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ON THE CONTINUITY OF THE SEMIVARIATION IN LOCALLY CONVEX SPACES

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ABSTRACT. If the introduced Condition (GB) is fulfilled, then everywhere convergence of nets of measurable functions implies convergence in semivariation on a set of finite variation of a measure $\mathbf{m}: \Sigma \to L(\mathbf{X}, \mathbf{Y})$ which is σ-additive in the strong operator topology (Σ is a σ-algebra of subsets, and $\mathbf{X}$, $\mathbf{Y}$ are both locally convex spaces). In the case of the purely atomic measure Condition (GB) is fulfilled.

Introduction

In the operator valued measure theory in Banach spaces pointwise convergence of sequences of measurable functions on a set of finite semivariation implies convergence in (continuous) semivariation of the measure $\mathbf{m}: \Sigma \to L(\mathbf{X}, \mathbf{Y})$, where Σ is a σ-algebra of subsets of a set $T \neq \emptyset$, and $\mathbf{X}$, $\mathbf{Y}$ are Banach spaces, cf. [1]. If $\mathbf{X}$ fails to be metrizable, the relation between these two convergences is quite unlike the classical situation, cf. [6, Example after Definition 1.11].

The importance of Condition (B) (for sequences) in the classical measure and integration theory was stressed by N. N. Luzin in his dissertation [5]. Condition (B) for nets in the classical setting was introduced and investigated by B. F. Goguadze, cf. [2]. We introduce Condition (GB), see Definition 1.2, which generalizes Condition (B) to the case of a measure $\mathbf{m}: \Sigma \to L(\mathbf{X}, \mathbf{Y})$ which is σ-additive in the strong operator topology, where Σ is a σ-algebra of subsets, and $\mathbf{X}$, $\mathbf{Y}$ are both locally convex spaces. If the introduced condition (GB) is fulfilled, then everywhere convergence of a net of measurable functions implies convergence in semivariation on a set of finite variation of the measure $\mathbf{m}$. The new condition concerns families of submeasures. If the measure $\mathbf{m}$ is purely atomic, then Condition (GB) is fulfilled.

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1. Preliminaries

By a *net* (with values in a set D) we mean a function from I to D, where I is a directed partially ordered set. To be more exact we will sometimes specify that, for instance, the net is an I-net to indicate that I is an index set for a given net. For terminology concerning the nets see [4]. $\mathbb{N} = \{1, 2, \dots\}$.

Let $\mathbf{X}$, $\mathbf{Y}$ be two Hausdorff locally convex topological vector spaces over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of seminorms which define the topologies on $\mathbf{X}$ and $\mathbf{Y}$, respectively. Let $L(\mathbf{X}, \mathbf{Y})$ denote the space of all continuous linear operators $L: \mathbf{X} \to \mathbf{Y}$.

Let $T \neq \emptyset$ be a set and let Σ be a σ-algebra of subsets of T. Denote by χ_E the characteristic function of the set E.

Let $\mathbf{m}: \Sigma \to L(\mathbf{X}, \mathbf{Y})$ be an operator valued measure σ-additive in the strong operator topology, i.e. if $E \in \Sigma$, then $\mathbf{m}(E)\mathbf{x}$ is an $\mathbf{Y}$-valued measure for every $\mathbf{x} \in \mathbf{X}$.

DEFINITION 1.1. *Let $p \in \mathcal{P}$, $q \in \mathcal{Q}$. Let $E \in \Sigma$.*

(a) *By the p,q-semivariation of a measure $\mathbf{m}$, cf.* [6], *we mean a set function $\hat{\mathbf{m}}_{p,q}: \Sigma \to [0, \infty]$, defined as follows:*

$$\hat{\mathbf{m}}_{p,q}(E) = \sup q\left(\sum_{n=1}^{N} \mathbf{m}(E_n)\mathbf{x}_n\right),$$

where the supremum is taken over all finite disjoint partitions $\{E_n \in \Sigma;\ E = \bigcup_{n=1}^{N} E_n,\ E_n \cap E_m = \emptyset,\ n \neq m,\ m, n = 1, 2, \dots, N\}$ of E and all finite sets $\{\mathbf{x}_n \in \mathbf{X};\ p(\mathbf{x}_n) \leq 1,\ n = 1, 2, \dots, N\}$, $N \in \mathbb{N}$.

