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LONGEST CIRCUITS IN TRIANGULAR AND QUADRANGULAR 3-POLYTOPES WITH TWO TYPES OF EDGES

STANISLAV JENDROĽ—ROMAN KEKEŇÁK

ABSTRACT. The paper deals with the longest circuits in triangular and quadrangular 3-polytopes with two types of edges. Hamiltonicity and shortness invariants for several families of the mentioned 3-polytopes are determined. Three relationships among some subfamilies of triangular and quadrangular 3-polytopes are given.

1. Introduction

There are many papers studying circuits in varied families of planar 3-connected graphs (or, equivalently, 3-polytopal graphs), see e.g. Ewald and others [3], Grünbaum [4, 5], Grünbaum and Malkevitch [6], Grünbaum and Walther [7], Harant and Walther [8], Jackson [9], Jucovič [14], Owens [16, 17, 18, 19], Zaks [22] and others. In [7], Grünbaum and Walther introduced several numbers that measure, in a certain sense, the size of the longest circuits in graphs belonging to this family of graphs. Let us mention two of these measures. For a graph G let $v(G)$ denote the number of vertices of G and $h(G)$ the maximum length of simple circuits in G. For an infinite family of graphs $\mathscr{F}$, the *shortness exponent*, $\sigma(\mathscr{F})$ or σ and the *shortness coefficients*, $\varrho(\mathscr{F})$ or ϱ, are defined by

$$\sigma(\mathscr{F}) = \liminf_{G \in \mathscr{F}} \frac{\log h(G)}{\log v(G)},$$

and

$$\varrho(\mathscr{F}) = \liminf_{G \in \mathscr{F}} \frac{\log h(G)}{\log v(G)}, \quad \text{respectively.}$$

Both σ and ϱ lie between 0 and 1 inclusive and $\varrho = 0$ when $\sigma < 1$.

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We recall that G is called *hamiltonian* if $v(G) = h(G)$. The family of graphs $\mathcal{F}$ is called *hamiltonian* provided that all its members are hamiltonian and $\mathcal{F}$ is called *nonhamiltonian* if it contains no hamiltonian graph.

For an infinite nonhamiltonian family of graphs $\mathcal{F}$ it is *important* to consider the *length coefficient*, $\tau(\mathcal{F})$ or τ, defined by

$$\tau(\mathcal{F}) = \limsup_{G \in \mathcal{F}} \frac{h(G)}{v(G)}.$$

Jendroľ and Tkáč [13] define an edge of the type $(a, b; p, q)$ in a planar graph to be an edge incident with vertices of valency a and valency b and faces with p and q edges. The present paper is devoted to a study of the longest circuits in 3-polytopal graphs G having k-gonal faces only, $k = 3, 4$, and exactly two types of edges. Notice that the vertices of such graphs G can have at most three different valencies because of the connectedness of G. So let us denote by $\mathcal{R}_k(a, b, c)$ the family of all 3-polytopal graphs all edges of which are either of the type $(a, b; k, k)$ or of the type $(b, c; k, k)$. In the sequel let $\mathcal{S}(a, b, c) = \mathcal{R}_3(a, b, c)$ and $\mathcal{Q}(a, b, c) = \mathcal{R}_4(a, b, c)$.

The present paper is organized as follows. In Section 2 we shall study the longest circuits in simplicial graphs from the families $\mathcal{S}(a, b, c)$. Section 3 contains our results showing some relationships between some subfamilies of triangular and quadrangular 3-polytopal graphs. Section 4 is devoted to the study of the numbers σ, ϱ and τ for some subfamilies of quadrangular 3-polytopal graphs with exactly two types of edges. In Section 5 we shall discuss some open problems.

2. Hamiltonicity of the family $\mathcal{S}(a, b, c)$

In [13], the first step in the study of the combinatorial structure of graphs to $\mathcal{S}(a, b, c)$ has been made. For all triples (a, b, c) of positive integers it has been decided whether the family $\mathcal{S}(a, b, c)$ is finite or not and for each finite family $\mathcal{S}(a, b, c)$, all polytopes belonging to $\mathcal{S}(a, b, c)$ have been constructed. This result is employed in the sequel. We note that the longest circuits in graphs of the families $\mathcal{S}^*(a, b, c)$ dual to those of $\mathcal{S}(a, b, c)$, have been studied in Owens [18, 19] and Jendroľ and Mihók [12].

The main result of this Section is contained in

Theorem 2.1.

(i) *The family $\mathcal{S}(a, b, c)$ is hamiltonian for every triple*
$(a, b, c) \in \{(4, 4, c), 3 \le c \ne 4; (5, 5, c), 3 \le c \ne 5; (6, 6, c), 3 \le c \le 5; (7, 7, 3); (7, 7, 4)\}$.

(ii) *There is an infinite hamiltonian subfamily of the family* $\mathscr{S}(8, 8, 3)$ and $\varrho(\mathscr{S}(8, 8, 3)) \le \frac{14}{15}$.

(iii) *The families* $\mathscr{S}(9, 9, 3)$ *and* $\mathscr{S}(10, 10, 3)$ *are nonhamiltonian* $\varrho(\mathscr{S}(9, 9, 3)) \le \frac{25}{28}$, $\varrho(\mathscr{S}(10, 10, 3)) \le \frac{25}{32}$ *and* $\tau(\mathscr{S}(10, 10, 3)) \le \frac{6}{7}$.

(iv) *Let* a, b, c *be integers such that* $a \ge 3$, $b \ge 3$, $c \ge 3$ *and at most two of them are equal to each other. If* $(a, b, c) \notin \{(4, 4, c), 3 \le c \ne 4; (5, 5, c), 3 \le c \ne 5; (6, 6, c), 3 \le c \le 5; (a, a, 3), 7 \le a \le 10; (7, 7, 4)\}$, then $\mathscr{S}(a, b, c)$ is empty.

The next two theorems will be useful in the proof of Theorem 2.1.

