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ON CHARACTERIZATION OF SUMMABILITY FIELDS BY INTEGRAL

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In paper [3] J. R. Edwards and S. G. Wayment defined a summability method using the technique of the integration theory. By this method the characterization of convergence fields of the $(C, 1)$ method and some strongly regular matrix methods are given. A summability integral defined by J. R. Edwards and S. G. Wayment was obtained by a non-negative set function defined on a logic of a set (see [4]).

This paper shows that for the characterization of the above mentioned convergence fields a usual integral is sufficient defined by an additive measure on an algebra of a subset of the set N (N means the set of all positive integers), which is in a certain sense maximal. Also the characterization of convergence fields of some matrix summability methods (not only strongly regular) is given. In paper X denotes a Banach space and sequences consist of elements of X.

1.

In paper [1] R. C. Buck used the new measure theoretic approach to the density of sets of positive integers. There is defined a system of measurables sets, which have the characteristic functions $(C, 1)$ summable (also a generalization for matrix summability methods, which are not weaker than the $(C, 1)$ method, is given).

Buck's approach to density is used for a regular matrix summability method, but the extension is made in another way.

The generalised density $\delta_T(A)$ of a set $A \subset N$ (in the following only density of A) given by the regular non-negative matrix $T=(a_{mn})$ we shall call the limit $\lim\limits_{m\to\infty} \sum\limits_{n=1}^{\infty} a_{mn}\chi_A(n)$ if this limit exists. A set A for which $\delta_T(A)$ exists, is called the set with the δ_T density. If T equals the matrix of the $(C, 1)$ method, we have the usual asymptotic density.

Let **D** be an algebra of a set with the δ_T density, i.e., a system of a set $A \subset N$ satisfying the following conditions:

(*i*) if $A \in \mathbf{D}$, then $N - A \in \mathbf{D}$

(*ii*) if $A, B \in \mathbf{D}$, then $A \cup B \in \mathbf{D}$

(*iii*) if $A \in \mathbf{D}$, then $\delta_T(A)$ exists.

Let $\mathscr{S}$ denote the set of all algebras **D** of set with the δ_T density. Then exist maximal elements in $\mathscr{S}$ with respect to the ordering given by the inclusion. By the Kuratowski—Zorn maximum principle it is sufficient to show that every chain $\mathscr{S}_1 \subset \mathscr{S}$ has the upper boundary in $\mathscr{S}$. Let $\mathscr{S}_1 \subset \mathscr{S}$ be a chain. Let us put $\mathbf{D}^* = \{\mathbf{D} : \mathbf{D} \in \mathscr{S}_1\}$. It is easy to see that $\mathbf{D}^*$ is an algebra of sets with the δ_T density and $\mathbf{D}^*$ the upper boundary of $\mathscr{S}_1$ in $\mathscr{S}$.

Remark 1. An example of such an algebra (see [1]) is the system **D** of sets, which are finite unions of aritmetical progresions of positive integers or which differ from these by finite sets (T is the matrix of the $(C, 1)$ method). An arithmetical progresion is a sequence of positive integers of the form $\{an + b\}_{n=1}^{\infty}$, where a, b are positive integers and b can be equal to zero. The characteristic function of the arithmetical progresion $\{an + b\}_{n=1}^{\infty}$ is $(C, 1)$ summable to the number $\frac{1}{a}$.

The algebra **D** of remark 1 was extended by the Carathéodory method to the system $\mathbf{D}\mu_T$ of all sets measurable with respect to the outer measure μ_T^* defined in the following way:

$$\mu_T^*(A) = \inf\{\delta_T(B) : B \in \mathbf{D} \text{ and } B \dot{\supset} A\} \quad \text{for} \quad A \subset N.$$

The symbol $\dot{\supset}$ means that if a finite set is deleted from A, we have $B \supset A$.

Theorem 1. *Let $\mathscr{S}$ be the system of all the algebras of sets with the δ_T density ordered by inclusion.* **D** *means a maximal element in $\mathscr{S}$ and $\mathbf{D}\mu_T$ means the system of all measurable sets in the Carathéodory sense with respect to the outer measure μ_T^*, where*

$$\mu_T^*(A) = \inf\{\delta_T(B) : B \in \mathbf{D}, B \dot{\supset} A\} \quad \textit{for} \quad A \subset N.$$

Then $\mathbf{D}\mu_T \subset \mathbf{D}$.