(b) *By the p,q-variation of a measure $\mathbf{m}$ we mean a set function $\mathbf{v}_{p,q}(\mathbf{m}, \cdot): \Sigma \to [0, \infty]$, defined as follows:*

$$\mathbf{v}_{p,q}(\mathbf{m}, E) = \sup \sum_{n=1}^{N} q_p\big(\mathbf{m}(E_n)\big), \qquad E \in \Sigma, \quad q_p\big(\mathbf{m}(E)\big) = \sup_{p(\mathbf{x}) \leq 1} q\big(\mathbf{m}(E)\mathbf{x}\big),$$

where the supremum is taken over all finite disjoint partitions $\{E_n \in \Sigma;\ E = \bigcup_{n=1}^{N} E_n,\ E_n \cap E_m = \emptyset,\ n \neq m,\ n, m = 1, 2, \dots, N,\ N \in \mathbb{N}\}$ of E.

The proof of the following lemma is trivial.

LEMMA 1.2. *The p,q-(semi)variation of $\mathbf{m}$ is a monotone and σ-additive (σ-subadditive) set function, and $\mathbf{v}_{p,q}(\emptyset) = 0$ ($\hat{\mathbf{m}}_{p,q}(\emptyset) = 0$) for every $p \in \mathcal{P}$ and $q \in \mathcal{Q}$.*

Note that $\hat{\mathbf{m}}_{p,q}(E) \leq \mathbf{v}_{p,q}(E)$ for every $q \in \mathcal{Q}$, $p \in \mathcal{P}$, $E \in \Sigma$.

DEFINITION 1.3. *We say that a set $E \in \Sigma$ is of positive variation of a measure $\mathbf{m}$ if there exist $q \in \mathcal{Q}$, $p \in \mathcal{P}$ such that $\mathbf{v}_{p,q}(\mathbf{m}, E) > 0$.*

We say that a set $E \in \Sigma$ is $\hat{\mathbf{m}}$-null if $\hat{\mathbf{m}}_{p,q}(E) = 0$ for every $q \in \mathcal{Q}$, $p \in \mathcal{P}$.

We say that a set $E \in \Sigma$ is of finite variation of a measure $\mathbf{m}$ if to every $q \in \mathcal{Q}$ there exists $p \in \mathcal{P}$ such that $\mathbf{v}_{p,q}(\mathbf{m}, E) < \infty$. We will denote this relation briefly $\mathcal{Q} \to_E \mathcal{P}$, or, $q \mapsto_E p$, $q \in \mathcal{Q}$, $p \in \mathcal{P}$.

Note that the relation $\mathcal{Q} \to_E \mathcal{P}$ in Definition 1.3 may be different for different sets $E \in \Sigma$ of finite variation of $\mathbf{m}$.

DEFINITION 1.4. *A measure $\mathbf{m}$ is said to satisfy Condition* (GB) *if for every $E \in \Sigma$ of finite variation and every net of sets $E_i \in \Sigma$, $E_i \subset E$, $i \in I$, there holds*

$$\limsup_{i \in I} E_i \neq \emptyset$$

whenever there exist real numbers $\delta(q, p, E) > 0$, $p \in \mathcal{P}$, $q \in \mathcal{Q}$, such that $\hat{\mathbf{m}}_{p,q}(E_i) \geq \delta(q, p, E)$ for every $i \in I$ and every couple $(p, q) \in \mathcal{P} \times \mathcal{Q}$ such that $q \mapsto_E p$.

DEFINITION 1.5. *We say that a set $A \in \Sigma$ of positive semivariation of a measure $\mathbf{m}$ is an $\hat{\mathbf{m}}$-atom if every subset E of A is either $\emptyset$ or $E \notin \Sigma$. We say that a measure $\mathbf{m}$ is purely atomic if each $E \in \Sigma$ can be expressed in the form $E = \bigcup_{k=1}^{\infty} A_k$, where A_k, $k \in \mathbb{N}$, are $\hat{\mathbf{m}}$-atoms.*

DEFINITION 1.6. *A function $\mathbf{f}\colon T \to \mathbf{X}$ is said to be measurable if*

$$\{t \in T;\ p(\mathbf{f}(t)) \geq \eta\} \in \Sigma$$

for every $\eta > 0$ and $p \in \mathcal{P}$.

LEMMA 1.7. *If there exists a nonmeasurable set E such that $E \subset E_0$, $E_0 \in \Sigma$, and every finite subset of E is measurable, then the set of all measurable functions is not closed with respect to pointwise limits of measurable functions.*

Proof. In [2, 10.1, p. 126], there is shown the assertion for increasing nets of measurable, real and uniformly bounded functions.