Theorem 2.2 (Pareek [20], a weaker result in Ewald [1]). *Let G be a triangular planar nonhamiltonian graph. Then* $\Delta(G) \ge 8$, *where* $\Delta(G)$ *is a maximum degree of G.*

Theorem 2.3. *Every graph G belonging to* $\mathscr{S}(5, 5, c)$, $3 \le c \ne 5$, *is* 4-*connected.*

Proof. Suppose that there is a graph G in $\mathscr{S}(5, 5, c)$ which is not 4-connected. It can be easily verified that in G every minimal separating set consists of three vertices which form a separating triangle C (i.e., there are vertices of G both inside and outside of C). We denote by H_1 the subgraph consisting of C and the edges of G lying in its interior, by H_2 the subgraph consisting of C and the edges in its exterior. We may assume that $v(H_1) \le v(H_2)$ and that H_1 is minimal, that is that no separating triangle C of G has H_1^* with $v(H_1^*) < v(H_1)$. Let x_1, x_2 and x_3 be vertices of C. At least two of them, e.g. x_1 and x_2 are 5-valent in G. It is easy to see that $3 \le \deg_{H_i}(x_j) \le 4$ for any $i = 1, 2$ and $j = 1, 2$. The assumption $\deg_{H_i} x_j = \deg_{H_i} x_k = 3$ for $i = 1$ or 2 and $j, k \in \{1, 2, 3\}, j \ne k$, leads to a contradiction with the 3-connectedness or the planarity of G, respectively. It is sufficient to consider the case $\deg_{H_1} x_1 \ne \deg_{H_1} x_2$. Evidently $\deg_{H_1} x_3 \ge 4$. Since H_1 is triangular too, there are vertices y_1 and y_2 in H_1 such that the vertices x_1, x_2, y_1 and x_2, x_3, y_2, respectively form a face. H_1 contains only one of the edges $x_1 y_2$ and $x_2 y_1$, therefore G has an edge $y_1 y_2$ too. Because $\deg_{H_1} y_i = \deg_G y_i \ge 5$, $i = 1, 2$, the vertices y_1, y_2 and x_3 create a separating triangle C_1 in G. For the subgraph H_3 consisting of C_1 and edges of G lying in its interior we have $v(H_3) < v(H_1)$, which is a contradiction with the minimality of H_1. □

The proof of the Theorem 2.1 in the case (i) for the triple $(a, b, c) \in \{(6, 6, c), 3 \le c \le 5; (7, 7, c), 3 \le c \le 4\}$ follows immediately from Theorem 2.2. By the well-known Tutte theorem (see, e.g., Ore [15]) every 4-connected planar graph is hamiltonian, therefore the family $\mathscr{S}(5, 5, c)$ for any $c \ge 3$, $c \ne 5$ is hamiltonian too. The family $\mathscr{S}(4, 4, c)$, $c \ge 3$, $c \ne 4$ consists of exactly one graph-c-sided bipyramid — which is hamiltonian.

The propositions of the case (iv) follow from |13|. □

The Proof of the Theorem 2.1 in the cases (ii) and (iii). Let $v_k(G)$ denote the number of k-valent vertices of G. The well-known Euler formula applied to triangular graphs leads to the following equality

$$\sum_{k \geq 3} (6 - k) v_k(G) = 12. \tag{2.0}$$

This equality and $v(G) = v_3(G) + v_c(G)$ for $G \in \mathscr{S}(c, c, 3)$, $8 \leq c \leq 10$, give

$$v(G) = 4 + \frac{c-3}{3} v_c(G). \tag{2.1}$$

Since, in G, edges connecting 3-valent vertices are not allowed, we have

$$h(G) \leq 2v_c(G). \tag{2.2}$$

From (2.1) and (2.2) it is easy to see that the families $\mathscr{S}(9, 9, 3)$ and $\mathscr{S}(10, 10, 3)$ are nonhamiltonian and that

$$\tau(\mathscr{S}(10, 10, 3)) = \limsup_{G \in \mathscr{S}(10, 10, 3)} \frac{h(G)}{v(G)} \leq \lim_{v_{10}(G) \to \infty} \frac{2v_{10}(G)}{4 + \frac{7}{3} v_{10}(G)} = \frac{6}{7}.$$

To prove the remaining part of the cases (ii) and (iii) we shall present methods based on inductive constructions of the sequences $\{G_n\}$, $n = 0, 1, 2, \ldots$, of graphs with the desired properties. In every case, the graph G_n $n = 1, 2, \ldots$ is obtained by replacing certain parts of G_{n-1} by new graphs of a suitable type.

The construction of a sequence of hamiltonian graphs from $\mathscr{S}(8, 8, 3)$ starts with a graph G_0 obtained from graph H in Fig. 2.1 by adding an edge $v_1 x_{28}$ (numerals in this and further figures denote indices of vertices). To obtain G_n from G_{n-1}, $n = 1, 2, \ldots$, we delete from G_{n-1} the edge $x_{11}x_{13}$ and place into a quadrangle thus vacated a copy of graph H shown in Fig. 2.1; in this we identify the vertices x_1, x_2, x_{28}, x_{29} of H with the vertices x_{14}, x_{11}, x_{12}, x_{13} of G_{n-1}, respectively and the corresponding edges. The labels of all the vertices of G_n except the labels of the vertices of the "last" subgraph H of G_n are deleted.

Now we show that G_n is hamiltonian if G_{n-1} is hamiltonian. A hamiltonian circuit in G_{n-1} pases throught the edges $\ldots x_{10}x_{11}$, $x_{11}x_{12}$, $x_{12}x_{13}$, $x_{13}x_{14}$, $x_{14}x_{15} \ldots$ of G_n. In H (and in G_0) a hamiltonian circuit passes through the edges $x_i x_{i-1}$, $i = 1, 2, \ldots, 29$ and $x_1 x_{29}$. A hamiltonian circuit in G_n consists of the part of the hamiltonian circuit of G_{n-1} between x_{14} and x_{11} and a hamiltonian path from x_2 to x_1 in H.

The proof of the bound of the shortness coefficient for the family $\mathscr{S}(8, 8, 3)$ is based on a construction of an infinite sequence of nonhamiltonian graphs of this class. The construction starts with a graph G_0 obtained from the graph H in Fig. 2.2 by adding an edge a b.

