta > 0$ such that $f(x) \le \varepsilon$ whenever $|x| \le \delta$. Put $n_m = mj$. Then, by Theorem 2, there are $m < l$ such that $f\big((n_l - n_m)x_i\big) \le \varepsilon$ for $i = 1,\dots,k$. Evidently $n_l - n_m = lj - mj \ge j$. □

COROLLARY 13. *If $f\colon \mathbb{T} \to [0,\infty)$ is a continuous function, $f(0) = 0$, and $f(x) > 0$ for some $x \in \mathbb{T}$ then*

(i) *every family* $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{N}_{0f}$, $\mathcal{B}_{0f}$, $\mathcal{N}_f$, $\mathcal{B}_f$, $\mathcal{A}_f$, $w\mathcal{D}_f$ *is a family of thin sets and the following inclusions hold:*

$$\begin{array}{ccccc} & & \mathcal{A}_f & \longrightarrow & w\mathcal{D}_f \\ & & \uparrow & & \uparrow \\ p\mathcal{D}_f & \longrightarrow & \mathcal{N}_{0f} & \longrightarrow & \mathcal{N}_f \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D}_f & \longrightarrow & \mathcal{B}_{0f} & \longrightarrow & \mathcal{B}_f \end{array}$$

(ii) *every family* $p\mathcal{D}_f$, $\mathcal{N}_{0f}$, $\mathcal{N}_f$, $\mathcal{A}_f$, $w\mathcal{D}_f$ *contains all countable subsets of* $\mathbb{T}$.

If, in addition, $f(x-y) \le f(x) + f(y)$ *for any* $x, y \in \mathbb{T}$ *then* $f(x) = f(-x)$ *for* $x \in \mathbb{T}$ *and*

(iii) *every family* $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{N}_{0f}$, $\mathcal{B}_{0f}$, $\mathcal{N}_f$, $\mathcal{B}_f$, $\mathcal{A}_f$ *is trigonometric like and therefore, contains only meager and Lebesgue measure zero sets;*

(iv) $f(x+y) \le f(x) + f(y)$ *for every* $x, y \in \mathbb{T}$, *and hence every family* $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{N}_{0f}$, $\mathcal{B}_{0f}$, $\mathcal{N}_f$, $\mathcal{B}_f$, $\mathcal{A}_f$, $w\mathcal{D}_f$ *is closed under the arithmetic sums*

$$A + A = \{x \in \mathbb{T};\ x = y + z \text{ for some } y, z \in A\}$$

and none of them is an ideal;

(v) *there are sets* $A, B \in \mathcal{D}_f$ *such that neither* $A - B$ *nor* $A + B$ *is in* $w\mathcal{D}_f$.

Proof. (i), (ii) and (iii) follow immediately by Theorem 4. We shall prove parts (iv) and (v). Evidently $f(x+y) \le f(x) + f(y)$. Thus every family considered is closed for the arithmetical sums.

We shall follow an idea of J. Marcinkiewicz [Ma]. Let $\{m_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of natural numbers. We denote $n_k = \sum_{i=0}^{k} m_i$. Any real $x \in \mathbb{T}$ has the unique infinite binary expansion $x = \sum_{i=1}^{\infty} x_i 2^{-i}$, where $x_i = 0, 1$ and there is an arbitrarily large i such that $x_i = 1$. We set

$$A = \left\{x \in \mathbb{T};\ (\forall k)(\forall i)(n_{2k} < i \le n_{2k+1} \to x_i = 0)\right\},$$
$$B = \left\{x \in \mathbb{T};\ (\forall k)(\forall i)(n_{2k+1} < i \le n_{2k+2} \to x_i = 0)\right\}.$$

Let ε be a positive real. Since the sequence $\{m_k\}_{k=0}^{\infty}$ is strictly increasing, there exists a k for which $2^{m_{2k+1}} < \varepsilon$. Then for arbitrary $x \in A$ we obtain (modulo 1):

$$2^{n_{2k}} x = \sum_{i=n_{2k}+1}^{\infty} x_i 2^{-i+n_{2k}} = \sum_{i=n_{2k+1}+1}^{\infty} x_i 2^{-i+n_{2k}} \le 2^{n_{2k}-n_{2k+1}} = 2^{m_{2k+1}} < \varepsilon .$$

Thus A is an $\mathfrak{f}$-Dirichlet set. Similarly, we may show that B is an $\mathfrak{f}$-Dirichlet set.