DEFINITION 1.8. *We say that a net $\mathbf{f}_i$, $i \in I$, of measurable functions is eventually $\hat{\mathbf{m}}$-convergent on $E \in \Sigma$ to a measurable function $\mathbf{f}$ if for every $q \in \mathcal{Q}$ there is $p \in \mathcal{P}$, such that for every $\eta > 0$,*

$$\lim_{i \in I} \hat{\mathbf{m}}_{p,q}\Big(\{t \in E;\ p(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \eta\}\Big) = 0. \tag{1}$$

2. Condition (GB) and purely atomic $L(\mathbf{X},\mathbf{Y})$-valued measures

In this section we show that a class of measures satisfying Condition (GB) is non empty. First, we prove a lemma.

LEMMA 2.1. *Let $E \in \Sigma$ be a set of positive and finite variation of a (countable) purely atomic measure $\mathbf{m}$.*

If A_k, $k \in \mathbb{N}$, is a class of $\hat{\mathbf{m}}$-atoms such that $A_k \subset E$, $k \in \mathbb{N}$, then

$$\mathbf{v}_{p,q}(\mathbf{m},E) = \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k)$$

for every couple $(p,q) \in \mathcal{P} \times \mathcal{Q}$ such that $q \mapsto_E p$.

Proof. Let $q \mapsto_E p$, $q \in \mathcal{Q}$, $p \in \mathcal{P}$. Then by Definition 1.1 and Lemma 1.2 we obtain

$$\begin{aligned}\mathbf{v}_{p,q}(\mathbf{m},E) &= \sum_{k=1}^{\infty} \mathbf{v}_{p,q}(\mathbf{m},A_k) = \sum_{k=1}^{\infty} q_p\big(\mathbf{m}(A_k)\big) \\ &= \sum_{k=1}^{\infty} \sup_{p(\mathbf{x})\le 1} q\big(\mathbf{m}(A_k)\mathbf{x}\big) = \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k),\end{aligned}$$

because A_k, $k \in \mathbb{N}$, are $\hat{\mathbf{m}}$-atoms.

THEOREM 2.2. *If $\mathbf{m}$ is a (countable) purely atomic $L(\mathbf{X},\mathbf{Y})$-valued measure, then $\mathbf{m}$ satisfies Condition* (GB).

Proof. Let $E \in \Sigma$ be an arbitrary set of finite variation. Let $E_i \in \Sigma$, $i \in I$, be an I-net of sets such that there are $\delta(q,p,E) > 0$ with $\delta(q,p,E) \le \hat{\mathbf{m}}_{p,q}(E_i)$ for every $i \in I$ and every couple $(p,q) \in \mathcal{P} \times \mathcal{Q}$, satisfying $q \mapsto_E p$.

Denote by $\{A_k;\ k \in \mathbb{N}\}$ the class of atoms of the measure $\mathbf{m}$ such that $A_k \subset E$, $k \in \mathbb{N}$. Clearly

$$\delta(q,p,E) \le \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) = \mathbf{v}_{p,q}(\mathbf{m},E) < \infty .$$

To prove the assertion, it is enough to show that for every cofinal J-subnet of the I-net $E_i \in \Sigma$, $i \in I$, $J \subset I$, there exists an atom A such that A is a subset of each element of a cofinal K-subnet of this J-net of sets, $K \subset J$.

Suppose this is not true for some J-subnet. Without loss of generality let it be the I-net E_i, $i \in I$, itself. So, for every atom A_k, $k \in \mathbb{N}$, there exists an index $i_k \in I$ such that $A_k \not\subset E_i$ for every $i \geq i_k$, $i \in I$. Take real numbers $\varepsilon(q,p,E) > 0$ such that $\varepsilon(q,p,E) < \delta(q,p,E)$. Then there are non-negative integers $N(q,p,E)$ such that

$$\sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) - \sum_{k=1}^{N(q,p,E)} \hat{\mathbf{m}}_{p,q}(A_k) < \varepsilon(q,p,E).$$

The existence of such $N(q,p,E)$ follows from the series convergence on the left hand side of the inequality.