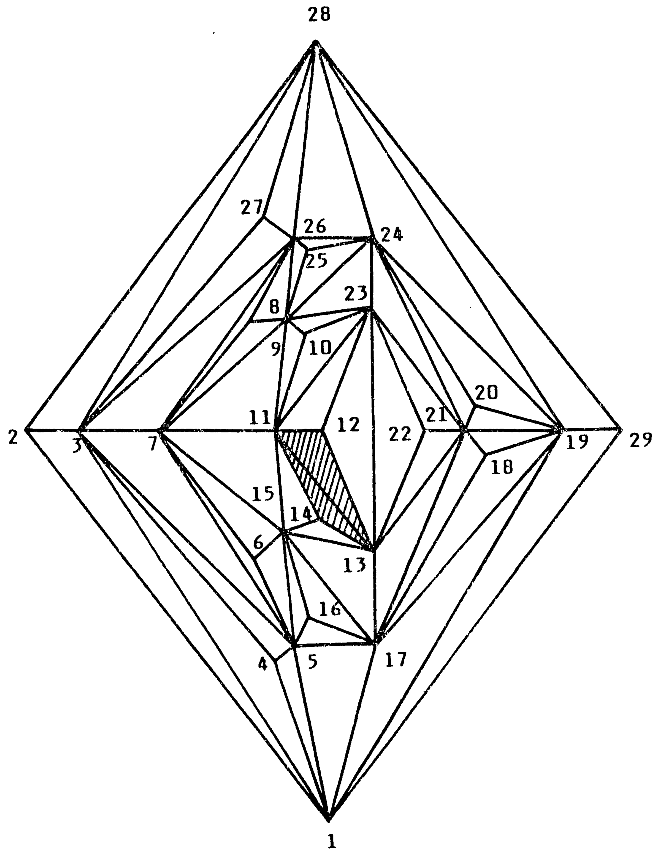
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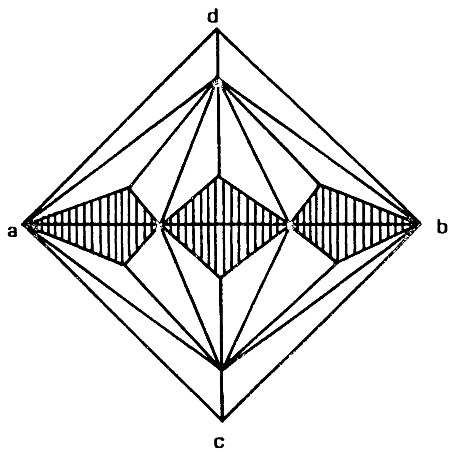
Fig. 2.1


d
a
b
c
Fig. 2.2

To obtain G_n, $n = 1, 2, \ldots$, from G_{n-1} each of the quadrangular parts of G_{n-1} marked dark is replaced by the graph H of Fig. 2.2 in such a way that the vertices a and b are identified with the trivalent vertices of the boundary of the marked part, the vertices c and d with 8-valent ones respectively, and the corresponding boundary edges are identified, too.

In G_{n-1}, $n = 1, 2, \ldots$ there are 3^n dark marked quadrangular parts, this means that there are at least 3^n subgraphs in G_n isomorphic to H. Any two different such subgraphs have at most one vertex in common.

It is easy to verify that G_n belongs to $\mathscr{S}(8, 8, 3)$ and that for the number of vertices $v(G_n)$ of G_n, $n = 0, 1, \ldots$

$$v(G_n) = 4 + 10 \sum_{k=0}^{n} 3^k = 5.3^{n+1} - 1.$$

On the other hand every longest circuit of G_n contains at most five trivalent vertices from the interior of each copy of H. Therefore

$$h(G_n) \leq v(G_n) - 3^n = -1 + 5.3^{n+1} - 3^n = 14.3^n - 1.$$

The above considerations yield

$$\varrho(\mathscr{S}(8, 8, 3)) = \liminf_{n \to \infty} \frac{h(G_n)}{v(G_n)} \leq \lim_{n \to \infty} \frac{-1 + 14.3^n}{-1 + 15.3^n} = \frac{14}{15}.$$

To establish an upper bound of shortness coefficient for the family $\mathscr{S}(9, 9, 3)$ (or $\mathscr{S}(10, 10, 3)$) we proceed similarly as above. The graph G_0 is obtained from the graph H in Fig. 2.3 (or Fig. 2.4) by adding an edge connecting the vertices a and b.

The graph G_n, $n = 1, 2, \ldots$, results from G_{n-1} by replacing each of the dark marked quadrangles of G_{n-1} by a copy of H in Fig. 2.3 (or Fig. 2.4) identifying the boundaries of the dark marked quadrangle and of H, respectively. Every longest circuit of G_n omits at least three vertices (seven vertices, respectively) of each copy of H of G_n. Since G_{n-1} contains 7^n (8^n, respectively) dark marked quadrangles, an easy computation shows that

$$v(G_n) = 4 + 24 \sum_{k=0}^{n} 7^n = 4.7^{n+1} \quad \text{and} \quad h(G_n) \leq v(G_n) - 3.7^n = 25.7^n$$

for $G_n \in \mathscr{S}(9, 9, 3)$ and

$$v(G_n) = 4 + 28 \sum_{k=0}^{n} 8^n = 4.8^{n+1}, \quad h(G_n) \leq 25.8^n \quad \text{for} \quad G_n \in \mathscr{S}(10, 10, 3),$$

respectively.