Any real $x \in \mathbb{T}$ can be written as $x = y + z$ (and $x = y - z$) for some $y \in A$ and $z \in B$. Thus $A \pm B = \mathbb{T}$ and therefore $A \pm B \notin w\mathcal{D}_f$.

Since $A - B \subseteq (A \cup B) - (A \cup B)$ we obtain $A \cup B \notin \mathcal{A}_f$, $A \cup B \notin \mathcal{N}_f$.

By [BL2; Theorem 13], the family $w\mathcal{D}_f$ is trigonometric like and therefore $A \cup B \notin w\mathcal{D}_f$. □

According to Rajchman's theorem ([Ba]), every A-set is an H_σ-set and therefore σ-porous — for the definition see e.g. [Za] or [BKR]. One can easily generalize this result[1] as follows:

THEOREM 14. *If $f\colon \mathbb{T} \to [0,\infty)$ is continuous, $f(0) = 0$ and $f(x) \neq 0$ for some $x \in \mathbb{T}$, then every $\mathcal{A}_f$-set is an H_σ-set and therefore σ-porous.*

Proof. Let $A = \left\{x \in \mathbb{T};\ \lim\limits_{k\to\infty} f(n_k x) = 0\right\}$. Since f is continuous there exists an interval (c,d) such that $f(x) > 0$ for $x \in (c,d)$. The sets

$$A_n = \{x \in \mathbb{T};\ (\forall k \geq n)(n_k x \notin (c,d))\}$$

are H-sets and

$$A \subseteq \bigcup_{n=0}^{\infty} A_n .$$

□

7. Adding a point to a thin set

Let $\mathcal{F}$ be a family of thin sets. According to [Ar], [BKR] and [Bk2] we say that a set $B \subseteq [0,1]$ is *permitted for the family* $\mathcal{F}$ if for every $A \in \mathcal{F}$ also $A \cup B \in \mathcal{F}$.

The classical result concerning trigonometric thin sets says that every finite subset of $\mathbb{T}$ is permitted for trigonometric families of thin sets. We show that similar results hold for our generalization.

THEOREM 15. *Let $f\colon \mathbb{T} \to [0,\infty)$ be a continuous function, $f(0) = 0$, $f(x) > 0$ for some $x \in \mathbb{T}$, and $f(x-y) \leq f(x) + f(y)$ for every $x,y \in \mathbb{T}$. Then every finite set is permitted for the families $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{N}_{0f}$, $\mathcal{B}_{0f}$, $\mathcal{A}_f$, and $w\mathcal{D}_f$.*

[1] I have proved the theorem assuming that $f(x) = 0$ for finitely many $x \in \mathbb{T}$. M. Repický and the referee have remarked that the theorem holds in the general form presented.

Proof. Evidently it suffices to prove the assertion for a one point set since then we may proceed by induction. Moreover, since all the proofs are very similar we shall provide them for just two cases: $\mathcal{N}_{0f}$ and $w\mathcal{D}_f$.

Let $A \in \mathcal{N}_{0f}$ and $x \in \mathbb{T}$. Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of natural numbers such that

$$\sum_{k=0}^{\infty} f(n_k y) < \infty \qquad \text{for} \quad y \in A .$$

Using Theorem 2, by induction, we may construct an increasing sequence $\{k_i\}_{i=0}^{\infty}$ such that

$$f\big((n_{k_{i+1}} - n_{k_i})x\big) < \frac{1}{2^i} . \tag{4}$$

Now put $m_i = n_{k_{i+1}} - n_{k_i}$. We may assume that $m_{i+1} > m_i$ for all i. By (4) we have

$$\sum_{i=0}^{\infty} f(m_i x) \le \sum_{i=0}^{\infty} \frac{1}{2^i} < \infty .$$

For $y \in A$ we have

$$\sum_{i=0}^{\infty} f(m_i y) = \sum_{i=0}^{\infty} f\big((n_{k_{i+1}} - n_{k_i})y\big) \le \sum_{i=0}^{\infty} f(n_{k_{i+1}} y) + \sum_{i=0}^{\infty} f(n_{k_i} y) < \infty .$$

Thus the series $\sum_{i=0}^{\infty} f(m_i y)$ converges on the set $A \cup \{x\}$ and therefore $A \cup \{x\} \in \mathcal{N}_{0f}$.