Proof. It is sufficient to show that $\delta_T(A)$ exists for every $A \in \mathbf{D}\mu_T$ (since **D** is a maximal element in $\mathscr{S}$, we have $\mathbf{D}\mu_T \subset \mathbf{D}$). Let there be $A \in \mathbf{D}\mu_T$. Then $\mu_T^*(E) = \mu_T^*(E \cap A) + \mu_T^*(E \cap A')$ holds for every $E \subset N$. Let there be $\varepsilon > 0$. To every $\varepsilon > 0$ there exists (from the definition of μ_T^*) a set $B \in \mathbf{D}$ such that $B \dot{\supset} A$ and $|\mu_T^*(B) - \mu_T^*(A)| < \varepsilon$.

Since $A \in \mathbf{D}\mu_T$ and $B \dot{\supset} A$, the following is valid

$$\mu_T^*(B) = \mu_T^*(A) + \mu_T^*(B \cap A'),$$

i.e.

$$|\mu_T^*(B) - \mu_T^*(A)| = \mu_T^*(B - A) \quad \text{and} \quad \mu_T^*(B - A) < \varepsilon.$$

The last inequality yields that for every $\varepsilon > 0$ there exists a set $C \in \mathbf{D}$ such that $C \dot{\supset} B - A$ and $\delta_T(C) < \varepsilon$. Then we have

$$\delta_T(B-A) = \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}\chi_{B-A}(n) \leq$$

$$\leq \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}\chi_C(n) = \delta_T(C) < \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, there holds $\delta_T(B-A) = \mu_T^*(B-A) = 0$ and therefore we have $(\mu_T^*(B) = \delta_T(B))$

$$|\mu_T^*(A) - \delta_T(A)| \leq |\mu_T^*(A) - \mu_T^*(B)| + \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}(\chi_B(n) -$$

$$-\chi_A(n)) = |\mu_T^*(A) - \mu_T^*(B)| + \delta_T(B-A) < \varepsilon .$$

The last relation gives rise to the existence of the $\delta_T(A)$.

In the rest of the paper $\mathbf{D}$ denotes a maximal element of $\mathcal{S}$. The density δ_T is an additive measure on $\mathbf{D}$ (shall be denoted also μ_T or shortly μ).

2.

Let $(X, \|\cdot\|)$ be a Banach space, T a regular matrix method given by a non-negative matrix and $\mathbf{D}$ a maximal element of $\mathcal{S}$. $(N, \mathbf{D}, \mu_T)$ is a measurable space. Let there be the set of all simple measurable functions denoted by $\mathcal{J}_0$. An $x \in \mathcal{J}_0$ iff there exist $x_1, x_2, \ldots, x_n \in X$ and $B_1, B_2, \ldots, B_n \in \mathbf{D}$ such that $\bigcup_{i=1}^{n} B_i = N$; $B_i \cap B_j = \emptyset$ for $i \neq j$ and $x = \sum_{i=1}^{n} x_i \chi_{B_i}$.

The validity of the following theorem can be easily verified. $\mathcal{J}_0$ can be accompanied by the sup norm $\|\cdot\|_\infty$ $\left(\|x\|_\infty = \sup_n \|x_n\| \text{ for } x \in \mathcal{J}_0\right)$.

Theorem 2. *$(\mathcal{J}_0, \|\cdot\|_\infty)$ is a normed linear space.*

Definition. *Suppose $x \in \mathcal{J}_0$, $x = \sum_{i=1}^{n} x_i \chi_{B_i}$. Then we define the integral of x by the formula*

$$\int x \, \mathrm{d}\mu_T = \sum_{i=1}^{n} x_i \mu_T(B_i).$$

It is easy to see that $\|\int x \, \mathrm{d}\mu_T\| \leq \|x\|_\infty$. $\mathcal{J}_0$ is a not complete normed space. A completion of $\mathcal{J}_0$ with respect to the sup norm will be denoted $(\mathcal{J}_1, \|\cdot\|_\infty)$. The $\mathcal{J}_1$

is the uniform hull of the $\mathcal{I}_0$. The integral can be extended to $\mathcal{I}_1$ in a natural way, i.e. suppose $x \in \mathcal{I}_1$, $x = \lim x^n$, $x^n \in \mathcal{I}_0$ and $\{x^n\}_{n=1}^{\infty}$ form a Cauchy sequence, then

$$\int x \, d\mu_T = \lim_{n \to \infty} \int x^n \, d\mu_T .$$

The following theorem gives the necessary and sufficient condition for $x \in \mathcal{I}_1$.