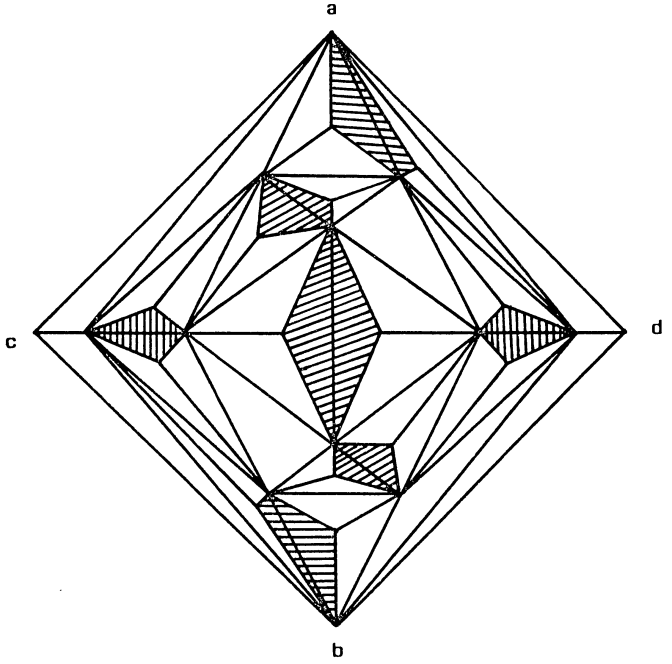


Fig. 2.3

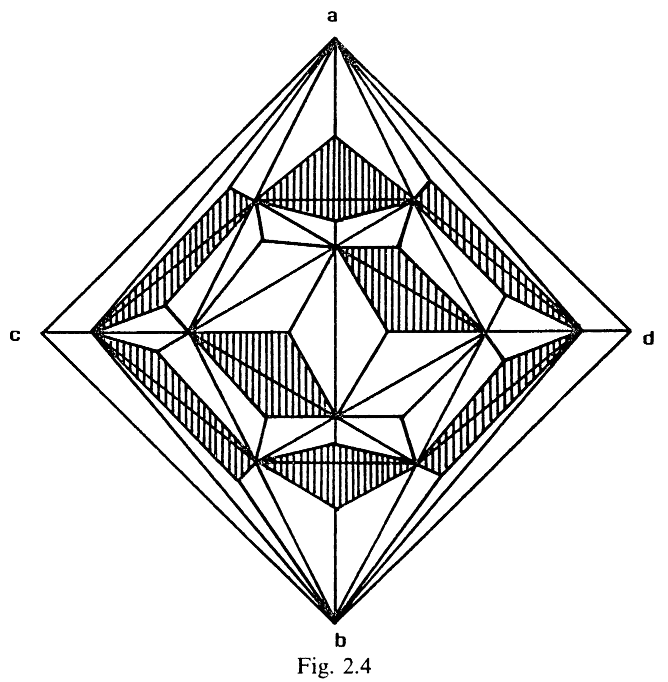


Fig. 2.4

So

$$\varrho(\mathscr{S}(9, 9, 3)) \le \frac{22}{28} \quad \text{and} \quad \varrho(\mathscr{S}(10, 10, 3)) \le \frac{25}{32}. \quad \square$$

3. Relationship among some families of triangular and quadrangular 3-polytopal graphs

Almost all considerations in the sequel use the notion of the *radial graph* $r(G)$ of a given planar graph G (see Jucovič [14], Ore [15]). Given a planar graph G we associate with G (with vertex-set $V(G)$, edge-set $E(G)$ and face-set $F(G)$) a graph $r(G)$ so that $V(r(G)) = V(G) \cup F(G)$; $e = xy \in E(r(G))$ if and only if $x \in V(G)$, $y \in F(G)$ and x is a vertex of the face y or $x \in F(G)$, $y \in V(G)$ and y is a vertex of the face x. As every edge $g \in E(G)$ is incident with two vertices and with two faces of G, g determines a quadrangular face of $r(G)$. So for every graph G, $r(G)$ is a quadrangular graph whose vertex-set $V(r(G))$ is partitioned into two disjoint sets. The valencies of vertices in one set are those of the vertices of $V(G)$, the valencies of the other second are equal to those of the faces from $F(G)$.

Theorem 3.1 (Jendroľ, Jucovič and Trenkler [11]). *If $H \in \mathscr{Q}(3, 3, c)$, then H is the radial graph of a c-gonal pyramid or of a triangular 3-polytopal graph G belonging to $\mathscr{S}(c, c, 3)$.* $\square$

It is easy to see that for every triangular 3-polytopal graph G the radial graph $H = r(G)$ of the graph G is a quadrangular one with the property that at least one of the end-vertices of any edge e of H is trivalent. If G does not contain trivalent vertices, then every edge of $r(G)$ has exactly one endvertex trivalent.

Theorem 3.2. *If H is a quadrangular 3-polytopal graph in which every edge has exactly one trivalent vertex, then there is a triangular 3-polytopal graph G without trivalent vertices such that*

$$H = r(G) \quad \textit{and} \quad v_k(H) = v_k(G) \quad \textit{for every } k, \quad k \neq 3.$$

Proof. For given H we shall construct a triangular 3-polytopal graph G. The vertex-set $V(G)$ of G consists of the vertices of H having valencies >3 in H. Two vertices x and y of G are connected by an edge provided that there is in H a face α incident to the vertices x and y. Let y be a k-valent vertex of H, $k \ge 3$. Let $x_0, x_2, \ldots, x_{k-1}$ be trivalent vertices of H adjacent to y such that the vertices x_i, y, x_{i+1} are incident to the same face β_i, $i = 0, 1, \ldots, k-1$. Let y_i be the fourth vertex of the face β_i. (Indices are taken modulo k.) By the assumption of the theorem $\deg_H y_i > 3$ and the vertices y_i, x_{i+1}, y_{i+1} are incident to a face γ_i. Therefore G also contains the edges yy_i, yy_{i+1} and $y_i y_{i+1}$. These edges form a triangular face in G. This means that every face of G is a triangle and there is an unambiguous correspondence between the vertices of $V(G)$ and the non-trivalent vertices of H and between the faces of $F(G)$ and the trivalent vertices of H, respectively. Obviously $H = r(G)$. G is clearly a 3-polytopal triangulation. □

Corollary 3.2. *To every graph $H \in \mathcal{Q}(a, 3, b)$, $a \neq 3 \neq b$, there is a triangular 3-polytopal graph G with vertices of valencies a and b only and such that $H = r(G)$.* □

Theorem 3.3. *For every triangular graph G*

$$h(r(G)) = 2h(G).$$

Proof. For the purpose of the proof let α_i denote a face of $F(G)$ and a vertex of $V(r(G))$ associated to α_i in $r(G)$. The indices below are taken modulo k.

