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AN ELEMENTARY PROOF OF THE FUBINI-STONE THEOREM

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0. The Fubini-Stone theorem is the analog of the Fubini theorem from the theory of integration for the Daniell integral, see 7-2 in [1], or § 23 Th. 2 in [2]. Its proofs (as far as it is known to the author) essentially exploit, besides elementary facts, the completeness of the class of summable functions. The proofs of the mentioned completeness are rather long and exploit the monotone and the dominated convergence theorem, see the proofs of Th. 6-4IV in [1] and of Th. 1 in § 16 in [2]. The purpose of this note is to give a short proof based only on few quite standard elementary facts. These facts, together with notations, are summarized in points 1 and 2 below.

1. $R=(-\infty, +\infty)$ and $R^*=\langle -\infty, +\infty\rangle$ with operations as in 4-1 in [1]. $(T, \mathscr{F}, I)$ denotes an elementary Daniell integral, see 6-1 in [1]. $\mathscr{F}^o$ is the class of over-functions of $\mathscr{F}$ and $I^o: \mathscr{F}^o \to R^*$ is the corresponding extension of I, see 6-2 in [1]. We point out the next simple fact, see Th. 6-2III(d) in [1]:

(1) If $f_n \in \mathscr{F}^o$, $n=1, 2, \ldots$, and $f_n \nearrow f$, then $f \in \mathscr{F}^o$ and $I^o(f_n) \nearrow I^o(f)$.

For each $f: T \to R^*$ we define its upper integral $\bar{I}(f)$ and its lower integral $\underline{I}(f)$ by equalities:

$$\bar{I}(f)=\inf\{I^o(h): h\in\mathscr{F}^o, h\geqq f\}, \ (\inf\{\emptyset\}=+\infty),$$

and

$$\underline{I}(f)=-\bar{I}(-f).$$

The class $\mathscr{L}$ of summable functions is determined by the equality:

$$\mathscr{L}=\{f: f: T\to R^*, -\infty<\underline{I}(f)=\bar{I}(f)<+\infty\}.$$

For $f\in\mathscr{L}$ the common value $\underline{I}(f)=\bar{I}(f)$ is denoted by $I(f)$.

The classes $\mathscr{N}$ and $\boldsymbol{N}$ of I-null functions and I-null sets respectively are defined by equalities:

$$\mathscr{N}=\{f: f: T\to R^*, \bar{I}(|f|)=0\},$$

and

$$\boldsymbol{N}=\{E: E\subset T, \chi_E\in\mathscr{N}\}.$$

If $h \in \mathscr{F}^o$, $H = \{t: t \in T, h(t) = +\infty\}$ and $I^o(h) < +\infty$, then $H \in \boldsymbol{N}(\chi_H \leqq \frac{1}{n}(h \vee 0)$ for each $n = 1, 2, \ldots,$ and $I^o(h \vee 0) = I^o(h) - I^o(h \wedge 0) < +\infty$, since $I^o(h \wedge 0) > -\infty \cdot)$. Hence,

(2) if $f: T \to R^*$, $B^+ = \{t: t \in T, f(t) = +\infty\}$, and $\bar{I}(f) < +\infty$, then $B^+ \in \boldsymbol{N}$.

The properties of I^o and the definition of $\bar{I}$ imply:

(3) If $f, g: T \to R^*$, and $\bar{I}(f) + \bar{I}(g)$ is not of the form $(+\infty) + (-\infty)$, or $(-\infty) + (+\infty)$, then $\bar{I}(f+g) \leqq \bar{I}(f) + \bar{I}(g)$.

Thus $A \cup B \in \boldsymbol{N}$, when $A, B \in \boldsymbol{N}$. If $A \subset T$, $B \in \boldsymbol{N}$, and $A \subset B$, then $A \in \boldsymbol{N}$ by the monotonicity of $\bar{I}$.

Using (1) we easily obtain:

(4) If $f: T \to R^*$ and $A = \{t: t \in T, f(t) \neq 0\}$, then $f \in \mathscr{N} \Leftrightarrow (+\infty)|f| \in \mathscr{N} \Leftrightarrow A \in \boldsymbol{N}$.

(3) and the definition of $\underline{I}$ implies:

(5) If $f, g: T \to R^*$, and $\underline{I}(f) + \underline{I}(g)$ is not of the form $(+\infty) + (-\infty)$, or $(-\infty) + (+\infty)$, then $\underline{I}(f+g) \geqq \underline{I}(f) + \underline{I}(g)$.

If $\bar{I}(f) < +\infty$, then $\bar{I}(-f) + \bar{I}(f)$ is not of the form $(-\infty) + (+\infty)$, hence $0 = \bar{I}(-f+f) \leqq \bar{I}(-f) + \bar{I}(f)$ by (3). Thus

$$\text{(6)} \qquad \underline{I}(f) \leqq \bar{I}(f) \quad \text{for each} \quad f: T \to R^*.$$

(3), (5), (6) and the definition of $\mathscr{L}$ imply:

(7) If $f, g \in \mathscr{L}$, then $-f, f+g \in \mathscr{L}$, and $I(f+g) = I(f) + I(g)$.

