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RELATIONSHIPS BETWEEN FAMILIES OF THIN SETS

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ABSTRACT. We show that the families of thin sets $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{B}_f$, $\mathcal{N}_f$, $\mathcal{A}_f$, and $w\mathcal{D}_f$ coincide with corresponding trigonometric families when $\mathbf{Z}(f)$ is a finite set of rationals. By a theorem of [BUKOVSKÁ, Z.—BUKOVSKÝ, L.: *Comparing families of thin sets*, Real Anal. Exchange **27** (2001/2002), 609–625] this is not true for the families $\mathcal{B}_{0f}$ and $\mathcal{N}_{0f}$.

We start with some notations. We work on the topological group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ often identified with intervals $\langle -1/2, 1/2\rangle$ or $\langle 0, 1\rangle$. Throughout the paper, f, g will denote continuous functions satisfying

$$f, g\colon \mathbb{T} \to \langle 0, +\infty), \qquad 0 \in \mathbf{Z}(f), \mathbf{Z}(g) \neq \mathbb{T}, \tag{1}$$

where

$$\mathbf{Z}(f) = \{x \in \mathbb{T} :\ f(x) = 0\}$$

is the zero set of the function f. The functions f, g are supposed to be periodically extended to the set $\mathbb{R}$. For convenience we define

$$f_n(x) = f(nx) \qquad \text{for} \quad x \in \mathbb{T}.$$

In our paper [BZ] we have introduced families of thin sets $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$, $\mathcal{B}_f$, $\mathcal{N}_f$, $\mathcal{A}_f$, and $w\mathcal{D}_f$ in a similar way as one defines the trigonometric families of thin sets $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{B}_0$, $\mathcal{N}_0$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{A}$, and $w\mathcal{D}$ in the case[1] $f(x) = \|x\|$ or equivalently $f(x) = |\sin(\pi x)|$. We have shown that those families possess properties similar to those of trigonometric ones. The first important problem is

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[1] $\|x\|$ is the distance of the real x from the nearest integer.

the relationships of those new introduced families to the corresponding trigonometric families. The main result of this paper, together with [BB; Theorem 16], gives an almost complete answer to this question in the case when $\mathbf{Z}(f)$ is a finite set of rationals. Moreover we present affirmative answers to problems a), b) and c) of [BB].

Since for a positive real c, we have $\mathcal{T}_f = \mathcal{T}_{c \cdot f}$ for any of the introduced families $\mathcal{T}$, throughout the paper we shall assume that

$$\max\{f(x) : \ x \in \mathbb{T}\} = \max\{g(x) : \ x \in \mathbb{T}\} = 1 .$$

Our main result is:

THEOREM 1. *If $m \cdot \mathbf{Z}(f) \subseteq \mathbf{Z}(g)$ for a positive integer m, then $\mathcal{T}_f \subseteq \mathcal{T}_g$ holds true for $\mathcal{T} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$.*

The Theorem is a strengthening of [BB; Theorems 7, 14]. For the proof we shall need auxiliary results.

LEMMA 2. *If $m \cdot \mathbf{Z}(f) \subseteq \mathbf{Z}(g)$ for a positive integer m, then for any $\varepsilon > 0$, $\varepsilon < 1$, there exists a $\delta > 0$ such that $g(mx) < \varepsilon$ for any $x \in \mathbb{T}$ such that $f(x) < \delta$.*

Proof. Let $U = \{x \in \mathbb{T} : \ g(mx) < \varepsilon\}$. Then $\mathbf{Z}(f) \subseteq U \neq \mathbb{T}$. Set

$$\delta = \min\{f(x) : \ x \in \mathbb{T} \setminus U\} .$$

Then $\delta > 0$. If $f(x) < \delta$, then $x \in U$ and therefore $g(mx) < \varepsilon$. □

In [BL] the author proves a generalization of a result of [HMP] saying that

$$\mathcal{N}_f \cap \mathcal{F}_\sigma = w\mathcal{D}_f \cap \mathcal{F}_\sigma \tag{2}$$

($\mathcal{F}_\sigma$ is the family of F_σ subsets of $\mathbb{T}$). We shall need a modification of this result.

