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ORIENTABLY SIMPLE GRAPHS

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The genus $\gamma(G)$ of a graph G is defined to be the minimum genus of the orientable surfaces on which G is embeddable. Every graph is embeddable in a nonorientable surface, just as it is in an orientable one (since we may add a crosscap to an appropriate orientable surface). We define the nonorientable genus $\tilde{\gamma}(G)$ of a graph G to be the minimum number of crosscaps needed (there should be at least one) on a sphere to achieve embeddability. By our definition, which is slightly different from the one given by White and Beineke [8] and White [7], all planar graphs have nonorientable genus equal to one.

Recall that an embedding of G into a surface S is called a 2-cell embedding if and only if all regions of the embedding are 2-cells. By a region we mean a connected set of the complement of G relative to S. An embedding of G into the orientable surface of genus $\gamma(G)$ is called (orientably) minimal; see for instance Youngs [9]. Here we extend this definition to nonorientable surfaces. We call an embedding of G into the nonorientable surface of genus $\tilde{\gamma}(G)$ nonorientably minimal. The well-known theorem of Youngs states the expected fact, namely that all orientably minimal embeddings of connected graphs are 2-cell embeddings.

Theorem 1 (Youngs). *An embedding of a connected graph G in a surface of genus $\gamma(G)$ is a 2-cell embedding.*

The purpose of this note is to clarify the situation with the nonorientable analog of this theorem which cannot be formulated without modifications. There are graphs G that have embeddings into the nonorientable surface of genus $\tilde{\gamma}(G)$ which are not 2-cell embeddings. For example, the complete graph K_7 has an embedding into torus whereas it has no embedding into the Klein bottle, see Franklin [2]. This means that its nonorientable genus is equal to 3 since we can attach a crosscap to the torus and thereby obtain a nonorientably minimal embedding that is not a 2-cell embedding.

In a certain sense Youngs has resolved the issue by introducing the notion of the simplest embedding; see Youngs [9], and also White [7], and White and Beineke [8]. But he did not consider specifically nonorientable embeddings. Before stating our result recall a folklore inequality

$$\tilde{\gamma}(G) \leqslant 2\gamma(G) + 1 \tag{1}$$

between the orientable and nonorientable genus of a connected graph G that was probably first formally stated and proved by Stahl [6].

The following theorem shows that the Euler formula lower bound that is valid only for 2-cell embeddings and is frequently used for orientable embeddings can be applied to almost all connected graphs in the nonorientable case as well.

Theorem 2. *Let G be a connected graph. Then*

a) All nonorientably minimal embeddings of G are 2-cell embeddings if and only if

$$\tilde{\gamma}(G) \leqslant 2\gamma(G) \tag{2}$$

b) All nonorientably minimal embeddings of G are non-2-cell embeddings if and only if G is a tree.

c) There are some 2-cell and some non-2-cell embeddings that are nonorientably minimal if and only if G *is not a tree and*

$$\tilde{\gamma}(G) = 2\gamma(G) + 1 \tag{3}$$

Proof. Clearly all three cases are disjoint and cover all the possibilities. If the inequality (2) holds, then the simplest embedding in the sense of Youngs is nonorientable and according to [9], Theorem 4.2) it is a 2-cell; hence we have case a). Assume now that the inequality (2) does not hold. In this case we may easily construct a nonorientably minimal embedding which is not a 2-cell embedding; we only have to add a crosscap to the minimal orientable embedding. So we cannot have case a).

To finish the proof we have to distinguish between cases b) and c). A graph embedding is 2-cell if and only if it can be described by a generalized embedding scheme of Stahl [6]. Furthermore, the embedding is nonorientable if and only if there exists a cycle in G which is 1-trivial; see [6, Theorem 5]. Therefore, the existence of a cycle in G is an obvious necessary condition for the existence of a nonorientable 2-cell embedding and hence for a nonorientably minimal 2-cell embedding. This condition is also sufficient. Namely, let G be a connected graph containing a cycle and let N be a nonorientable surface of genus $\tilde{\gamma}(G) = 2\gamma(G) + 1$. Furthermore, let S be the orientable surface of genus $\gamma(G)$. There is an obvious non-2-cell embedding of G into N which is obtained by adding a crosscap to the minima. embedding of G into S. Since G contains a cycle its orientably minimal embedding into S has at least two faces and there exists an edge e of G that lies on a common boundary of two distinct faces. By making a perturbation in the sense of Stahl of the edge e we obtain a 2-cell embedding of G into N. This proves the distinction between cases b) and c).

In the above proof we observed that a connected graph admits a 2-cell nonorientable embedding if and only if it is not a tree.

White and Beineke [8] have introduced the notion of an orientably simple graph, that is a graph for which the equality (3) holds. In order to stimulate research along these lines we propose that terms orientably neutral for graphs that satisfy the equality

$$\tilde{\gamma}(G) = 2\gamma(G) \tag{4}$$

and orientably complicated for graphs that are neither orientably simple nor orientably neutral. Moreover, we call the difference $n(G) = 2\gamma(G) - \tilde{\gamma}(G)$ the nonorientablity excess of G. Let us mention some known facts in this new terminology.

All planar graphs are orientably simple. Note that they would be orientably neutral if we adopted White's definition of the nonorientable genus.

Auslander, Brown and Youngs have proved that there exist graphs embeddable in a projective plane with arbitrarily large nonorientability excess.

A connected graph of girth c having both an orientable and a nonorientable c-gonal embedding is orientably neutral. In particular, this is true for simplicial graphs having triangular orientable and nonorientable embeddings, or for bipartite graphs possessing both orientable and nonorientable quadrilateral embeddings.

Ringel and others have considered minimal embeddings of several well-known families of graphs. The complete graph K_n is orientably simple if and only if $n = 1, 2, 3, 4, 7$; it is orientably neutral if and only if n is congruent to 0, 3, 4, 7, 8, 11 modulo 12. The complete bipartite graph $K_{q,p}$ is orientably simple if and only if $p \leqslant 2$ or $q \leqslant 2$, it is orientably neutral if and only if p or q is congruent to 2 modulo 4 or if $p + q$ is congruent to 0 modulo 4. The n-cube graph Q_n is orientably simple if and only if $n = 1, 2, 3, 4, 5$ and is orientably neutral otherwise.

It would be of interest to know whether there exists a characterization of orientably simple graphs, that would not involve topological notions. But from the research into the nonexistence of graph embeddings it seems plausible that the characterization is difficult, see for instance Ringel [4]. Note that the problem of characterization of orientably simple graphs is related to the problem of calculating the genus and the nonorientable genus of a graph. If the genera can be computed in polynomial time, then the characterization problem is solvable in polynomial time.

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ГРАФЫ, ПРОСТЫЕ ОТНОСИТЕЛЬНО ОРИЕНТИРУЕМОСТИ

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Резюме

В 1963-ем году Й. В. Т. Янгс показал, что минимальные ориентируемые вложения графов всегда являются двухклеточными. Нами получены необходимые и достаточные условия для неориентируемого аналога теоремы Янгса.