First we show that $h(r(G)) \leq 2h(G)$. Obviously $r(G)$ is the bipartite graph with a vertex-set $V(r(G)) = V(G) \cup F(G)$. Let x_i and α_i denote the member of $V(G)$ and $F(G)$, respectively. Let $C = x_0, \alpha_0, x_1, \alpha_1, x_2, \ldots, x_{k-1}, \alpha_{k-1}, x_k = x_0$ be a longest circuit in $r(G)$. Since G is triangular one, the vertices x_i and x_{i+1} are incident to the face α_i in G. Therefore the vertices x_i and x_{i+1} are adjacent in G, this means that $C' = x_0, x_1, \ldots, x_k = x_0$ is a circuit of the length k in G.

Let $C' = x_0, \mathrm{e}_0, x_1, \mathrm{e}_1, x_2, \ldots, x_{h-1}, \mathrm{e}_{h-1}, x_h = x_0$ be the longest circuit in G with $h = h(G)$ and $\mathrm{e}_i = x_i x_{i+1}$. Let $E(C')$ be a set of edges of C', $E(\alpha)$ be a set of edges incident to the face α and $F(\mathrm{e})$ be a couple of faces incident to the edge e in G, respectively. If a vertex x and a face α are incident in G, then the corresponding vertices x and α of $r(G)$ are adjacent. Let φ be a mapping which maps every edge e to a face belonging to $F(\mathrm{e})$. If the mapping φ from $E(C')$ to $F(G)$ is an injection, then the sequence $x_0, \varphi(\mathrm{e}_0), x_1, \varphi(\mathrm{e}_1), x_2 \ldots x_{h-1}, \varphi(\mathrm{e}_{h-1})$, x_0 forms a circuit of the length $2h$ in $r(G)$. To finish the proof it is sufficient to show that the mapping can always be chosen in such a way that φ is an injection. The following two facts are evident

$$F(\mathrm{e}_i) \cap F(\mathrm{e}_j) = \emptyset \quad \text{for} \quad j \notin \{i-1, i, i+1\}, \tag{3.1}$$

$$|F(\mathrm{e}_i) \cap F(\mathrm{e}_{i+1})| \leq 1 \quad \text{for every} \quad i = 0, 1, \ldots, h-1. \tag{3.2}$$

If for every $i = 0, 1, \ldots, h-1$ $F(\mathrm{e}_i) \cap F(\mathrm{e}_{i+1}) = \emptyset$, then the required mapping φ can be easily chosen. If this is not true, it is sufficient to suppose $F(\mathrm{e}_0) \cap F(\mathrm{e}_1) \neq \emptyset$. In this case φ is defined as follows

$$\varphi(\mathrm{e}_0) = F(\mathrm{e}_0) \cap F(\mathrm{e}_1).$$

Let $F_i = \{\varphi(\mathrm{e}_t), t = 0, 1, \ldots, i-1\}$, then we put $\varphi(\mathrm{e}_i) = \alpha \in F(\mathrm{e}_i) - F_i$, α arbitrary. (We can do it because $F(\mathrm{e}_i) - F_i$ is always nonempty.) In the opposite case there is a minimum i_0 such that $F(\mathrm{e}_{i_0}) - F_{i_0} = \emptyset$. Let $F(\mathrm{e}_{i_0}) = \{\alpha_1, \alpha_2\}$, then there must be indices $j, l < i_0$ such that $F(e_{i_0}) \cap F(e_j) = \{\alpha_1\}$ and $F(e_{i_0}) \cap F(e_1) = \{\alpha_2\}$; however, (3.1) and (3.2) imply $j = l = i_0 - 1$, which is a contradiction. □

4. The longest circuits in the families $\mathscr{Q}(a, b, c)$

Basic combinatorial properties of graphs of the families $\mathscr{Q}(a, b, c)$ have been investigated in Jendroľ and Jucovič [10]. In the sequel we shall consider only triples (a, b, c) for which the families $\mathscr{Q}(a, b, c)$ are nonempty.

Theorem 4.1. (i) *In the family* $\mathscr{Q}(3, 3, c)$, $c \geq 4$, *there is a unique hamiltonian graph — a radial graph of a c-sided pyramid* M(c).

(ii) *the family* $\mathscr{Q}(3, 3, c) - \{M(c)\}$, $4 \leq c \leq 4$, *contains a unique nonhamiltonian graph.*

(iii) *For every graph* $H \in \mathscr{Q}(3, 3, c) - \{M(c)\}$, $6 \leq c \leq 7$, *there is*

$$h(H) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(H) \quad \textit{and}$$

$$\varrho(\mathscr{Q}(3, 3, c)) = \tau(\mathscr{Q}(3, 3, c)) = \frac{2}{3}, \quad \sigma(\mathscr{Q}(3, 3, c)) = 1.$$

(iv) $$\varrho(\mathscr{Q}(3, 3, 8)) \leq \frac{28}{45} \quad \text{and} \quad \tau(\mathscr{Q}(3, 3, 8)) = \frac{2}{3}.$$

(v) $$\varrho(\mathscr{Q}(3, 3, 9)) \leq \frac{25}{42} \quad \text{and} \quad \tau(\mathscr{Q}(3, 3, 9)) \leq \frac{2}{3}.$$

(vi) $$\varrho(\mathscr{Q}(3, 3, 10)) \leq \frac{25}{48} \quad \text{and} \quad \tau(\mathscr{Q}(3, 3, 10)) \leq \frac{4}{7}.$$

(vii) *For every* $c > 10$ *the family* $\mathscr{Q}(3, 3, c) - \{M(c)\}$ *is empty.*

