Mathematica Slovaca

Ján Ohriska

Asymptotic properties of solutions of a second order nonlinear delay differential equation

Mathematica Slovaca, Vol. 31 (1981), No. 1, 83--90

Persistent URL: http://dml.cz/dmlcz/129269

Math. Slovaca 31, 1981, No. 1, 83—90

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF A SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATION

JÁN OHRISKA

1. Introduction

Consider the equation

$$u''(t)+p(t)u^{\alpha}(\tau(t))=0 \tag{1}$$

on some half-line $[t_0, \infty)$.

For this equation the following conditions are assumed to hold throughout the paper

(*i*) $p(t)\in C_{[t_0,\infty)}$, $p(t)$ is nontrivial in every neighbourhood of infinity,

(*ii*) $\tau(t)\in C_{[t_0,\infty)}$, $\tau(t)\leqq t$, $\lim_{t\to\infty}\tau(t)=\infty$,

(*iii*) $\alpha=r/s$, where r and s are odd natural numbers.

We restrict our attention to solutions of (1) which exist on $[t_0, \infty)$ and are nontrivial in every neighbourhood of infinity. A solution is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory.

It is well known (cf. [2]) that nonoscillatory solutions of equation (1) for $p(t)\geqq 0$ can be of the following three types:

a) $|u(t)|\to c$, $u'(t)\to 0$ $(0<c)$ for $t\to\infty$,
b) $|u(t)|\to\infty$, $u'(t)\to c$ $(0<c)$ for $t\to\infty$,
c) $|u(t)|\to\infty$, $u'(t)\to 0$ for $t\to\infty$.

Necessary and sufficient conditions for the existence of a nonoscillatory solution of (1) of the type *a*) and *b*) may be found in [2], [5]. For the existence of a nonoscillatory solution of (1) of the type *c*) we know only the necessary conditions (cf. [2]) like the conditions

$$\int^{\infty} tp(t)\,dt=\infty,\quad \int^{\infty}\tau^{\alpha}(t)p(t)\,dt<\infty \quad \text{for the case } 0<\alpha<1,$$

and the conditions

$$\int^{\infty} t^{\alpha} p(t)\,\mathrm{d}t = \infty, \quad \int^{\infty} \tau(t) p(t)\,\mathrm{d}t < \infty \quad \text{for the case} \quad \alpha > 1.$$

These conditions (for the case $0<\alpha<1$) are contained in Theorem 3 of this paper and in Theorem 1 of [3].

Many authors have studied the asymptotic and oscillatory properties of equation (1). We shall consider the asymptotic properties of equation (1) for the special case $0<\alpha<1$ in part 2 and for the general case $\alpha>0$ in part 3.

2. Asymptotic theorems for the case $0<\alpha<1$

We first mention the following

Definition 1. *Let* $\gamma(t)=\sup\{s \geqq t_0 \mid \tau(s) \leqq t\}$ *for* $t \geqq t_0$.

From this definition we see that $t \leqq \gamma(t)$ and $\tau(\gamma(t))=t$. Another quality of function $\gamma(t)$ is contained in the following lemma (proved in [3]).

Lemma 1. *For every t such that* $t_0 \leqq t < \infty$, *the value* $\gamma(t)$ *is finite.*

Theorem 1. *Supposte that* $0<\alpha<1$ *and* $\int^{\infty} \tau^{\alpha}(t)|p(t)|\,\mathrm{d}t<\infty$. *Then for every solution* $u(t)$ *of* (1) *there exists* $\lim_{t\to\infty} u'(t)$.

Proof. The proof is obtained by modification of Belohorec [1]. Let us consider a solution $u(t)$ of (1) which satisfies the initial conditions

$$u(t_1)=\varphi(t_1)(=u_0), \quad u'(t_1+0)=u_1,$$

$$u(\tau(t))=\varphi(\tau(t)) \quad \text{for} \quad \tau(t)<t_1 \ (t_1 \in [t_0, \infty),\ t_1 \geqq 0),$$

where $\varphi(t)$ is an initial function and u_1 is a real number.