THEOREM 3. *A closed set $A \subseteq \mathbb{T}$ is a $\mathcal{B}_f$-set if and only if A is a weak f-Dirichlet set, i.e.*

$$\mathcal{B}_f \cap \text{Closed} = w\mathcal{D}_f \cap \text{Closed} . \tag{3}$$

The starting point of the proof (see [HMP], compare [BL; Lemma 11]) is:

LEMMA 4. (Host-Méla-Parreau) *If $K \subseteq \mathbb{T}$ is a closed weak f-Dirichlet set, then for any positive integer m, the zero function belongs to the closed convex hull of $\{f_n : \ n \geq m\}$ in the Banach space $C(K)$ with the uniform topology.*

Proof. Let H_m be the closed convex hull of $\{f_n : \ n \geq m\}$, m being a positive integer. Assume $0 \notin H_m$. Then by the Hahn-Banach theorem (see e.g. [RW1]) there exists a linear functional $F \colon C(K) \to \mathbb{R}$ and a positive real ε such that $F(f) \geq \varepsilon$ for any $f \in H_m$. By Riesz theorem (see e.g. [RW2]) there exists a Borel measure μ on K such that $F(f) = \int_K f(x)\, \mathrm{d}\mu(x)$. Then

$$\liminf_{n\to\infty} \int_K f_n(x)\, \mathrm{d}\mu(x) \geq \varepsilon\,,$$

contradicting the fact that K is weak f-Dirichlet. □

Proof of Theorem 3. If $K \subseteq \mathbb{T}$ is a closed weak f-Dirichlet set, then using the lemma one can easily define by induction nonnegative reals a_i and an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that

$$\sum_{i=n_k}^{n_{k+1}-1} a_i = 1\,, \qquad \sum_{i=n_k}^{n_{k+1}-1} a_i f_i(x) \leq 2^{-k} \quad \text{for any } x \in K\,.$$

Then $\sum_{i=0}^{\infty} a_i = \infty$ and $\sum_{i=0}^{\infty} a_i f_i(x) \leq 2$ for any $x \in K$. Thus, K is a $\mathcal{B}_f$-set. □

From the proof we can obtain another proof of a result by P. Eliaš [EP; p. 1114] for trigonometric thin sets. We introduce a notion.

A set $A \subseteq \mathbb{T}$ is said to be *uniformly $\mathcal{B}_f$-set* if there are nonnegative reals a_i and a positive real c such that $A \subseteq \left\{x \in \mathbb{T}:\ \sum_{i=0}^{\infty} a_i f_i(x) \leq c\right\}$ and the series $\sum_{i=0}^{\infty} a_i f_i(x)$ converges uniformly on the set A.

In notation of [EP], a $\mathcal{B}$-set is an $\mathcal{C}_{BS}^{i}$ set and a uniformly $\mathcal{B}$-set is an $\mathcal{C}_{US}^{i}$ set (compare [EP]). By the proof of Theorem 3, every $\mathcal{B}_f$-set is uniformly $\mathcal{B}_f$-set. Compare also [BL; Corollary 12(i)].

Proof of Theorem 1. For $\mathcal{T} = \mathcal{D}, p\mathcal{D}, \mathcal{A}, w\mathcal{D}$, the proof is based on Lemma 2.

By Lemma 2 there exists a sequence of positive reals $\{\delta_k\}_{k=0}^{\infty}$ such that $g(mx) \leq 2^{-k-1}$ for any x satisfying the inequality $f(x) \leq \delta_k$. We can assume that $\lim_{k\to\infty} \delta_k = 0$.

Now, let $A \subseteq \mathbb{T}$ be an f-Dirichlet (pseudo f-Dirichlet) set, the sequence $\{n_k\}_{k=0}^{\infty}$ being such that $f(n_k x) \leq \delta_k$ for every $x \in A$ and every (almost every) $k \in \mathbb{N}$. Then $g(mn_k x) \leq 2^{-k-1}$ for every $x \in A$ and every (almost every) $k \in \mathbb{N}$. Thus the set A is also g-Dirichlet (pseudo g-Dirichlet).