Proof. It is easy to see that the graph M(c) — a radial graph of a c-sided pyramid — is hamiltonian. By Theorem 3.1 and Corollary 3.2 there is for every graph $H \in \mathscr{Q}(3, 3, c) - \{M(c)\}$ a graph $G \in \mathscr{S}(3, 3, c)$ such that $H = r(G)$. Let $f(M)$ denote the number of faces of a planar graph M. Since G is triangular we have

$$v(G) = v_3(G) + v_c(G) = 4 + \left(\frac{c}{3} - 1\right)v_c(G) \tag{4.1}$$

and

$$f(G) = 4 + 2\left(\frac{c}{3} - 1\right)v_c(G). \tag{4.2}$$

By Theorems 3.2 and 3.3, (4.1) and (4.2) there is

$$h(H) = 2h(G) \leq 2v(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(H) \quad (4.3)$$

and $\quad v(H) = v(G) + f(G) = 8 + (c-3)\,v_c(G) = 8 + 3(c-3)\,v_c(H).$

From (4.3) and (4.4) it follows that all graphs belonging to the family $\mathscr{Q}(3, 3, c) - \{\mathrm{M}(c)\}$ are nonhamiltonian and that $\tau(\mathscr{Q}(3, 3, c)) \leq \frac{2}{3}$. This finishes the proof in the case (i).

By Theorem 2.1 the families $\mathscr{S}(6, 6, 3)$ and $\mathscr{S}(7, 7, 3)$ are hamiltonian. For every graph $H \in \mathscr{Q}(3, 3, c)$, $6 \leq c \leq 7$, the Theorems 3.1 and 3.3 imply

$$h(H) = 2h(G) = 2v(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(G) = 8 + 2\left(\frac{c}{3} - 1\right)v_c(H).$$

The inequalities for $\varrho(\mathscr{Q}(3, 3, c))$, $8 \leq c \leq 10$, are obtained by using Theorems 3.1 and 3.3, the relations (4.1), (4.2), (4.3) and (4.4) and sequences of triangular nonhamiltonian graphs belonging to $\mathscr{S}(c, c, 3)$ which were used in the proof of the Theorem 2.1 (ii) and (iii). For the cases (ii) and (vii) see [10]. □

Lemma 4.1. *For $a \neq b \neq c \neq a$ the family $\mathscr{Q}(a, b, c)$ is nonhamiltonian.*

Proof. Every graph H belonging to $\mathscr{Q}(a, b, c)$ is bipartite. Its one coloured class of vertices consists of all vertices of the valencies a and c, while its other class contains all b-valent vertices of H. This implies

$$av_a(H) + cv_c(H) = bv_b(H). \quad (4.5)$$

The Euler polyhedral formula for the quadrangular graph P gives

$$\sum_{k \geq 3} (4 - k)\,v_k(P) = 8. \quad (4.6)$$

For H belonging to the family $\mathscr{Q}(a, b, c)$ the equality (4.6) provides

$$(4 - a)\,v_a(H) + (4 - b)\,v_b(H) + (4 - c)\,v_c(H) = 8. \quad (4.7)$$

An assumption of hamiltonicity of H implies

$$v_a(H) + v_c(H) = v_b(H). \quad (4.8)$$

From (4.5), (4.7) and (4.8) it is easy to obtain a contradiction. □

Theorem 4.2. (i) *The family $\mathscr{Q}(4, 3, c)$, $c \geq 5$, is nonhamiltonian.*

(ii) *The family $\mathscr{Q}(4, 3, 5)$ contains exactly four graphs.*

(iii) *For every graph $H \in \mathscr{Q}(4, 3, c)$, $6 \leq c \leq 7$ there is*

$$h(H) = 12 + (c - 4)\,v_c(H) \quad \textit{and}$$

$$\varrho(\mathcal{Q}(4, 3, c)) = \tau(\mathcal{Q}(4, 3, c)) = \frac{2}{3}, \quad \sigma(\mathcal{Q}(4, 3, c)) = 1.$$

(iv) *For every* $c \geq 8$ *there is* $\tau(\mathcal{Q}(4, 3, c)) = \frac{2}{3}$.

Proof. By Corollary 3.2 for every graph $H \in \mathcal{Q}(4, 3, c)$ there exists a triangular graph G with vertices of valencies 4 and c only and such that $H = r(G)$. For G from (2.0) we can easily obtain

$$f(G) = 8 + (c - 4)\,v_c(G) \quad \text{and} \quad v(G) = 6 + \frac{1}{2}(c - 4)\,v_c(G).$$

Since
$$h(H) = 2h(G) \leq 2v(G) = 12 + (c - 4)\,v_c(H) \tag{4.9}$$

and
$$v(H) = v(G) + f(G) = 12 + \frac{3}{2}(c - 4)\,v_c(H)$$

we can easily obtain $\tau(\mathcal{Q}(4, 3, c)) \leq \frac{2}{3}$.

By Theorem 2.2 all triangular planar graphs with maximum degree ≤ 7 are hamiltonian. therefore for $6 \leq c \leq 7$ there is an equality in (4.9). The equality for ϱ, σ and τ can now be easily obtained. The case (iii) is exhausted.

To prove the equation in (iv) it is sufficient to construct an infinite sequence of triangular hamiltonian graphs G_n with 4-valent and c-valent vertices only. A construction of such sequence begins with a graph G_0 of a c-sided bipyramid. Let G_{n-1}, $n = 1, 2, \ldots$ be a triangular hamiltonian graph with the required property. Choose three 4-valent vertices x, y, z in such a way that the distance between x and z is two and y is a vertex adjacent to both of them. Let w be a vertex of G_{n-1} adjacent to y, $w \neq x, z$. We add $c - 4$ new vertices $z_1, z_2, \ldots, z_{c-4}$ in the edge yw and join them with the vertices x and z. A graph G_n thus obtained has two c-valent vertices and $c - 2$ 4-valent vertices more than the graph G_{n-1}. It can be verified that G_n is hamiltonian provided that G_{n-1} is. The cases (i) and (ii) follow from Lemma 4.1 and [10], respectively. □

Theorem 4.3. (i) *The family* $\mathcal{Q}(5, 3, c)$, $c \geq 6$, *is nonhamiltonian.*

(ii) *For every graph* $H \in \mathcal{Q}(5, 3, c)$, $6 \leq c \leq 7$,

$$h(H) = 24 + 2(c - 5)\,v_c(H) \quad \textit{and}$$

$$\varrho(\mathcal{Q}(5, 3, c)) = \tau(\mathcal{Q}(5, 3, c)) = \frac{2}{3}, \quad \sigma(\mathcal{Q}(5, 3, c)) = 1$$

(iii) *For every* $c \geq 12$, $\tau(\mathcal{Q}(5, 3, c)) = \frac{2}{3}$.

