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ON THE ADDITIVITY OF GUDDER INTEGRAL

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1. Introduction

S. Gudder in [1] and [2] developed integration of observables, which are functions mapping a non-empty set equipped with a σ-class of its subsets and a probability measure into the real line. In the present paper we prove some sufficient conditions for the additivity of this integral, which are more general than the conditions established by Gudder in [2]. However, we show that the integral need not be additive in general, thus answering Gudder's question in [1] and [2] in the negative. We also show that the method used to construct the counterexample cannot produce an example of bounded observables on which the integral would not be additive.

2. Notation and preliminaries

Let X be a non-empty set. A *σ-class* C is a nonempty collection of subsets of X satisfying:

(i) The complement a' of each $a \in C$ is in C;

(ii) If $a_1, a_2, \ldots$ is a sequence of pairwise disjoint elements in C, then $\cup_i a_i$ is in C.

Evidently every σ-class C on X contains X.

A state m on C is a map m: $C \to \langle 0, 1\rangle$ such that

(i) $m(\emptyset) = 0$, $m(X) = 1$

(ii) If a_i $(i = 1, 2, \ldots)$ are mutually disjoint elements, of C, then $m(\cup_i a_i) = \Sigma_i m(a_i)$.

A generalized measure space is a triple (X, C, m) where C is a σ-class of subsets of X and m is a state on C.

The idea of studying such spaces was introduced by P. Suppes [4]. Further studies are in [1], [2], [3], and [5].

Two elements $a, b \in C$ are called compatible iff $a \cap b \in C$. A family $A \subset C$ is said to be compatible provided any finite intersection of sets in A belongs to C.

An observable is a function f: $X \to R$ for which $f^{-1}(b) \in C$ for every Borel set b in the real line R. If f is an observable, we use the notation $A_f = \{f^{-1}(b) : b$ is a Borel set in the real line}. We say that two observables f and g are compatible iff $A_f \cup A_g$ is compatible. It is well known (see [2]) that the sum of any two compatible observables is an observable. However, there are non-compatible observables whose sum is an observable. If f, g and $f+g$ are observables, the functions f and g are said to be summable.

If f is an observable, then the state m restricted to A_f is a probability measure on a σ-algebra. Thus (X, A_f, m) becomes a probability space and the integral $\int f \, dm$ is defined as the usual integral of f in this probability space.

If f and g are summable observables, the question arises whether or not $\int (f+g) \, dm = \int f \, dm + \int g \, dm$. In [2], S. Gudder gives a sufficient condition for the additivity of his integral (Theorem 1 below). For stating Gudder's result we need some definitions.

An observable f is called simple iff its range is finite. Let f and g be summable simple observables with the values $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, respectively. If we have $a_i + b_j = a_p + b_q$ whenever $i \neq p$ and $j \neq q$, then $(a_i, b_j; a_p, b_q)$ is called a degeneracy for f, g. If the values of f and g can be reordered in such a way that whenever $(a_i, b_j; a_p, b_q)$ is a degeneracy we have $i = q$ and $j = p$, then f, g are called symmetric.

Theorem 1. (Theorem 3.1 of [2]). *Let (X, C, m) be a generalized measure space. If f and g are symmetric, then*

$$\int (f+g) \, dm = \int f \, dm + \int g \, dm.$$

3. Results

It is known that m need not be even subadditive on C and so, in general, cannot be extended to the σ-algebra $A(C)$ generated by C. However, if it can be, then the answer to the above question is obvious.

Theorem 2. *Let (X, C, m) be a generalized measure space. Suppose that m can be extended to the σ-algebra $A(C)$ generated by C. Then the integral is additive on summable observables.*

Proof. Denote by μ the extension of m. Then

$$\int f \, dm + \int g \, dm = \int f \, d\mu + \int g \, d\mu = \int (f+g) \, d\mu = \int (f+g) \, dm,$$

where we have used the additivity of the usual integral $\int d\mu$.

Corollary 3. Let the state m be a point measure on C, that is, there exists an $\omega \in X$ with $m(a) = 1$ if $\omega \in a$, $m(a) = 0$ if $\omega \notin a$ for all a in C. Then the integral with respect to m is additive on summable observables.

Clearly the assumption of the last theorem is too strong. In fact, its proof makes no use of the values of μ on the whole $A(C)$ and requires the state m to be extended to a smaller σ-algebra only.

Theorem 4. *Let (X, C, m) be a generalized measure space, let f and g be summable observables. If there exists a measure μ defined on the σ-algebra generated by $A_f \cup A_g \cup A_{f+g}$ such that μ coincides with m on A_f, A_g, and A_{f+g}, then*

$$\int f \, \mathrm{d}m + \int g \, \mathrm{d}m = \int (f+g) \, \mathrm{d}m.$$

Conjecture 5. *The condition in the last theorem is necessary.*

At the Symposium Probastat '77 in Smolenice, Czechoslovakia, an example suggested by the authors was presented of two summable observables on a suitable generalized measure space such that the integral is not additive on them. The two observables were unbounded functions.

In the following we describe in general the method of construction of such examples and show that the observables in any example produced by this method are necessarily unbounded.