For an $\mathcal{A}$-set the proof is similar. We prove the inclusion for weak Dirichlet sets.

We assume that $A \in w\mathcal{D}_f$ is a Borel set[2], μ is a Borel measure on $\mathbb{T}$, $\lim_{k\to\infty} \int_A f(n_k x)\, \mathrm{d}\mu(x) = 0$, and $\mu(\mathbb{T}) = 1$. For given $\varepsilon > 0$ there exists a real $\delta > 0$ such that the assertion of Lemma 2 holds true. For $k \in \mathbb{N}$ we set

$$B_k = \{x \in \mathbb{T} :\ f(n_k x) < \delta\}\,.$$

Let k_0 be such that $\int_A f(n_k x)\, \mathrm{d}\mu(x) < \delta \cdot \varepsilon$ for $k \geq k_0$. Then $\mu(A \setminus B_k) < \varepsilon$ and

$$\begin{aligned}\int_A g(mn_k x)\, \mathrm{d}\mu(x) &\leq \int_{A\setminus B_k} g(mn_k x)\, \mathrm{d}\mu(x) + \int_{B_k} g(mn_k x)\, \mathrm{d}\mu(x) \\ &< \varepsilon + \mu(B_k)\varepsilon \leq 2\varepsilon\,.\end{aligned}$$

Thus $A \in w\mathcal{D}_g$.

Now, let $A \subseteq \mathbb{T}$ be a $\mathcal{B}_f$-set (an $\mathcal{N}_f$-set). Then there exists a closed $\mathcal{B}_f$-set (an F_σ $\mathcal{N}_f$-set) B such that $A \subseteq B$. Thus B is a weak f-Dirichlet set and consequently a weak g-Dirichlet set. By (3) (or (2)), B is also a $\mathcal{B}_f$-set ($\mathcal{N}_f$-set). □

Corollary 5. *For any continuous function f satisfying condition* (1) *the inclusion $\mathcal{T} \subseteq \mathcal{T}_f$ holds true for $\mathcal{T} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$.*

Corollary 6. *If $\mathbf{Z}(f)$ is a finite set of rationals, then $\mathcal{T} = \mathcal{T}_f$ for $\mathcal{T} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$.*

In [BB; Theorem 10], we have shown that for $\mathcal{A}$-sets the opposite implication holds true, i.e. $\mathcal{A} = \mathcal{A}_f$ if and only if $\mathbf{Z}(f)$ is a finite set of rationals. In spite of Corollary 6 a similar result does not hold true for $\mathcal{T} = \mathcal{D}$, $p\mathcal{D}$. For to construct a counterexample, let us recall [BB; Theorem 4]:

> *If $\mathbf{Z}(f)$ is a g-Dirichlet (pseudo g-Dirichlet) set, then $\mathcal{D}_f \subseteq \mathcal{D}_g$ ($p\mathcal{D}_f \subseteq p\mathcal{D}_g$).*

Now, let $A \subseteq \mathbb{T}$ be a perfect Dirichlet set, $0 \in A$. The existence of such a set follows e.g. by [Ka]. Thus, A is not a finite set of rationals. Let $f \colon \mathbb{T} \to \mathbb{R}^+$ be such that $\mathbf{Z}(f) = A$. Then by above cited theorem we obtain $\mathcal{D}_f \subseteq \mathcal{D}$ and $p\mathcal{D}_f \subseteq p\mathcal{D}$. Moreover, we can assume that $f(x) \leq |x|$ for any $x \in \langle -1/2, 1/2\rangle$. Then even $\mathcal{D}_f = \mathcal{D}$ and $p\mathcal{D}_f = p\mathcal{D}$.

The problem is open for $\mathcal{B}$, $\mathcal{N}$ and $w\mathcal{D}$.

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[2]There are different definitions of a weak Dirichlet set B in the literature. In [BZ] we ask that there is a Borel set $A \supseteq B$ satisfying the condition, that for any Borel measure μ on A there is a sequence $\{n_k\}$ such that $\lim_{k\to\infty} \int_A f(n_k x)\, \mathrm{d}\mu(x) = 0$. In [HMP], [BL], the authors ask the existence of an analytic set A with this property. In our proof, one can replace the word "Borel" by "analytic" and everything will work.