Taking the atom A_1 we find an index $i_1 \in I$ such that $A_1 \not\subset E_i$ for every $i \geq i_1$, $i \in I$. Thus, from the σ-subadditivity of the p,q-semivariation of the measure $\mathbf{m}$, for $i \geq i_1$, we obtain

$$\hat{\mathbf{m}}_{p,q}(E_i) \leq \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) - \hat{\mathbf{m}}_{p,q}(A_1).$$

Further, we find an index $i_2 \in I$, $i_2 \geq i_1$, such that $A_2 \not\subset E_i$ for every $i \geq i_2$, $i \in I$, and

$$\hat{\mathbf{m}}_{p,q}(E_i) \leq \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) - \hat{\mathbf{m}}_{p,q}(A_1) - \hat{\mathbf{m}}_{p,q}(A_2)$$

for every $i \geq i_2$, $i \in I$. Repeating this procedure by induction we can write:

$$\hat{\mathbf{m}}_{p,q}(E_i) \leq \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) - \sum_{k=1}^{N(q,p,E)} \hat{\mathbf{m}}_{p,q}(A_k) < \varepsilon(q,p,E)$$

for every $i \geq i_{N(q,p,E)}$, $i, i_{N(q,p,E)} \in I$.

This contradicts $\hat{\mathbf{m}}_{p,q}(E_i) \geq \delta(q,p,E)$, $i \in I$. The theorem is proved, cf. also [3].

3. Condition (GB) and eventual $\hat{\mathbf{m}}$-convergence of measurable functions on measurable sets

In this section we show that Condition (GB) is a necessary and sufficient condition for the assertion that everywhere convergence of measurable functions implies eventual $\hat{\mathbf{m}}$-convergence in locally convex setting. Further, as an appendix, we show that a Egorov theorem cannot hold for arbitrary nets of measurable functions without some restrictions putting on the measure, net convergence of functions, or class of measurable functions.

THEOREM 3.1. *Let a measure* $\mathbf{m}$ *satisfy Condition* (GB). *If a net of measurable functions* $\mathbf{f}_i$, $i \in I$, *converges everywhere on a set* $E \in \Sigma$ *of finite variation to a measurable function* $\mathbf{f}$, *then it eventually* $\hat{\mathbf{m}}$*-converges to* $\mathbf{f}$ *on* E.

Proof. Let $\mathbf{f}$ be a measurable function and $\mathbf{f}_i$, $i \in I$, be a net of measurable functions such that for every $p \in \mathcal{P}$ the equality

$$\lim_{i \in I} p(\mathbf{f}_i(t) - \mathbf{f}(t)) = 0 \tag{2}$$

is true for every $t \in E$. Show that the net $\mathbf{f}_i$, $i \in I$, is eventually $\hat{\mathbf{m}}$-convergent on E to $\mathbf{f}$.

Let us denote

$$E_i(p, \eta) = \{t \in E;\ p(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \eta\} \in \Sigma,$$

for every $\eta > 0$, $p \in \{p \in \mathcal{P};\ q \mapsto_E p,\ q \in \mathcal{Q}\}$, $i \in I$.

Now, suppose that there are $q_0 \in \mathcal{Q}$, $p_0 \in \mathcal{P}$, $\eta_0 > 0$, $\delta_0 > 0$, such that

$$\hat{\mathbf{m}}_{p_0,q_0}(E_i(p_0, \eta_0)) \geq \delta_0 \tag{3}$$

holds for a cofinal J-subnet $E'_j(p_0, \eta_0)$, $j \in J$, $J \subset I$ of the I-net $E_i(p_0, \eta_0)$, $i \in I$. Consider the J-net $E'_j(p_0, \eta_0)$, $j \in J$. From (3) and Condition (GB) we see that there is a cofinal K-subnet $E''_k(p_0, \eta_0)$, $k \in K$ of the net $E'_j(p_0, \eta_0)$, $j \in J$, $K \subset J$, such that

$$E'' = \bigcap_{k \in K} E''_k(p_0, \eta_0) \neq \emptyset.$$

Take a point $t_0 \in E''$ and $k \in K$. Then clearly

$$p_0(\mathbf{f}_k(t_0) - \mathbf{f}(t_0)) \geq \eta_0. \tag{4}$$

Pointwise convergence (2) of the net $\mathbf{f}_i$, $i \in I$, to $\mathbf{f}$ implies pointwise convergence of every subnet of the net $\mathbf{f}_i$, $i \in I$, to the same function $\mathbf{f}$. Thus, the net $\mathbf{f}_k(t_0)$, $k \in K$, converges to the point $\mathbf{f}(t_0)$. This is a contradiction with (4).