Let $f \in \mathscr{N}$. Then by the monotonicity of $\underline{I}$ and $\bar{I}$ and (6) $0 = -\bar{I}(|f|) = \underline{I}(-|f|) \leqq \underline{I}(f) \leqq \bar{I}(f) \leqq \bar{I}(|f|) = 0$. Thus

$$\text{(8)} \qquad \mathscr{N} \subset \mathscr{L}.$$

2. Let $(T_1, \mathscr{F}_1, I_1)$ and $(T_2, \mathscr{F}_2, I_2)$ be two elementary Daniell integrals and denote by $\mathscr{L}_1$ and $\mathscr{L}_2$ the corresponding classes of summable functions. Put $T_3 = T_1 \times T_2$. By $\mathscr{F}_1 * \mathscr{F}_2$ we denote the class of all $f: T_3 \to R$ such that $f(t_1, \cdot) \in \mathscr{F}_2$ for each $t_1 \in T_1$ and such that $I_2 f(\cdot, \cdot) \in \mathscr{F}_1$ (from now on we use If instead of $I(f)$).

Suppose $\mathscr{F}_3$ to be a vector lattice of functions $f: T_3 \to R$ such that $\mathscr{F}_3 \subset \mathscr{F}_1 * \mathscr{F}_2$, and for $f \in \mathscr{F}_3$ put $I_3 f = I_1 I_2 f(\cdot, \cdot)$. Then clearly $(T_3, \mathscr{F}_3, I_3)$ is an elementary Daniell integral. By $\mathscr{L}_3$ we denote the corresponding class of summable functions.

If $f: T_3 \to R^*$, $h_n \in \mathscr{F}_3$, $n = 1, 2, \ldots$, and $h_n \nearrow h \geqq f$, then using (1) we easily obtain that $\bar{I}_1 \bar{I}_2 f(\cdot, \cdot) \leqq \bar{I}_1 \bar{I}_2 h(\cdot, \cdot) = I^o I^o h(\cdot, \cdot) = \lim I_1 I_2 h_n(\cdot, \cdot) = \lim I_3 h_n = I_3^o h$. Thus

$$\text{(9)} \qquad \bar{I}_1 \bar{I}_2 f(\cdot, \cdot) \leqq \bar{I}_3 f \quad \text{for each} \quad f: T_3 \to R^*.$$

From (9), (6) and the definition of the lower integral we immediately have our basic inequality:

$$\text{(10)} \qquad \underline{I}_3 f \leqq \underline{I}_1 \underline{I}_2 f(\cdot, \cdot) \leqq \bar{I}_1 \bar{I}_2 f(\cdot, \cdot) \leqq \bar{I}_3 f \quad \text{for each} \quad f: T_3 \to R^*.$$

We define $\mathscr{L}_1 * \mathscr{L}_2$ to be the class of all $f\colon T_3 \to R^*$ such that there exist an I_1-null set $E \subset T_1$ and a $\varphi \in \mathscr{L}_1$ such that $f(t_1, \cdot) \in \mathscr{L}_2$ and $I_2 f(t_1, \cdot) = \varphi(t_1)$ if $t_1 \in T_1 - E$. For such an f with corresponding φ we write $I_2 f(\cdot, \cdot) = \varphi$. By this definition $I_2 f(\cdot, \cdot)$ does not have a unique meaning as an element of $\mathscr{L}_1$. Since the ambiguity involves only an I_1-null set $E \subset T_1$, however, by (4), (7) and (8) the numerical value $I_1 I_2 f(\cdot, \cdot) = I_1 \varphi$ is unique.

3. The Fubini-Stone theorem. Suppose that $\mathscr{F}_3 \subset \mathscr{F}_1 * \mathscr{F}_2$ and that $I_3 f = I_1 I_2 f(\cdot, \cdot)$ for each $f \in \mathscr{F}_3$. Then $\mathscr{L}_3 \subset \mathscr{L}_1 * \mathscr{L}_2$ and $I_3 f = I_1 I_2 f(\cdot, \cdot)$ for each $f \in \mathscr{L}_3$.

Proof. Let $f \in \mathscr{L}_3$. Then by (6) and (10)
$-\infty < I_3 f = \underline{I}_1 \underline{I}_2 f(\cdot, \cdot) = \bar{I}_1 \underline{I}_2 f(\cdot, \cdot) = \underline{I}_1 \bar{I}_2 f(\cdot, \cdot) = \bar{I}_1 \bar{I}_2 f(\cdot, \cdot) = I_3 f < +\infty$, hence $\underline{I}_2 f(\cdot, \cdot)$, $\bar{I}_2 f(\cdot, \cdot) \in \mathscr{L}_1$, and $I_1[\bar{I}_2 f(\cdot, \cdot) - \underline{I}_2 f(\cdot, \cdot)] = 0$ by (7). Thus owing to (6) and (4) there is an I_1-null set $A \subset T_1$ such that $\bar{I}_2 f(t_1, \cdot) = \underline{I}_2 f(t_1, \cdot)$ for each $t_1 \in T_1 - A$. Since $\bar{I}_2 f(\cdot, \cdot) \in \mathscr{L}_1$, according to (2) there is an I_1-null set $B \subset T_1$ such that $|\bar{I}_2 f(t_1, \cdot)| < +\infty$ for each $t_1 \in T_1 - B$. Thus $f(t_1, \cdot) \in \mathscr{L}_2$ for each $t_1 \in T_1 - (A \cup B)$. Taking $\varphi = \bar{I}_2 f(\cdot, \cdot)$ and $E = A \cup B$ we see that $f \in \mathscr{L}_1 * \mathscr{L}_2$ and that $I_3 f = I_1 I_2 f(\cdot, \cdot)$. The theorem is proved.

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ЭЛЕМЕНТАРНОЕ ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ ФУБИН-СТОУНА

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Резюме

Теорема Фубини-Стоуна является аналогом теоремы Фубини для интеграла Даниэлля, см. (1, отдел 7-2) или (2, § 23 Теор. 2). В заметке дается короткое доказательство этой теоремы основано на простом неравенстве (10) и на самых элементарных свойствах интеграла Даниэлля.