COROLLARY 7. *If* $\mathbf{Z}(f) \subseteq \mathbf{Z}(g)$, *then* $\mathcal{T}_f \subseteq \mathcal{T}_g$ *for* $\mathcal{T} = \mathcal{D}, p\mathcal{D}, \mathcal{A}, \mathcal{B}, \mathcal{N}, w\mathcal{D}$.

COROLLARY 8. *If* $\mathbf{Z}(f) = \mathbf{Z}(g)$, *then* $\mathcal{T}_f = \mathcal{T}_g$ *for* $\mathcal{T} = \mathcal{D}, p\mathcal{D}, \mathcal{A}, \mathcal{B}, \mathcal{N}, w\mathcal{D}$.

Let us remark the following. Our proofs are constructive in the case of $\mathcal{T} = \mathcal{D}, p\mathcal{D}, \mathcal{A}, w\mathcal{D}$ in the following sense: if the sequence $\{n_k\}_{k=0}^{\infty}$ witnesses that $A \in \mathcal{T}_f$, then the sequence $\{m \cdot n_k\}_{k=0}^{\infty}$ does so for $A \in \mathcal{T}_g$. Maybe that the speed of convergence has been changed. In the case of $\mathcal{T} = \mathcal{B}, \mathcal{N}$ this is not true. If, e.g.

$$A = \Big\{x \in \mathbb{T} : \sum_{k=0}^{\infty} a_k f_k(x) < \infty\Big\}, \qquad \sum_{k=0}^{\infty} a_k = \infty, \quad a_k \geq 0,$$

is an $\mathcal{N}_f$-set, we have no explicit method for finding nonnegative reals b_k such that

$$A \subseteq \Big\{x \in \mathbb{T} : \sum_{k=0}^{\infty} b_k g_k(x) < \infty\Big\}, \qquad \sum_{k=0}^{\infty} b_k = \infty.$$

According to [BB; Theorem 16], the assertions of Corollaries 6, 7 and 8 for the families $\mathcal{B}_0$ and $\mathcal{N}_0$ do not hold true.

In [BZ] we have introduced and investigated the condition

$$f(x-y) \leq f(x) + f(y) \qquad \text{for any} \quad x, y \in \mathbb{T}. \tag{4}$$

If the function f satisfies this condition, then the considered families of thin sets posses some additional properties. On the other hand, if condition (4) is satisfied, then one can easily see that the set $\mathbf{Z}(f)$ is a closed subgroup of $\mathbb{T}$ and therefore $\mathbf{Z}(f)$ is a finite set of rationals. So the results of [BZ] and [BL] about the families $\mathcal{T}_f$, $\mathcal{T} = \mathcal{D}, p\mathcal{D}, \mathcal{A}, w\mathcal{D}$ assuming condition (4) follow immediately by Corollary 6. By [BB; Theorem 16] this is not the case of the families $\mathcal{B}_{0f}$ and $\mathcal{N}_{0f}$. However, we have the following reduction.

THEOREM 9. *If the function f satisfies condition* (4), *then there exists a continuous function* $g\colon \mathbb{T} \to \mathbb{R}^+$ *satisfying* (4) *and such that* $\mathbf{Z}(g) = \{0\}$, $\mathcal{B}_{0f} = \mathcal{B}_{0g}$ *and* $\mathcal{N}_{0f} = \mathcal{N}_{0g}$.

Proof. Assume that condition (4) holds true. Then either $\mathbf{Z}(f) = \{0\}$ (and everything is trivial) or there exists a positive integer k such that

$$\mathbf{Z}(f) = \{0, h, 2h, \dots, (k-1)h\},$$

where $h = 1/k$. Using (4) one can easily show that f is periodic with the period h, i.e. $f(x+h) = f(x)$ for any $x \in \mathbb{T}$. We set

$$g(x) = f(x/k) \qquad \text{for} \quad x \in \langle 0, 1 \rangle.$$

According to the periodicity of the function f, the function g is well defined and $f(x) = g(kx)$ for any $x \in \mathbb{T}$. Moreover, g satisfies condition (4).