Integrating (1) twice from t_1 to t $(t \geqq t_1)$ we have

$$u(t)=u_0+u_1(t-t_1)-\int_{t_1}^{t} (t-x)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x$$

and for $t-t_1 \geqq 1$ we get

$$|u(t)| \leqq (t-t_1)\left[|u_0|+|u_1|+\int_{t_1}^{t} |p(x)|\,|u(\tau(x))|^{\alpha}\,\mathrm{d}x\right].$$

Now for $t \geqq \gamma(t_1+1)$ the last inequality yields

$$|u(\tau))| \leqq \tau(t)\left[|u_0|+|u_1|+\int_{t_1}^{t} |p(x)|\,|u(\tau(x))|^{\alpha}\,\mathrm{d}x\right]. \tag{2}$$

Raising both sides of the inequality (2) to the power α and multiplying by $|p(t)|$, we obtain

$$\frac{|p(t)|\,|u(\tau(t))|^{\alpha}}{\left[|u_0|+|u_1|+\int_{t_1}^{t}|p(x)|\,|u(\tau(x))|^{\alpha}\,\mathrm{d}x\right]^{\alpha}}\leqq\tau^{\alpha}(t)\,|p(t)|.$$

Integrating the last inequality from $\gamma(t_1+1)$ to t $(t\geqq\gamma(t_1+1))$ yields

$$|u_0|+|u_1|+\int_{t_1}^{t}|p(x)|\,|u(\tau(x))|^{\alpha}\,\mathrm{d}x\leqq\left[K_1+(1-\alpha)\int_{\gamma(t_1+1)}^{t}\tau^{\alpha}(x)\,|p(x)|\,\mathrm{d}x\right]^{1/(1-\alpha)},$$

where

$$K_1=\left[|u_0|+|u_1|+\int_{t_1}^{\gamma(t_1+1)}|p(x)|\,|u(\tau(x))|^{\alpha}\,\mathrm{d}x\right]^{1-\alpha}.$$

From this we have by (2)

$$|u(\tau(t))|\leqq K\tau(t)\quad\text{for}\quad t\geqq\gamma(t_1+1),\tag{3}$$

where

$$K=\left[K_1+(1-\alpha)\int_{\gamma(t_1+1)}^{\infty}\tau^{\alpha}(x)\,|p(x)|\,\mathrm{d}x\right]^{1/(1-\alpha)}.$$

Finally, integrating (1) from t_2 to t $(\gamma(t_1+1)\leqq t_2\leqq t)$, we get

$$u'(t)=u'(t_2)-\int_{t_2}^{t}p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x.\tag{4}$$

According to (3) we obtain

$$\left|\int_{t_2}^{t}p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x\right|\leqq K^{\alpha}\int_{t_2}^{\infty}\tau^{\alpha}(x)\,|p(x)|\,\mathrm{d}x\quad\text{for}\quad t\geqq t_2.$$

It follows from this that the integral $\int_{t_2}^{\infty}p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x$ exists. Futher by (4) the $\lim_{t\to\infty}u'(t)$ exists and our proof is completed.

Theorem 2. *Suppose that* $0<\alpha<1$ *and* $\int^{\infty}t\tau^{\alpha}(t)\,|p(t)|\,\mathrm{d}t<\infty$. *Then any solution* $u(t)$ *of* (1) *is of the form*

$$u(t)=c_2t+c_1+o(1),$$

where c_1 *and* c_2 *are suitable constants.*

Proof. The proof is obtained again by modification of Belohorec [1]. Let us suppose that

$$\int^{\infty} t\tau^{\alpha}(t)\,|p(t)|\,\mathrm{d}t<\infty .$$

Then according to Theorem 1 the $\lim_{t\to\infty} u'(t)$ exists. Denote it by c_2. Let $t_1\in[t_0,\infty)$, $t_1\geqq 0$ such that for $t\geqq t_1$ the inequality (3) holds.