Theorem 3. *Let $x = \{x_n\}$ be a sequence of elements of X. $H(x)$ denotes the set of all the limit points of the sequence x. Then $x \in \mathcal{I}_1$ iff the following condition is fulfilled:*

For every $\varepsilon > 0$ there exists a finite ε-net $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ in $H(x)$ and such a finite system of sets $\{B_1, B_2, \ldots, B_r\} \in \mathbf{D}$ that $\bigcup_{i=1}^{n} B_i = N$ and $B_i \subset \{n \in N: \|x_n - \alpha_i\| < \varepsilon\}$.

The proof immediately follows from the fact that $\mathcal{I}_0$ is dense in $\mathcal{I}_1$.

Remark 2. The compactness of $H(x)$ is a necessary condition for $x \in \mathcal{I}_1$.

From Theorem 3 it follows that the operator defined by the integral is a generalized limit. This generalized limit is the same as the usual limit for every convergent sequence.

3.

In this section there will be shown the relation between the summability field $W(T)$ of the regular summability method given by the non-negative matrix and $\mathcal{I}_1$.

In paper [3] a charakterization of the summability fields of the $(C, 1)$ method and strongly regular methods which are not weaker than the $(C, 1)$ method is given. This characterization is given by the integral constructed for this purpose. We can show that the mentioned summability fields can be characterized by the above mentioned integral on $\mathcal{I}_1$. Proofs are done in the same way as in [3].

Let **D** denote a maximal element of $\mathcal{S}$ which contains all arithmetical progressions. Let $x = \{x_n\}$ be a sequence of elements of X. We can define the sequence $x^{\mathrm{mod}\, m} = \{x_n^{\mathrm{mod}\, m}\}$, where $x_n^{\mathrm{mod}\, m} = x_n$ for $1 \leqslant n \leqslant m$ and $x_n^{\mathrm{mod}\, m} = x_i$ for $1 \leqslant i \leqslant m$, $n \equiv i \bmod m$, for every positive integer m. The sequence $x^{\mathrm{mod}\, m}$ can be written in the form

$$x^{\mathrm{mod}\, m} = \sum_{i=1}^{m} x_i \chi_{B_i}, \quad \text{where} \quad B_i = \{mk + i: k = 0, 1, 2, \ldots\}$$

and thus $x^{\mathrm{mod}\, m}$ belongs to $\mathcal{I}_1$ for every m.

The following theorem characterizes $W((C, 1))$ by the integral on $\mathcal{I}_1$.

Theorem 4. *A sequence x belongs to $W((C, 1))$ iff $\lim_{m \to \infty} \int x^{\mathrm{mod}\, m} \, d\mu_{(C,1)}$ exists.*

Then there holds

$$(C,1)-\lim x=\lim_{m\to\infty}\int x^{\bmod m}\,\mathrm{d}\mu_{(C,1)}.$$

Proof. Let be $x^{\bmod m}=\sum_{i=1}^{m}x_i\chi_{B_i}$, $B_i=\{mk+i:\ k=0,1,\dots\}$. Then we have

$$\int x^{\bmod m}\,\mathrm{d}\mu_{(C,1)}=\sum_{i=1}^{m}x_i\mu_{(C,1)}(B_i)=\frac{1}{m}\sum_{i=1}^{m}x_i.$$

Therefore the following holds

$$(C,1)-\lim x=\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m}x_i=\lim_{m\to\infty}\int x^{\bmod m}\,\mathrm{d}\mu_{(C,1)}.$$

An example of a regular transformation which is not strongly regular and for which the summability field can be characterized by integral, is a summability method given by a matrix $D=(d_{mn})$. The matrix D arises from the matrix of the $(C,1)$ method putting $(r-1)$ columns consisting of zeros between two neighbouring columns of the $(C,1)$ method, i.e. $d_{mn}=\frac{1}{m}$ for $n=rk+1$ $(k=0,1,\dots,m-1)$ and $d_{mn}=0$ for other n.