Proof. The proof in the cases (i) and (ii) is similar to the proof of the parts (i) and (iii) of the previous Theorem 4.2. We omit it. The equality $\tau(\mathcal{Q}(5, 3, c)) = \frac{2}{3}$ can be obtained by using (2.0), Theorems 3.2 and 3.3, Corollary 3.2 and the fact that the family $\mathcal{S}(5, 5, c)$, $c \geq 12$, is hamiltonian. □

Theorem 4.4. (i) *In the family* $\mathcal{Q}(3, 4, 4)$ *there is an infinite hamiltonian subfamily and an infinite nonhamiltonian subfamily.*

(ii) $$\sigma(\mathcal{Q}(3, 4, 4)) = 1$$

(iii) *The family* $\mathcal{Q}(3, 4, c)$, $c \geq 5$, *is nonhamiltonian and*

$$\tau(\mathcal{Q}(3, 4, c)) = 1.$$

Proof. A construction of an infinite sequence of hamiltonian graphs starts with a graph G_0 in Fig. 4.1. A circuit $C_0 = u_1, u_2, \ldots, u_7, u_{0,1}, u_{0,2}, u_{0,0}, u_{0,3}, \ldots, u_{0,8}, u_8, u_9, u_1$ is a hamiltonian circuit in G_0.

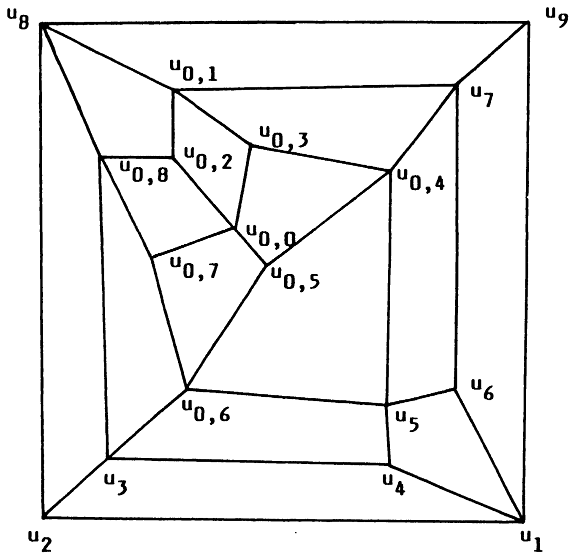


Fig. 4.1

To obtain a graph G_n from the graph G_{n-1} we delete from G_{n-1} the vertex $u_{n-1,0}$ (and edges incident with it) and fill an 8-gon $u_{n-1,1}, u_{n-1,3}, u_{n-1,4}, \ldots, u_{n-1,8}, u_{n-1,2}$, thus vacated in the manner as shown in Fig. 4.2.

A hamiltonian circuit C_n of G_n is obtained from the hamiltonian circuit C_{n-1} of G_{n-1} by replacing its part $u_{n-1,2}, u_{n-1,0}, u_{n-1,3}$ by the path $u_{n-1,2}, u_{n,1}, u_{n,2}, u_{n,0}, u_{n,3}, \ldots, u_{n,8}, u_{n-1,3}$.

To prove the existence of an infinite nonhamiltonian subfamily of the family $\mathscr{Q}(3, 4, 4)$ it is sufficient to consider the family of all 4-regular 3-polytopal graphs with triangular and quadrangular faces only. For every graph G from this family there is $v(G) \neq f(G)$, therefore the graph $r(G) \in \mathscr{Q}(3, 4, 4)$ and it is nonhamiltonian (see, e.g., Jucovič [14]).

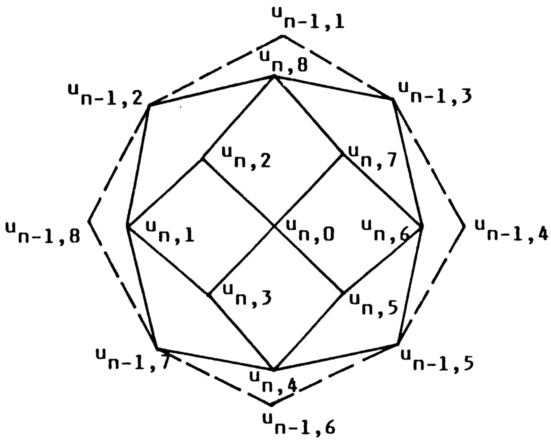


Fig. 4.2

The proof that $\sigma(\mathscr{Q}(3, 4, 4)) = 1$ may be found in Ewald [2]. The family $\mathscr{Q}(3, 4, c)$, $c \geq 5$ is nonhamiltonian by Lemma 4.1. In order to prove the second part of (iii) it is sufficient to construct an infinite sequence $\{G_n\}$ of 4-regular 3-polytopal graphs with triangular and c-gonal faces only. It can be verified that $r(G_n) \in \mathscr{Q}(3, 4, c)$ and $f(G_n) = v(G_n) + 2$.

Since the graph $r(G_n)$ is bipartite, the vertices of the one colour class of $r(G_n)$ correspond to the vertices of G_n and the vertices of the second correspond to the faces of G_n. Therefore it is sufficient to show that in G_n there exists an alternating sequence C'_n of vertices and faces of G_n, $x_0, \alpha_0, x_1, \alpha_1, x_2, \ldots, x_m, \alpha_m, x_0$ such that $m = v(G_n)$, $\alpha_i \neq \alpha_j$, $x_i \neq x_j$ if $i \neq j$ and α_i is incident to x_i and x_{i+1}, α_m is incident to x_m and x_0 for every $i = 0, 1, \ldots, m$.