Integrating (1) from s to ∞ ($s\geqq t_1$) and then from t_1 to t ($t\geqq t_1$), we get

$$\begin{aligned} u(t)&=c_2t+u(t_1)-c_2t_1+\\ &+\int_{t_1}^{\infty}(x-t_1)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x+\int_{t}^{\infty}(t-x)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x . \end{aligned} \tag{5}$$

Now, using (3) and the assumption of the theorem, we have

$$\left|\int_{t_1}^{t}(x-t_1)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x\right|\leqq K^{\alpha}\int_{t_1}^{\infty}x\tau^{\alpha}(x)\,|p(x)|\,\mathrm{d}x<\infty,\quad t\geqq t_1 .$$

From this we see that $\int_{t_1}^{\infty}(x-t_1)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x$ exists and that

$$\int_{t}^{\infty}(t-x)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x=o(1).$$

If we put

$$c_1=u(t_1)-c_2t_1+\int_{t_1}^{\infty}(x-t_1)p(x)u^{\alpha}(\tau(x))\,\mathrm{d}x ,$$

then from (5) we have

$$u(t)=c_2t+c_1+o(1)$$

and the theorem is proved.

We shall assume in the sequel that $p(t)\geqq 0$.

Theorem 3. *Let* $0<\alpha\leqq 1$ *and* $p(t)\geqq 0$. *Let*

$$\int^{\infty} tp(t)\,\mathrm{d}t<\infty . \tag{6}$$

Then any nonoscillatory solution $u(t)$ *of* (1) *is either bounded or of the form* $u(t)\sim ct$ $(c\neq 0)$.

Proof. The proof is obtained again by modification of Belohorec [1]. Let $u(t)$ be a nonoscillatory solution of (1). We may assume that $u(t)>0$ for $t\geqq t^*\geqq t_0$, since a parallel argument holds if $u(t)<0$. Then $u(\tau(t))>0$, $u|(t)\leqq 0$, $u'(t)>0$ for $t\geqq T\geqq t^*$, and $\lim_{t\to\infty} u'(t)\geqq 0$ (cf. [4]).

Assume that $u(t)$ is an unbounded solution of (1). Then there exists $t_1 \geqq T$, $t_1 > 0$ such that $u(t) > 1$ for $t \geqq t_1$.

Let us take an arbitrary ε such that $0 < \varepsilon < 1/6$. Then by the condition (6) we know that there exists $t_2 \geqq T$ such that

$$\int_{t_2}^{\infty} xp(x)\, \mathrm{d}x < \varepsilon . \tag{7}$$

Let $t_3 = \max\{t_1, t_2\}$. Integrating (1) from s to t ($t \geqq s \geqq t_3$) and then from t_3 to t (with respect to s), we get

$$u(t) = u(t_3) + (t - t_3)u'(t) + \int_{t_3}^{t} (x - t_3)p(x)u^{\alpha}(\tau(x))\, \mathrm{d}x ,$$

whence

$$1 < \frac{u(t_3)}{u(t)} + \frac{tu'(t)}{u(t)} + \int_{t_3}^{t} xp(x)\, \mathrm{d}x \tag{8}$$

because $u(x) > 1$ and $u^{\alpha}(\tau(x)) \leqq u^{\alpha}(x) \leqq u(x)$. Now it follows from (7) and (8) that

$$1 - \varepsilon < \frac{u(t_3)}{u(t)} + \frac{tu'(t)}{u(t)}, \qquad t \geqq t_3 ,$$

whence

$$\liminf_{t\to\infty} \frac{tu'(t)}{u(t)} \geqq 1 - 2\varepsilon .$$

From the last inequality we know that there exists $t_4 \geqq t_3$ such that

$$tu'(t) \geqq (1 - 3\varepsilon)u(t) \quad \text{for} \quad t \geqq t_4 . \tag{9}$$

Integrating (1) from t_4 to t ($t \geqq t_4$), using (9) and the fact that $u^{\alpha}(\tau(x)) \leqq u(x)$, we see that

$$(1 - 3\varepsilon)(u'(t_4) - u'(t)) \leqq \int_{t_4}^{t} xp(x)u'(x)\, \mathrm{d}x \leqq$$

$$\leqq u'(t_4) \int_{t_4}^{\infty} xp(x)\, \mathrm{d}x < \varepsilon u'(t_4),$$

whence

$$0 < \frac{1 - 4\varepsilon}{1 - 3\varepsilon}\, u'(t_4) < u'(t) .$$

The last inequality implies $\lim_{t\to\infty} u'(t) = c > 0$, i.e. $u(t) \sim ct$. The theorem is proved.

3. Asymptotic theorems for the case $\alpha > 0$

In this section we shall state conditions which imply that a nonoscillatory solution of (1) is one of the types *b*), *c*) or *a*), *c*).