Lemma 1. *Let $B_i=\{rkm+i:\ k=0,1,\dots,\ 1\leqslant i\leqslant rm\}$. Then $\mu_D(B_i)=\frac{1}{m}$ for $i=rs+1$, $0\leqslant s<m$ and $\mu_D(B_i)=0$ for other i.*

Proof. The set B_i can be written in the form $B_i=\{k:\ k\equiv i \bmod rm\}$. In the first place we determine the sum $\sum_{j=1}^{n}\chi_{B_i}(j)$. It is easy to see that we obtain $\sum_{j=1}^{n}\chi_{B_i}(j)=0$ for $n<i$ and $\sum_{j=1}^{n}\chi_{B_i}(j)=k+1$ for $krm+i\leqslant n<(k+1)rm+i$. It can be simply written

$$\sum_{j=1}^{n}\chi_{B_i}(j)=\left[\frac{n+rm-i}{rm}\right]$$

($[a]$ denotes the integral part of a). Since $d_{nj}=\frac{1}{n}$ for $j\equiv i \bmod rm$, $i=rs+1$, $0\leqslant s<m$, we have

$$\mu_D(B_i)=\lim_{n\to\infty}\sum_{j=1}^{r(n-1)+1}\mathrm{d}_{nj}\chi_{B_i}(j)=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{r(n-1)+1}\chi_{B_i}(j)=\frac{1}{m}.$$

For other i and $j\equiv i \bmod rm$ there holds $d_{nj}=0$ and therefore $\mu_D(B_i)=0$.

Theorem 5. *A sequence x belongs to $W(D)$ iff $\lim_{m\to\infty}\int x^{\bmod rm}\,\mathrm{d}\mu_D$ exists. Then*

$$\mathrm{D}-\lim x=\lim_{m\to\infty}\int x^{\bmod rm}\,\mathrm{d}\mu_D.$$

Proof. Since $x^{\mathrm{mod}\, rm}$ belongs to $\mathcal{J}_0$, we have

$$\int x^{\mathrm{mod}\, rm}\, \mathrm{d}\mu_D = \sum_{n=1}^{rm} x_n \mu_D(\{k\colon k \equiv n \bmod rm\}) = \sum_{s=0}^{m-1} \frac{1}{m} x_{rs+1}$$

$\left(d_{mn} = \frac{1}{m} \text{ for } n = rk+1,\ 0 \leq k \leq m-1\right)$.

For the D limit of x there holds

$$D - \lim x = \lim_{m\to\infty} \sum_{n=1}^{r(m-1)+1} d_{mn} x_n = \lim_{m\to\infty} \sum_{s=0}^{m-1} \frac{1}{m} x_{rs+1}$$

and therefore

$$D - \lim x = \lim_{m\to\infty} \int x^{\mathrm{mod}\, rm}\, \mathrm{d}\mu_D .$$

Remark 3. We can taken a Schur matrix instead of a regular non-negative matrix. Then the system $\mathcal{S}$ has exactly one maximal element (the system of all subsets of the set N). In this case the summability field can be characterized analogously to that in Theorem 4.

The set of all integrable sequences with respect to the measure μ_T cannot be used for the characterization of the summability field in the same way as in Theorem 4 (see [3] page 87).

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О ХАРАКТЕРИЗОВАНИИ ПОЛЯ СХОДИМОСТИ ПРИ ПОМОЩИ ИНТЕГРАЛА

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Резюме

В работе [4] определен «интеграл суммирования» при помощи меры определенной на логике подмножеств множества всех натуральных чисел а также характеризованы поля сходимости для некоторых сильно регулярных методов суммирования.

В настоящей работе показано, что обыкновенное определение интеграла на алгебре подмножество множества всех натуральных чисел тоже допускает характеризовать поля сходимости тем самым образом как в [4], но, не только для сильно регулярных методов суммирования.