If $A = \left\{x \in \mathbb{T} : \sum\limits_{i=0}^{\infty} f(n_i x) < \infty\right\}$ is an $\mathcal{N}_{0f}$-set, then A is also an $\mathcal{N}_{0g}$-set — just replace $f(n_i x)$ by $g(kn_i x)$.

Now, let $A = \left\{x \in \mathbb{T} : \sum\limits_{i=0}^{\infty} g(n_i x) < \infty\right\}$ be an $\mathcal{N}_{0g}$-set. Let

$$L_j = \{i \in \mathbb{N} : n_i = j \mod k\}, \qquad j = 0, 1, \dots, k-1.$$

One can easily see that there is an $0 \le l < k$ such that the set L_l is infinite. Let $\{m_i\}_{i=0}^{\infty}$ be an increasing enumeration of the set $\{n_i : i \in L_l\}$. Then

$$f\left(\tfrac{m_{i+1} - m_i}{k} x\right) = g\big((m_{i+1} - m_i)x\big) \le g(m_{i+1}x) + g(m_i x)$$

and therefore

$$\sum_{i=0}^{\infty} f\left(\tfrac{m_{i+1} - m_i}{k} x\right) \le 2 \cdot \sum_{i \in L_l} g(n_i x) < \infty$$

for any $x \in A$. Thus A is also an $\mathcal{N}_{0f}$-set.

For $\mathcal{B}_0$-sets one can proceed in same way. □

In [BB] we formulated as an open problem the following question: Does there exist a function f such that $\mathcal{T}_f \ne \mathcal{T}$ for $\mathcal{T} = \mathcal{B}$, $\mathcal{N}$ and $w\mathcal{D}$? Actually, we did not realize that we have essentially answered this question there.

THEOREM 10. *There exists a continuous function f satisfying condition* (1) *such that $\mathcal{T} \subseteq \mathcal{T}_f$, $\mathcal{T} \ne \mathcal{T}_f$, for $\mathcal{T} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{B}_0$, $\mathcal{N}_0$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$.*

Proof. Let $\mathbf{C}$ be Cantor middle-thirds set

$$\left\{x \in \mathbb{T} : \left(\exists \{x_i\}_{i=1}^{\infty}\right)\left((\forall i \in \mathbb{N})(x_i = 0, 2) \ \wedge \ x = \sum_{i=1}^{\infty} x_i \cdot 3^{-i}\right)\right\}.$$

Let f be a continuous function such that $f(x) \le |x|$ for $x \in \langle -1/2, 1/2\rangle$ and $\mathbf{Z}(f) = \mathbf{C}$. Since $3^n x \in \mathbf{C}$ for any $x \in \mathbf{C}$, we obtain $f(3^n x) = 0$ for $x \in \mathbf{C}$, i.e. $\mathbf{C} \in \mathcal{D}_f$.

Let $\mathcal{T}$ be any of the symbols $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{B}_0$, $\mathcal{N}_0$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$. Then $\mathbf{C} \in \mathcal{T}_f$. H. Steinhaus [St] has shown that the arithmetic sum $\mathbf{C} + \mathbf{C}$ is the whole space $\mathbb{T}$. Since every trigonometric family of thin sets is closed for the arithmetic sum (compare [Ba], [Zy]), we obtain that $\mathbf{C} \notin \mathcal{T}$. Moreover, since $f(x) \le |x|$, we have $\mathcal{T} \subseteq \mathcal{T}_f$. □

Now, let $f(x) = \sqrt{\|x\|}$ for $x \in \mathbb{T}$. In [BB] we have asked whether the inclusions $\mathcal{B} \subseteq \mathcal{B}_f$, $\mathcal{N} \subseteq \mathcal{N}_f$ hold true. Since $\mathbf{Z}(f) = \{0\}$, by Theorem 1 the equalities $\mathcal{B} = \mathcal{B}_f$ and $\mathcal{N} = \mathcal{N}_f$ hold true.

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