Example 6. Let X be a non-empty set, let f, g be real valued functions on x satisfying the following conditions

(1) $f+g$ attains at least two distinct values

(2) If b_1, b_2 and b_3 are any Borel sets of reals such that $a_1 = f^{-1}(b_1)$, $a_2 = g^{-1}(b_2)$ and $a_3 = (f+g)^{-1}(b_3)$ are non-empty, then all the intersections $a_1 \cap a_2$, $a_2 \cap a_3$ and $a_1 \cap a_3$ are non-empty.

As an example we can take $X = R^2$, $f(x_1, x_2) = x_1$, $g(x_1, x_2) = x_2$. Under the said assumptions, the family $C = A_f \cup A_g \cup A_{f+g}$ is a σ-class. In fact, for each $a \in C$ there is $h \in \{f, g, f+g\}$ with $a \in A_h$ and hence $X \backslash a$ is also in the σ-algebra A_h, yielding that C is closed under complementation. Now for any disjoint family $\{a_n\}_{n=1}^{\infty}$ of elements of C all a_n are in the same A_h, h being one of the functions f, g, $f+g$, and hence $\bigcup_{n=1}^{\infty} a_n \in A_h \subset C$. With respect to C, the functions f and g are summable observables. It suffices now to define a stane m on (X, C) in such a way that $\int (f+g) \, \mathrm{d}m \neq \int f \, \mathrm{d}m + \int g \, \mathrm{d}m$. By (1), there exists $\alpha, \beta \in X$ with $f(\alpha) + g(\alpha) \neq f(\beta) + g(\beta)$. For $a \in A_f \cup A_g$ let $m(a)$ equal 1 or 0 according to whether a includes α or not. We obtain

$$\int f \, \mathrm{d}m + \int g \, \mathrm{d}m = f(\alpha) + g(\alpha).$$

Putting for $a \in A_{f+g}$ $m(a) = 1$ or 0 according to whether $\beta \in a$ or $\beta \notin a$, we complete the definition of m and, after easily checking that m is in fact a state on C, we see that

$$\int (f+g) \, \mathrm{d}m = f(\beta) + g(\beta) \neq f(\alpha) + g(\alpha) = \int f \, \mathrm{d}m + \int g \, \mathrm{d}m.$$

In the concrete example mentioned we may take $\alpha = [0, 0]$, $\beta = [1, 1]$.

The assumption (2) is fairly strong and although it is easy to find many pairs of functions satisfying (1) and (2), the assumption (2) implies that both f and g in the counterexample are unbounded functions.

Suppose that in an example based on (1) and (2) the observables f and g are bounded, i.e. there exist finite $p_1 = \inf f$, $p_2 = \sup f$, $q_1 = \inf g$, and $q_2 = \sup g$. We are going to prove that $\inf (f+g) = p_1 + q_1$. For $\delta > 0$ we have

$$f^{-1}\langle p_1, p_1 + \delta/2) \neq \emptyset \quad \text{and} \quad g^{-1}\langle q_1, q_1 + \delta/2) \neq \emptyset,$$

which by (2) implies

$$f^{-1}\langle p_1, p_1 + \delta/2) \cap g^{-1}\langle q_1, q_1 + \delta/2) \neq \emptyset.$$

In other words, there is $\xi \in X$ with $p_1 \leqq f(\xi) < p_1 + \delta/2$ and $q_1 \leqq g(\xi) < q_1 + \delta/2$, implying $f(\xi) + g(\xi) < p_1 + q_1 + \delta$. This together with the evident inequality $\inf (f+g) \geqq p_1 + q_1$ gives what has been claimed. Analogously $\sup (f+g) = p_2 + q_2$. Due to the last equality we have, for any $\varepsilon > 0$, $(f+g)^{-1}(p_2 + q_2 - \varepsilon/2, p_2 + q_2\rangle \neq \emptyset$. Since evidently $f^{-1}\langle p_1, p_1 + \varepsilon/2) \neq \emptyset$, by (2) we obtain that there is $\eta \in X$ with $p_2 + q_2 - \varepsilon/2 < f(\eta) + g(\eta) \leqq p_2 + q_2$ and $p_1 \leqq f(\eta) < p_1 + \varepsilon/2$. Substracting these inequalities we obtain $p_2 + q_2 - p_1 - \varepsilon < g(\eta) \leqq p_2 - p_1 + q_2$. We infer that $q_2 = \sup g \geqq p_2 + q_2 - p_1 - \varepsilon$, hence $p_1 + \varepsilon \geqq p_2$ for every $\varepsilon > 0$. We have obtained that $\inf f = p_1 \geqq p_2 = \sup f$, i.e. f is a constant function. Analogously g can be proved to be constant but in this case $\int (f+g)\, dm = \int f\, dm + \int g\, dm$. Thus we have shown that in every counterexample based on (1) and (2) the observables f and g are necessarily unbounded. This leads us to the following

Conjecture 7. *The Gudder integral is additive on bounded summable observables.*

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О АДДИТИВНОСТИ ИНТЕГРАЛА ГАДДЕРА

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Резюме

В статье доказываются некоторые достаточные условия аддитивности интеграла, введенного Гаддером для функций, измеримых на обобщенном пространстве с мерой. Конструируется пример, показывающий, что интеграл Гаддера в общем не аддитивен и показывается, что эта конструкция всегда приводит к неограниченным функциям.