Theorem 4. *Let* $\alpha > 0$ *and* $p(t) \geqq 0$. *Let either* $\limsup_{t\to\infty} t\int_t^\infty p(x)\,\mathrm{d}x = \infty$ *or* $\int^\infty tp(t)\,\mathrm{d}t = \infty$. *Then for every nonoscillatory solution* $u(t)$ *of* (1) *the condition* $\lim_{t\to\infty} |u(t)| = \infty$ *holds true.*

Proof. Let $u(t)$ be a nonoscillatory solution of (1). We may assume that $u(t) > 0$ and also $u(\tau(t)) > 0$ for $t \geqq T \geqq t_0$. Then $u\,(t) \leqq 0$, $u'(t) > 0$ for $t \geqq T$ and $\lim_{t\to\infty} u'(t) \geqq 0$.

Integrating (1) from t_1 to t $(\gamma(T) \leqq t_1 \leqq t)$, we have

$$u'(t) - u'(t_1) + \int_{t_1}^t p(x) u^\alpha(\tau(x))\,\mathrm{d}x = 0.$$

From this we see that there exists $\int_{t_1}^\infty p(x)u^\alpha(\tau(x))\,\mathrm{d}x$ and thus we can integrate (1) from t to ∞ $(t \geqq t_1)$. It follows that

$$u'(t) \geqq \int_t^\infty p(x)u^\alpha(\tau(x))\,\mathrm{d}x\,.$$

Integrating the last inequality from t_1 to t $(t \geqq t_1)$ we get

$$u(t) \geqq u(t_1) + \int_{t_1}^t (x - t_1)p(x)u^\alpha(\tau(x))\,\mathrm{d}x + (t - t_1)\int_t^\infty p(x)u^\alpha(\tau(x))\,\mathrm{d}x\,. \quad (10)$$

Since $u(t)$ is an increasing function and $\tau(x) \geqq T$ for $x \geqq t_1 \geqq \gamma(T)$, it is clear that (10) yields

$$u(t) \geqq u(t_1) + u^\alpha(T)\Big[\int_{t_1}^t (x - t_1)p(x)\,\mathrm{d}x + (t - t_1)\int_t^\infty p(x)\,\mathrm{d}x\Big].$$

Since the functions $u(t)$ and $F(t) = \int_{t_1}^t (x - t_1)p(x)\,\mathrm{d}x$ are increasing, $\limsup_{t\to\infty} u(t) = \lim_{t\to\infty} u(t)$ and $\limsup_{t\to\infty} F(t) = \liminf_{t\to\infty} F(t)$. Now from the last inequality we have

$$\lim_{t\to\infty} u(t) \geqq u(t_1) + u^\alpha(T)\Big[\int_{t_1}^\infty (x - t_1)p(x)\,\mathrm{d}x + \limsup_{t\to\infty}(t - t_1)\int_t^\infty p(x)\,\mathrm{d}x\Big].$$

This completes the proof.

Example 1. The hypotheses of Theorem 4 are satisfied for the equation

$$u''(t)+\frac{1}{4}\frac{1}{t^2}u^{7/3}(t^{3/7})=0.$$

This equation has a nonoscillatory solution $u(t)=t^{1/2}$. Analogously the hypotheses of Theorem 4 are satisfied for the equation

$$u''(t)+\frac{3}{16}\frac{1}{t^{59/40}}u^{3/5}(t^{1/2})=0.$$

This equation has a nonoscillatory solution $u(t)=t^{3/4}$.

Theorem 5. *Let $\alpha>0$ and $p(t)\geqq 0$. Let there exist a number $\beta\leqq 1$ such that the function $P(t) = p(t)\tau^\alpha(t)t^\beta$ is nondecreasing (for all sufficiently large t). Then for every nonoscillatory solution $u(t)$ of (1) the condition $\lim_{t\to\infty} u'(t)=0$ holds true.*

Proof. Let $u(t)$ be a nonoscillatory solution of (1). As before we may assume that $u(t)>0$ and also $u(\tau(t))>0$ for $t\geqq T\geqq t_0$. Then $u''(t)\leqq 0$, $u'(t)>0$ for $t\geqq T$ and $\lim_{t\to\infty} u'(t)\geqq 0$.

