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CONTINUITY OF THE SPECTRUM OF NORM-NORMAL MATRICES

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Let X be a Banach space. We denote by X^d the dual space of X. By an operator on X we always mean a bounded linear operator. We denote by C the set of all complex numbers. Let T be an operator on X. We denote by $V(T)$ the numerical range of T [2], i.e.

$$V(T)=\{f(Tx): x\in X, f\in X^d, |f|=|x|=f(x)=1\}\ .$$

An operator is said to be norm-Hermitian if $V(T)$ is real. An operator is said to be norm-normal if it is of the form $T=H+iK$ with commuting norm-Hermitian H, K [2]. If T is an operator, we denote by $\sigma(T)$ its spectrum and by $|T|_\sigma$ its spectral radius. It is well-known that the equality $|T|=|T|_\sigma$ holds for every normal operator on the Hilbert space. This equality holds also for every norm-Hermitian operator (see [2], [3], [8]), but it need not hold if T is norm-normal. Indeed, Crabb [4] has given a norm-normal operator T for which $|T|=\sqrt{2}|T|_\sigma$. We shall give a simple proof of the fact that the inequality $|A|\leqq 2|A|_\sigma$ holds for every norm-normal complex $n\times n$ matrix. This inequality holds for every normal element of a Banach algebra [5, p. 138]. It can be seen [2, p. 8] that a norm-normal operator T on X is a normal element of the Banach algebra $B(X)$ of all operators on X in the sense of [2, p. 54, definition 13].

Proposition 1. *Let v be a norm on C^n. If A is a v-normal complex $n\times n$ matrix, then*

$$|A|_\sigma\leqq|A|\leqq 2|A|_\sigma$$

($|\cdot|$ *denotes the operator norm induced by* v).

Proof. The first inequality holds for every operator. Let us prove the other. $A=U+iV$ with commuting v-Hermitian U, V. According to [6, prop. 5.11] A is diagonalizable and if

$$A=\lambda_1E_1+\ldots+\lambda_kE_k$$

is its spectral resolution and if $\alpha_i=\operatorname{Re}\lambda_i$, $\beta_i=\operatorname{Im}\lambda_i$, then

$$U=\alpha_1E_1+\ldots+\alpha_kE_k\ ,$$

$$V=\beta_1E_1+\ldots+\beta_kE_k\ .$$

Hence there exists an integer m, $1\leqq m\leqq k$, such that

$$|U|_\sigma=\max\{|\alpha_1|,\ \ldots,\ |\alpha_k|\}=|\alpha_m|\ .$$

Since U is v-Hermitian, $|U|=|U|_\sigma=|\alpha_m|$.
Hence $|A|_\sigma\geqq|\alpha_m+i\beta_m|\geqq|\alpha_m|=|U|$
and similarly $|A|_\sigma\geqq|V|$.
Hence $|A|=|U+iV|\leqq|U|+|V|\leqq 2|A|_\sigma$.

Pták and Zemánek [7] have proved that the spectrum of a normal operator on a Hilbert space, as a set valued function, is Lipschtzian in the Hausdorff metric. We shall prove a similar fact for a norm-normal complex $n\times n$ matrix. Let (E, d) be a metric space. For $x\in E$, $M\subset E$ we define $d(x, M)\ =\ \inf\{d(x, m)\colon m\in M\}$. If $M\subset E$ and r is a positive number, we set $V(m, r)\ =\ \{x\in E\colon d(x, M)\leqq r\}$.

Proposition 2. *Let v be a norm on C^n. Let A and T be complex $n\times n$ matrices. Let A be v-normal. Then there exists a positive number K such that*

$$\sigma(T)\subset V(\sigma(A),\ K|T-A|).$$

Proof. A is diagonalizable. Let $A=\lambda_1E_1+\ldots+\lambda_kE_k$ be its spectral resolution. Let $K=|E_1|+\ldots+|E_k|$. Let λ be a complex number such that

$$d=d(\lambda,\sigma(A))>K|T-A|.$$

It is easy to see that

$$(\lambda-A)^{-1}=(\lambda-\lambda_1)^{-1}E_1+\ldots+(\lambda-\lambda_k)^{-1}E_k\,.$$

Hence $|(\lambda-A)^{-1}|\leqq K/d$. It holds

$$\lambda-T=\lambda-A-(T-A)=(\lambda-A)\,(1-(\lambda-A)^{-1}(T-A))$$

and $|(\lambda-A)^{-1}(T-A)|\ \leqq\ (K/d)\,|T-A|<1$. Hence $(\lambda-T)^{-1}$ exists. This fact completes the proof.

Remark. Professor Pták has informed the author of this paper that the proposition 2 can be obtained by an application of one so far unpublished result due to B. Aupetit [1].

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НЕПРЕРЫВНОСТЬ СПЕКТРА НОРМАЛЬНЫХ В СМЫСЛЕ НОРМЫ МАТРИЦ

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Резюме

В статье определяется понятие эрмитовского и нормального в смысле нормы оператора в пространстве Банаха. В конечномерном случае доказывается, что для каждого нормального в смысле нормы оператора A имеет метсо неравенство $|A|\leqq 2|A|_\sigma$, где $|A|$ — норма и $|A|_\sigma$ — спектральный радиус оператора A. Рассматривая спектр матрицы как функцию принимающую значения в множестве всех подмножеств комплексной плоскости, доказывается непрерывность спектра в каждой нормальной в смысле нормы матрице.