THEOREM 3.2. *Let* $E \in \Sigma$ *be a set of positive and finite variation. Let* $E_i \in \Sigma$, $E_i \subset E$, $i \in I$, *be a net of subsets such that for every couple* $(p, q) \in \mathcal{P} \times \mathcal{Q}$, $q \mapsto_E p$, *there is* $\delta = \delta(q, p, E) > 0$ *such that the inequality* $\hat{\mathbf{m}}_{p,q}(E_i) \geq \delta$ *is true for every* $i \in I$, *but* $\limsup_{i \in I} E_i = \emptyset$.

Then there exists a net of uniformly bounded measurable functions such that it converges everywhere on the set E to a measurable function, but it does not eventually $\hat{\mathfrak{m}}$-converge to this function on E.

Proof. Let $\mathbf{x} \in \mathbf{X}$ be an arbitrary nonzero element. Put $\mathbf{f}(t) = 0 \in \mathbf{X}$ for every $t \in E$. It is easy to see that

$$\lim_{i \in I} p\big(\mathbf{x} \cdot \chi_{E_i}(t) - \mathbf{f}(t)\big) = 0$$

for every $t \in E$ and $p \in \mathcal{P}$. Indeed, let $t_0 \in E$. So, there is $i_0 \in I$ such that $t_0 \in E_i' = E \setminus E_i$ for every $i \geq i_0$, $i \in I$. Thus $\mathbf{x} \cdot \chi_{E_i}(t_0) = 0$ for every $i \geq i_0$, $i \in I$, and

$$\lim_{i \in I} \mathbf{x} \cdot \chi_{E_i}(t_0) = 0 .$$

On the other hand, for every $i \in I$ we have

$$\hat{\mathfrak{m}}_{p,q}\Big(\Big\{t \in E\,;\ p\big(\mathbf{x} \cdot \chi_{E_i}(t) - \mathbf{f}(t)\big) \geq \frac{1}{2}\Big\}\Big) \geq \delta ,$$

and the I-net $\mathbf{x} \cdot \chi_{E_i}$, $i \in I$, of functions does not eventually $\hat{\mathfrak{m}}$-converge to $\mathbf{f}$.

THEOREM 3.3. *Let $E \in \Sigma$ be a set of finite variation. Everywhere convergence of an I-net $\mathbf{f}_i$, $i \in I$, of measurable functions to a measurable function $\mathbf{f}$ on E implies eventual $\hat{\mathfrak{m}}$-convergence of the net $\mathbf{f}_i$, $i \in I$, to $\mathbf{f}$ on E if and only if the measure $\mathbf{m}$ satisfies Condition* (GB).

Proof. Combining Theorem 3.1 and Theorem 3.2 we obtain this criterion directly.

THEOREM 3.4. *Let $E \in \Sigma$ be a set of positive and finite variation. Let $\{t\} \in \Sigma$, $\hat{\mathfrak{m}}_{p,q}(\{t\}) = 0$ for every $t \in E$, $p \in \mathcal{P}$, $q \in \mathcal{Q}$.*

Then there exists a net of uniformly bounded measurable functions $\mathbf{f}_i$, $i \in I$, such that $\lim_{i \in I} p\big(\mathbf{f}_i(t) - \mathbf{f}(t)\big) = 0$ for every $t \in E$ and for every $F \subset E$, $F \in \Sigma$, of positive semivariation, pointwise convergence of the net $\mathbf{f}_i$, $i \in I$, on F is not uniform.

(We consider the uniform convergence with respect to the system of seminorms $\|\mathbf{f}\|_{F,p} = \sup_{t \in F} p\big(\mathbf{f}(t)\big)$, $p \in \mathcal{P}$, $F \subset F$, $F \in \Sigma$.)

Proof. Let I denote the direction given by the inclusion of sets. Let $\mathbf{x} \in \mathbf{X}$ be an non-zero element. Let $E_i \subset E$, $i \in I$, be a net of complements of finite subsets of the set E to E. It is easy to see that $\mathbf{x} \cdot \chi_{E_i}$, $i \in I$, is a (decreasing) I-net of functions converging to $0 \in \mathbf{X}$ at each point of the set E. But there does not exist an infinite subset $F \subset E$ such that the I-net $\mathbf{x} \cdot \chi_{E_i}$, $i \in I$, would converge uniformly. It follows from the fact that $\mathbf{x} \cdot \chi_{E_i}(t) = 0$ only on a finite subset of the set E.

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