The sequence C'_n specifies a circuit C_n in $r(G_n)$ of the length $h(r(G_n)) = 2v(G_n)$. Since $v(r(G_n)) = v(G_n) + f(G_n) = 2v(G_n) + 2$ we have $\tau(\mathscr{Q}(3, 4, c)) = 1$ for every $c \geq 5$. The construction of a required sequence G_n begins with a graph of c-sided antiprisma in Fig. 4.3 taken as G_0. to obtain the graph G_n we delete the edge $x_{n-1,1}y_{n-1,2}$ of G_{n-1} and add $c - 3$ new vertices $x_{n,1}, \ldots, x_{n,c-3}$ into the edge $x_{n-1,1}y_{n-1,1}$ and $c - 3$ new vertices $y_{n,1}, \ldots, y_{n,c-3}$ into the edge $x_{n-1,2}y_{n-1,2}$ and connect by an edge the couples of vertices $x_{n-1,1}$ and $y_{n,1}$; $x_{n,c-3}$ and $y_{n-1,2}$, for every $i = 1, 2, \ldots, c - 3$ the couples $x_{n,i}$ and $y_{n,i}$, for every $i = 1, 2, \ldots, c - 4$ the couples $x_{n,i}$ and $y_{n,i+1}$, respectively. See Fig. 4.4.

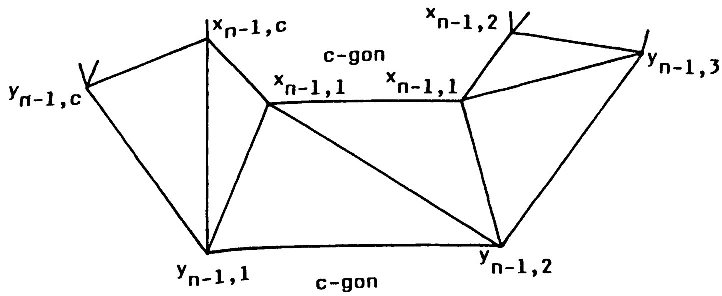


Fig. 4.3

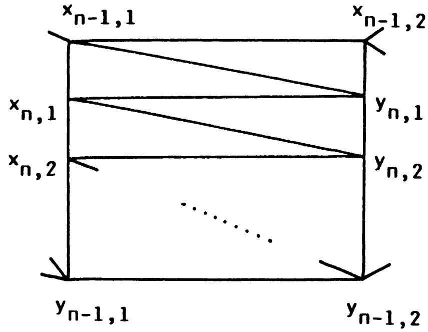


Fig. 4.4

To find a required sequence C'_n in G_n is easy and is left to the reader. □

Theorem 4.5. (i) *The family* $\mathscr{Q}(3, 5, c)$, $c \geq 5$ *is nonhamiltonian.*

(ii) $$\varrho(\mathscr{Q}(3, 5, 5)) \leq \frac{4}{5}.$$

(iii) $$\tau(\mathscr{Q}(3, 5, c)) \leq \frac{4}{5} \quad \textit{for every } c \geq 6.$$

Proof. Nonhamiltonicity of the family $\mathscr{Q}(3, 5, c)$ for $c \geq 6$ follows from Lemma 4.1. The proof of nonhamiltonicity of the family $\mathscr{Q}(3, 5, 5)$ is based on the fact that no graph H from $\mathscr{Q}(3, 5, 5)$ contains an edge with both end-vertices trivalent. This and (4.6) implies

$$h(H) \leq 2v_5(H) \leq v(H) = v_3(H) + v_5(H) = 2v_5(H) + 8.$$

To prove the case (ii) consider 5-regular polyhedral graphs G containing triangles and pentagons only. (For an existence of an infinite family of such graphs see Jucovič [14] or Trenkler [21]). Clearly $r(G) \in \mathscr{Q}(3, 5, 5)$. Denote by $f_k(P)$ the number of k-gonal faces of a 3-polytopal graph P. Using the Euler polyhedral formula we have

$$f(G) = 20 + 6f_5(G) \quad \text{and} \quad v(G) = 12 + 4f_5(G).$$

Since $v(r(G)) = v(G) + f(G) = 32 + 10f_5(G)$ and

$$h(r(G)) \leq 2v(G) = 24 + 8f_5(G),$$

we can easily obtain the proposition of (ii).

For every graph $H \in \mathcal{Q}(3, 5, c)$, $c \geq 6$, there is

$$3v_3(H) + cv_c(H) = 5v_5(H) \quad \text{and}$$

$$h(H) \leq 2\min\{v_3(H) + v_c(H), v_5(H)\}$$

because of the biparticity of H. The relation (4.6) implies

$$v_3(H) - v_5(H) + (4 - c)\,v_c(H) = 8.$$

These three relations lead to

$$h(H) \leq 24 + 4(c - 3)\,v_c(H)$$

and

$$v(H) \leq 32 + 5(c - 3)\,v_c(H),$$

from which we easily obtain $\tau(\mathcal{Q}(3, 5, c)) \leq \frac{4}{5}$. □

5. Remarks

The results presented leave many open questions, in particular for families of quadrangular 3-polytopal graphs with exactly two types of edges. Some of them concern the cases of the families of triangular graphs $\mathcal{S}(a, a, 3)$, $8 \leq a \leq 10$, too. We believe (in agreement with the conjecture of Grünbaum and Walther [7]) that in all these cases the shortness exponentis equal to 1; more precisely we state

Conjecture 1. $\sigma(\mathcal{S}(a, a, 3)) = \sigma(\mathcal{Q}(3, 3, c)) = 1$ *for every* c, $8 \leq c \leq 10$.

The following question would be interesting: What is the minimum number c_0 such that $\sigma(\mathcal{Q}(a, 3, c_0)) < 1$, $4 \leq a \leq 5$?

Theorem 4.2 (and Theorem 4.3 if $a = 5$) implies $c_0 \geq 8$. A similar question can be posed for the families $\mathcal{Q}(3, b, c)$ $4 \leq b \leq 5$.

Conjecture 2. $\sigma(\mathcal{Q}(3, b, c)) = 1$ for any $c \leq 7$ and $4 \leq b \leq 5$.

We should like to remind the reader that many problems concerning shortness parameters for various families of 3-polytopal graphs formulated by Grünbaum and Walther [7] are still open.

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