Now, let $A \in w\mathcal{D}_f$ and $x \in \mathbb{T}$. We may assume that A is a Borel set. Let μ be a positive Borel measure on $\mathbb{T}$. To simplify our computation we assume that $\mu(\mathbb{T}) = 1$. Put $\varepsilon = \mu(\{x\})$. Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of natural numbers such that

$$\lim_{k\to\infty} \int_A f_{n_k}(x) \, \mathrm{d}\mu(x) = 0 .$$

We construct the sequence $\{m_i\}_{i=0}^{\infty}$ in the same way as above. Then

$$\int_{A\cup\{x\}} f(m_i y) \, \mathrm{d}\mu(y) \le \int_A f(m_i y) \, \mathrm{d}\mu(y) + \frac{\varepsilon}{2^i} .$$

Since

$$\lim_{i\to\infty} \int_A f(m_i y) \, \mathrm{d}\mu(y) \le \lim_{i\to\infty} \int_A f(n_{k_{i+1}} y) \, \mathrm{d}\mu(y) + \lim_{i\to\infty} \int_A f(n_{k_i} y) \, \mathrm{d}\mu(y) = 0$$

we have

$$\lim_{i\to\infty} \int_{A\cup\{x\}} f(m_i y) \, \mathrm{d}\mu(y) = 0 .$$

□

As in [BB] using Theorem 9, we can easily prove:

THEOREM 16. *Let $f\colon \mathbb{T} \to [0,\infty)$ be a continuous function, $f(0) = 0$, $f(x) > 0$ for some $x \in \mathbb{T}$, and $f(x-y) \le f(x) + f(y)$ for every $x, y \in \mathbb{T}$. Then every set $A \subseteq \mathbb{T}$ of cardinality smaller than $\mathfrak{p}$ is permitted for $p\mathcal{D}_f$.*

Now, we extend Salem's theorem [Sa], [Ba], [Zy] (finite sets are permitted for $\mathcal{N}$) and the Arbault-Erdös theorem [Ar], [Ba], [Zy] (countable sets are permitted for $\mathcal{N}$). We need some restrictions in the form of a generalization of the classical Lipschitz and Hölder condition. Let $\psi\colon (0,\infty) \to (0,\infty)$. A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be *ψ-continuous* if for every $\delta > 0$ we have $|f(x) - f(y)| \le \psi(\delta)$ whenever $|x - y| \le \delta$, $x, y \in \mathbb{T}$. If $\psi(x) = Mx^\alpha$ with some positive real M we obtain the α-Hölder condition. If $\alpha = 1$ we obtain the Lipschitz condition. In fact, an arbitrary continuous function $f\colon \mathbb{T} \to \mathbb{R}$ is ψ-continuous for a suitable function ψ.

LEMMA 17. *Let $\varphi\colon [0,\infty) \to (0,\infty)$ be an increasing unbounded function such that*

$$\sum_{n=0}^{\infty} \frac{1}{\varphi(n)} < \infty. \tag{5}$$

Let

$$\psi(x) = \frac{1}{x\varphi(\frac{1}{x})}.$$

Assume that $f\colon \mathbb{T} \to [0,\infty)$ is a ψ-continuous function, $f(0) = 0$, $f(x) > 0$ for some $x \in \mathbb{T}$, and $f(x - y) \le f(x) + f(y)$ for every $x, y \in \mathbb{T}$. Assume that $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of nonnegative reals such that $\sum_{n=0}^{\infty} a_n = \infty$, $c > 0$. Let

$$A = \Big\{x \in \mathbb{T};\ \sum_{n=0}^{\infty} a_n f(nx) < \infty\Big\}, \qquad A_c = \Big\{x \in \mathbb{T};\ \sum_{n=0}^{\infty} a_n f(nx) \le c\Big\}.$$

If $b_0, \dots, b_l \in \mathbb{T}$ then there are nonnegative reals c_k, $k = 0, 1, \dots$, such that

(i) $\sum_{k=0}^{\infty} c_k = \infty$;

(ii) $\sum_{k=0}^{\infty} c_k f(kx) < \infty$ *for every* $x \in A \cup \{b_0, \dots, b_l\}$;

(iii) *for every n and $k \le n$ there exists an m such that* $\sum_{i=k}^{n} c_i f(ix) \le \sum_{i=k}^{m} a_i f(ix)$ *for all* $x \in A \cup \{b_0, \dots, b_l\}$;

(iv) *there is $c' > 0$ such that* $\sum_{k=0}^{\infty} c_k f(kx) \le c'$ *for every* $x \in A_c \cup \{b_0, \dots, b_l\}$;

(v) *moreover, for given $\varepsilon > 0$ we can assume that* $\sum_{k=0}^{\infty} c_k f(kb_i) \le \varepsilon$ *for* $i = 0, \dots, l$.