Suppose that the assertion of the theorem is not valid. Then $\lim_{t\to\infty} u'(t) = b>0$ and $u'(t)\geqq b$ for $t\geqq T$. It follows from this that $u(t)-u(T) \geqq b(t-T)$, respectively

$$u(t)\geqq\frac{b}{2}t \quad \text{for} \quad t\geqq 2T.$$

This means that for $t\geqq\gamma(2T)$ we have

$$u(\tau(t))\geqq\frac{b}{2}\tau(t)$$

and also

$$-u''(t)=p(t)u^\alpha(\tau(t))\geqq\left(\frac{b}{2}\right)^\alpha p(t)\tau^\alpha(t).$$

Let us choose $t_1\geqq\gamma(2T)$ such that $P(t_1)\neq 0$ (it is clear that then $P(t_1)>0$). Integrating the last inequality from t_1 to t ($t\geqq t_1$) we get

$$u'(t_1)-u'(t)\geqq\left(\frac{b}{2}\right)^\alpha\int_{t_1}^{t} p(x)\tau^\alpha(x)\,\mathrm{d}x =$$

$$=\left(\frac{b}{2}\right)^\alpha\int_{t_1}^{t} P(x)x^{-\beta}\,\mathrm{d}x\geqq\left(\frac{b}{2}\right)^\alpha P(t_1)\int_{t_1}^{t} x^{-\beta}\,\mathrm{d}x$$

or

$$u'(t)\leqq u'(t_1)-\left(\frac{b}{2}\right)^\alpha P(t_1)\int_{t_1}^{t} x^{-\beta}\,\mathrm{d}x. \tag{11}$$

The function on the right-hand side of (11) is decreasing and tends to $-\infty$ as $t \to \infty$. From this we see by (11) that there exists a value $t_2 \geqq t_1$ such that $u'(t) < 0$ for $t \geqq t_2$, which yields a contradiction and completes the proof of the theorem.

Example 2. Consider the equation

$$u\ (t) + \frac{2^{7/10}}{4} \frac{1}{t^{11\ 5}} u^{7/5}\left(\frac{1}{2} t\right) = 0.$$

The function $P(t) = p(t)\tau^{\alpha}(t)t^{\beta}$ is nondecreasing if $\beta = 9/10$. This equation has a nonoscillatory solution $u(t) = t^{1/2}$.

Acknowledgment. The author would like to express his heartfelt thanks to Professor Marko Švec for his interest in this work and for many helpful suggestions.

REFERENCES

[1] BELOHOREC, Š.: On some properties of the equation $y\ (x) + f(x)y^{\alpha}(x) = 0$, $0 < \alpha < 1$. Mat. Čas., 17, 1967, No. 1, 10—19.

[2] ОДАРИЧ, О. Х., ШЕВЕЛО, В. Х.: Некоторые вопросы асимптотического поведения решений нелинейных дифф. урав. с запаздываюьим аргументом. Дифф. урав. Том 9, 1973, №4, 637—646.

[3] OHRISKA, J.: The argument delay and oscillatory properties of differential equation of n-th order. Czech. Math. J., 29 (104), 1979, 268—283.

[4] OHRISKA, J.: The influence of argument delay on oscillatory properties of a second-order differential equation. Math. Slovaca, 28, 1978, No. 1, 41—56.

[5] ONOSE, H.: Oscillation and nonoscillation of delay differential equations. Ann. Math. pur. appl. (IV), 1976, Vol. 107, 159—168.

Received January 9, 1979

Katedra matematickej analýzy
Prírodovedeckej fakulty Univerzity P. J. Šafárika
Komenského 14
041 54 Košice

АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

Ян Огриска

Резюме

В работе рассматривается дифференциальное уравнение

$$u''(t) + p(t)u^{\alpha}(\tau(t)) = 0. \tag{1}$$

Предполагается, что

$$p(t) \in C_{[t_0, \infty)}, \quad \tau(t) \in C_{[t_0, \infty)}, \quad \tau(t) \leqq t, \quad \lim_{t \to \infty} \tau(t) = \infty.$$

В предлагаемой статье сформулированы аналоги некоторых теорем Ш. Белогорца для нелинейного дифференциального уравнения (1) в случае $0 < \alpha < 1$, и приведены некоторые результаты, касающиеся асимптотического поведения решений уравнения (1) для $\alpha > 0$.