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ON MEAN VALUES OF SUBADDITIVE PROCESSES

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During the last decade the theory of subadditive processes has developed and deepend. The existence of the mean value of a subadditive process plays a key role in this theory.

Definition. *Let T be a subset of the real line closed under addition, with the property: if s, $t \in T$, $s < t$, then also $t - s \in T$. A subadditive process on T is a family $(X_{s,t};\ s,\ t \in T,\ s < t)$ of random variables satisfying the following conditions:*

S 1. *Whenever $s < t < u$, $X_{s,u} \leqslant X_{s,t} + X_{t,u}$.*

S 2. *For all $u \in T$, the joint distributions of $(X_{s+u,t+u})$ are the same as those of $(X_{s,t})$.*

S 3. *For all positive $t \in T$, the expectation $E(X_{0,t})$ exists and satisfies $E(X_{0,t}) \geqslant -At$ for some constant $A \geqslant 0$.*

Assertion. (Kingman, [1, assertion 1.4.1.]) *Let $(X_{s,t})$ be a subadditive process on T. Then the finite limit*

$$\gamma = \lim_{\substack{t \to \infty \\ t \in T}} E(X_{0,t})/t = \inf_{\substack{t > 0 \\ t \in T}} E(X_{0,t})/t$$

exists.

The finite limit γ is called the mean value of a subadditive process.

Example. Let $T = Q^+$ be the set of all nonnegative rational numbers. Let $g(m/n) = \log_2 n$ for natural m, n, $GCD(m, n) = 1$. Then $X_{s,t} = g(t - s)$ for s, $t \in T$, $s < t$, is a subadditive process (on every probability space). This subadditive process has no mean value γ, i.e. the finite limit

$$\lim_{\substack{t \to \infty \\ t \in T}} E(X_{0,t})/t$$

does not exist.

Proof. As the process $(X_{s,t})$ is degenerate, $E(X_{s,t}) = X_{s,t}$, the condition S 2 is satisfied. As $\log_2 n$ for natural n is nonnegative, the condition S 3 is satisfied with $A = 0$. The subadditivity condition S 1 will be satisfied, if the function g is subadditive, i.e. if $g(p + r) \leqslant g(p) + g(r)$ for all p, $r \in Q^+ - \{0\}$. Let $p = m/n$,

$r=i/j$ for natural m, n, i, j, $GCD(m,n)=1$, $GCD(i,j)=1$. Then $p+r=(mj+ni)/jn=k/d$ for k, d natural, $GCD(k,d)=1$. Of course $jn \geq d$, so that $g(p+r) = \log_2 d \leq \log_2 jn = \log_2 j + \log_2 n = g(p)+g(r)$. This proves the fact that $(X_{s,t})$ is a subadditive process.

Let $t_n = n = n/1$ for $n=1, 2, \ldots$. Then $E(X_{0,t_n})=0$,

$$\lim_{n\to\infty} E(X_{0,t_n})/t_n = 0$$

Let $s_n = (n2^n+1)/2^n$ for $n=1, 2, \ldots$. Then $E(X_{0,s_n})=n$, so that

$$\lim_{n\to\infty} E(X_{0,s_n})/s_n = 1.$$

As we have two subsequences with different limits, the mean value of this process cannot exist.

As our example contradicts the assertion of Kingman, a subadditive process with a mean value necessarily satisfies stronger conditions than S 1, S 2, S 3. If T is the set of all integers or of all nonnegative integers, the conditions S 1, S 2, S 3 are strong enough to guarantee the existence of a mean value γ [2, Theorem 1.1.]. The following theorem solves the general case.

Theorem. Let $(X_{s,t})$ be a subadditive process on T satisfying the following condition:

there exists $r \in T$, $r>0$ such that

$$B = \sup_{\substack{t\in T\\ 0<t\leq r}} E(X_{0,t}) < \infty.$$

Then the finite limit

$$\gamma = \lim_{\substack{t\to\infty\\ t\in T}} E(X_{0,t})/t$$

exists.

Proof. As $(X_{s,t})$ is a subadditive process, it satisfies the condition S 2. So we have $E(X_{0,t})=E(X_{u,u+t})$ for all u, $t\in T$, $t>0$. Let r from the condition in our theorem be given. From the condition S 1 it follows that

$$X_{0,t} \leq X_{0,nr} + X_{nr,t}$$

for $t\in T$, $t>0$, where n is the integer part of t/r $(X_{t,t}\equiv 0)$. Then $E(X_{0,t}) \leq E(X_{0,nr}) + B$. Since $\lim_{\substack{t\to\infty\\ t\in T}} t/nr = 1$, and the condition S 3 implies $-Ar \leq B$, we get

$$\lim_{\substack{t\to\infty\\ t\in T}} E(X_{0,t})/t \leq \lim_{n\to\infty} E(X_{0,nr})/nr = \gamma_r.$$

The existence of the finite limit γ_r follows from [2, Theorem 1.1.], since $(X_{nr, mr})$, where n, m are nonnegative integers, $n < m$, is a subadditive process with discrete parameters.

Similarly

$$X_{0,(n+1)r} \leq X_{0,t} + X_{t,(n+1)r},$$

so that

$$\gamma_r \leq \lim_{\substack{t \to \infty \\ t \in T}} E(X_{0,t})/t.$$

This proves our theorem. Moreover, we have $\gamma = \gamma_r$.

Remark. For every $r \in Q^+$, $r > 0$, we have $\sup_{\substack{t \in Q \\ 0 < t \leq r}} g(t) = \infty$, where g is the function from our example. So the subadditive process from our example does not satisfy the condition from our theorem. The assertion 1.4.1. in [1] about existence of a finite mean value γ for every subadditive process is not correct. Of course, the condition imposed in Theorem 4 of [1] (namely 1.4.7.) is strong enough to eliminate the difficulty, but it si much stronger than the condition of our theorem.

REFERENCES

[1] KINGMAN, J. F. C.: Subadditive ergodic theory. Ann. Prob., 1, 1973, 883—909.

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О СРЕДНИХ СУБАДДИТИВНЫХ ПРОЦЕССОВ

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Резюме

В предлагаемой работе исследуются средние субаддитивных процессов с общими параметрами. Пример показывает, что утверждение 1.4.1. в [1] неверно.