Proof. It suffices to prove the lemma for $l = 0$ and then to proceed by induction.

By Corollary 13(iv), $f(x+y) \le f(x) + f(y)$ for any $x, y \in \mathbb{T}$, and therefore for any natural number n we have $f(nx) \le nf(x)$. Put $s_n = \sum_{k=0}^{n} a_k$. We may assume that $s_0 \ge 1$. By (5) and Theorem 1 we obtain

$$\sum_{k=0}^{\infty} \frac{a_k}{s_k} = \infty, \qquad \sum_{k=0}^{\infty} \frac{a_k}{\varphi(s_k)} < \infty.$$

By Theorem 2, for every natural number k there exists a positive natural number $q_k < s_k + 1 \le 2s_k$ such that $f(q_k k b_0) \le \psi(1/s_k)$.

Thus

$$\sum_{k=0}^{\infty} \frac{a_k}{s_k} f(q_k k b_0) \le \sum_{k=0}^{\infty} \frac{a_k}{s_k} \psi\left(\frac{1}{s_k}\right) = \sum_{k=0}^{\infty} \frac{a_k}{s_k} \frac{s_k}{\varphi(s_k)} = \sum_{k=0}^{\infty} \frac{a_k}{\varphi(s_k)} < \infty.$$

For any $x \in A$ we obtain

$$\sum_{k=0}^{\infty} \frac{a_k}{s_k} f(q_k kx) \le \sum_{k=0}^{\infty} \frac{a_k}{s_k} q_k f(kx) \le 2 \sum_{k=0}^{\infty} \frac{a_k}{s_k} s_k f(kx) \le 2 \sum_{k=0}^{\infty} a_k f(kx) < \infty.$$

Therefore, if we denote

$$c_n = \sum_k \left\{ \frac{a_k}{s_k};\ n = q_k k \right\},$$

(the sum of the empty set is 0) then we have immediately (i), (ii) and (iii).

Taking $c' = \max\left\{2c, \sum_{k=0}^{\infty} c_k f(kb_0)\right\}$ we obtain (iv) (for $l = 0$).

Let $d_i = \sum_{k=0}^{\infty} c_k f(kb_i) > 0$ for $i = 0, 1, \dots, l$. Denote $d = \max\{d_0, \dots, d_l\}$. Replacing every c_k by $\varepsilon c_k / d$ we obtain (v). □

Using the lemma one can now easily prove:

THEOREM 18. *Let $\varphi\colon [0,\infty) \to (0,\infty)$ be an increasing unbounded function such that*

$$\sum_{n=0}^{\infty} \frac{1}{\varphi(n)} < \infty. \tag{6}$$

Let

$$\psi(x) = \frac{1}{x\varphi(\frac{1}{x})}.$$

Assume that $f\colon \mathbb{T} \to [0,\infty)$ is a ψ-continuous function, $f(0) = 0$, $f(x) > 0$ for some $x \in \mathbb{T}$, and $f(x-y) \le f(x) + f(y)$ for every $x, y \in \mathbb{T}$. Then every finite set is permitted for the families $\mathcal{N}_f$ and $\mathcal{B}_f$.

If $\psi(x) = Mx^\alpha$, $M > 0$, $\alpha > 0$ then condition (6) is fulfilled.

THEOREM 19. *Let f, φ, and ψ be as in the Theorem* 18. *If $A \in \mathcal{N}_f$, B being a countable subset of $\mathbb{T}$, then $A \cup B \in \mathcal{N}_f$.*

Proof. We shall follow the proof of P. Erdös as presented in [Zy].

Let $\{b_n;\ n = 0, 1, \dots\}$ be an enumeration of the set B. Assume that $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of nonnegative reals such that

$$\sum_{n=0}^{\infty} a_n = \infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} a_n f(nx) < \infty \quad \text{for every } x \in A.$$

We construct a sequence of sequences of nonnegative reals $\{a_n^i\}_{n=0}^{\infty}$ and two increasing sequences of integers $\{n_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ such that for every $i \in \mathbb{N}$

$$n_{i+1} > m_i \geq n_i, \qquad \sum_{n=m_i+1}^{n_{i+1}} a_n^i \geq 1,$$

$$\sum_{n=m_i+1}^{n_{i+1}} a_n^i f(nb_j) \leq \frac{1}{2^i} \qquad \text{for} \quad j = 0, \dots, i,$$

and

$$\sum_{n=m_i+1}^{n_{i+1}} a_n^i f(nx) \leq \sum_{n=m_i+1}^{m_{i+1}} a_n f(nx) \qquad \text{for} \quad x \in A. \tag{7}$$

The construction is easy. Set $a_n^0 = a_n$ for every $n \in \mathbb{N}$, $m_0 = n_0 = -1$ and take $n_1 \geq 0$ such that $\sum_{n=m_0+1}^{n_1} a_n^0 \geq 1$ and $\sum_{n=n_0+1}^{n_1} a_n^0 f(nb_0) \leq \frac{1}{2^0} = 1$. Set $m_1 = n_1$. Assume we have already defined a_n^i, n_i and m_i. Let n_{i+1} be such that $\sum_{n=m_i+1}^{n_{i+1}} a_n^i \geq 1$ and we apply Lemma 17 to the series $\sum_{n=m_i+1}^{\infty} a_n f(nx)$, i.e. we assume that $a_n = 0$ for $n \leq m_i$, $\varepsilon = 2^{-i-1}$ and $l = i$. We obtain the reals a_n^{i+1} $(= c_n)$ and let m_{i+1} be such that (7) holds. The existence of such an m_{i+1} follows from Lemma 17.

Now, we define the sequence $\{c_n\}_{n=0}^{\infty}$ of nonnegative reals as follows:

$$c_n = \begin{cases} a_n^i & \text{if } m_i < n \leq n_{i+1} \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum_{n=m_{i-1}+1}^{m_i} c_n = \sum_{n=m_{i-1}+1}^{n_i} a_n^i \geq 1$ for every i, we have $\sum_{n=0}^{\infty} c_n = \infty$.

For any i and $n \geq i$ we have $c_n f(nb_i) \leq 2^{-n}$. Therefore

$$\sum_{n=0}^{\infty} c_n f(nb_i) = \sum_{n=0}^{m_i} c_n f(nb_i) + \sum_{n=m_i+1}^{\infty} c_n f(nb_i) \leq \sum_{n=0}^{m_i} c_n f(nb_i) + \sum_{n=i}^{\infty} \frac{1}{2^n} < \infty.$$

If $x \in A$ then we obtain

$$\sum_{n=0}^{\infty} c_n f(nx) = \sum_{i=0}^{\infty} \sum_{n=m_i+1}^{n_{i+1}} a_n^i f(nx) \leq \sum_{i=0}^{\infty} \sum_{n=m_i+1}^{m_{i+1}} a_n f(nx) < \infty .$$

□

Other results on trigonometric thin sets can be generalized in a similar way, for example, the reasoning presented in [BB] can be taken over almost literally to obtain:

THEOREM 20. *Let f, φ and ψ be as in Theorem 17. If $A \in \mathcal{N}_f$, $B \subseteq \mathbb{T}$ has cardinality smaller than $\mathfrak{p}$, then $A \cup B \in \mathcal{N}_f$, i.e. every subset of $\mathbb{T}$ of cardinality smaller than $\mathfrak{p}$ is permitted for $\mathcal{N}_f$.*

However, note that this theorem is also a corollary of Theorem 18 of [Bk2].

8. Some open problems

In the trigonometric case $f(x) = |\sin 2\pi x|$ all inclusions in Corollary 13(i) are proper. We do not know what happens in more general cases.

S. V. Konyagin [Za] has constructed an N-set which is not σ-porous. We are not able to generalize his construction. Thus:

PROBLEM. *To find conditions on $\mathfrak{f}$ which imply that*

(i) *the inclusions in Theorem* 4(i), (ii) *are proper;*
(ii) *there exists a non-σ-porous $\mathcal{N}^{\mathfrak{f}}$-set.*

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