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CRITICALITY CONCEPTS IN GEODETIC GRAPHS

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1. Introduction

A graph G is said to be geodetic if any pair of vertices of G is connected by a unique shortest path. Since a graph is geodetic iff each of its blocks is geodetic (see e.g. [7]), further interest has centred on the study of geodetic blocks. A survey on geodetic graphs was given by Bosák [1] (see also [5]). Criticality concepts of geodetic blocks were studied for the first time by Parthasarathy and Srinivasan in [4].

A geodetic graph G is said to be upper [lower] geodetic critical if $G+e$ [$G-e$] is not a geodetic graph, for any new edge e [edge $e\in E(G)$; respectively].

In [4], conjectures were raised that (1) *all geodetic blocks are upper geodetic critical and* (2) *those different from the cycles C_{2d+1} are lower geodetic critical.* Furthermore in [4], the upper geodetic criticality of geodetic blocks of diameters 2 and 3, and the lower geodetic criticality of the above blocks other than C_5 and C_7 were proved. In the present paper the two above conjectures slightly strengthened are settled in the affirmative for an arbitrary diameter d.

2. Definitions and previous results

We use the general notation and terminology of Harary [2] (the graphs considered are simple and undirected). If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. The *distance* between two vertices $u, v\in V(G)$ is denoted by $\varrho_G(u, v)$. A unique shortest u—v path in G is called a *geodesic* and is denoted by $\Gamma_G[u, v]$. Obviously, any subpath of a geodesic is also a geodesic. If P is a path or cycle, $|P|$ will denote the length of P. Clearly, if $\Gamma_G[u, v]$ exists, then $\varrho_G(u, v)=|\Gamma_G[u, v]|$. The supremum of all distances in G is the *diameter* of G, $d(G)$. A suspended path in G is a nontrivial path P with the property that every internal vertex of P has degree 2 (relative to G), i.e. if $P=[v_1, v_2, \dots, v_k]$, then $\deg_G v_i=2$ whenever $1<i<k$.

Theorem A. (Stemple [8, Theorem 2]) *In a geodetic block different from a cycle, any suspended path is a geodesic.*

If G is an even cycle (i.e. of even length), then we shall say that vertices $u, v \in V(C)$ are *C-opposite* if C can be decomposed into two sections $C[u, v]$ and $C[v, u]$ or length $|C|/2$.

Theorem B. (Stemple and Watkins [7, Theorem 2]) *A graph G is geodetic iff G does not contain any even cycle C such that for some pair of C-opposite vertices u, v $\varrho_G(u, v) = |C|/2$.*

For any $v \in V(G)$ the *i-th neighbourhood* of v is defined by

$$N_i(v) = \{u \in V(G) | \varrho_G(u, v) = i\}.$$

For any vertex $u \in N_i(v)$, a vertex $w \in N_{i-1}(v)$ such that $uw \in E(G)$ is called a *predecessor* of u; and a vertex $t \in N_{i+1}(v)$ such that $ut \in E(G)$ is called a *successor* of u. All these terms are meant relative to a fixed $v \in V(G)$.

Theorem C. (Parthasarathy and Srinivasan [3, Theorem 5]) *A graph G is geodetic iff for every $v \in V(G)$ each vertex of $N_i(v)$ is adjacent to a unique vertex of $N_{i-1}(v)$ for each i with $2 \leq i \leq d$.*

Theorem D. (Srinivasan [6, Theorem 2.7]) *Let G be a geodetic graph. Let $v \in V(G)$, $x, y \in N_i(v)$, and $xy \in E(G)$.* Let $a, b \in N_j(v)$, $j \neq i$, such that $a \neq b$ and $\varrho_G(x, a) = \varrho_G(y, b) = |j-i|$. Then $ab \notin E(G)$.

3. Main results

If P is a suspended path of a graph G, then $G-P$ denotes the graph obtained from G by deleting all internal vertices and all edges of P.

Lemma 1. *Let G be a geodetic block and let P be a maximal suspended path in G. Then G—P is a block.*

Proof. If G is an odd cycle, then $G-P$ is an edge. Therefore we can assume that G is not a cycle. Let P be a u—v path and assume that $G_1 = G - P$ is not a block. Then there exists a cutvertex $x \in V(G_1)$, such that graph G_1 can be edge-decomposed into two nontrivial subgraphs B_1, B_2 which meet only in x, and $u \in V(B_1)$ and $v \in V(B_2)$.

Let us consider a cycle C consisting of the paths $P_1 = \Gamma_{B_1}[u, x]$, $P_2 = \Gamma_{B_2}[x, v]$ and P (see Figure 1). Obviously, C is odd because of theorem B. Since G is not an odd cycle and P is a maximal suspended path, the graphs B_1 and B_2 are different from the paths P_1, P_2, respectively. Without loss of generality assume $|P_1| \leq |P_2|$.

Let Q be a second shortest x—u path in B_1 (i.e. a shortest x—u path but P_1). Note that the paths Q and P_1 can have common sections. From the choice of the path Q it follows that $|Q| - |P_1|$ is odd, because otherwise the set of edges $E(P_1) \cup E(Q)$ contains an even cycle C_1 consisting of the paths $P_1[w_1, w_2]$ and

$Q[w_2, w_1]$ (see Figure 1) and then using Theorem B we have a contradiction to the geodeticity of G (we can always find a pair w_1, w_1' of C_1-opposite vertices).

Now we shall consider a cycle C' consisting of the paths Q, P_2 and P. It is obvious from the above that C' is an even cycle. Let $w \in V(Q)$ and $w \in V(P_1)$ such that $\varrho_G(w_1, w) - \varrho_G(w_2, w) = 0$ or 1 (see Figure 1). From the construction of C' and the choice of $w \in C'$ there follows the existence of a pair w, w' of C'-opposite vertices with $\varrho_G(w, w') = |C'|/2$ and it is a contradiction to the geodeticity of G, because of Theorem B. Q. E. D.

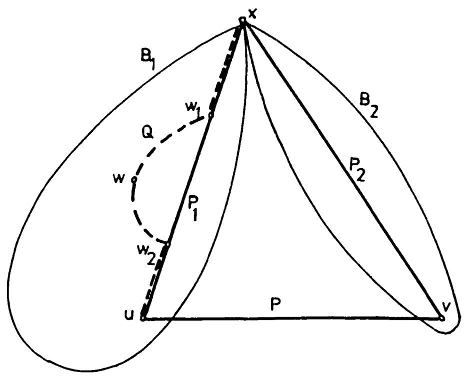

Fig. 1

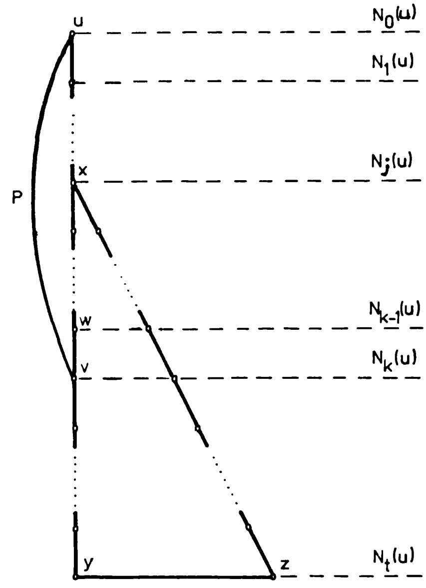

Fig. 2

Theorem 1. *Let G be a geodetic block with at least three vertices and $u, v \in V(G)$. If we join u, v by a new suspended path P, then the graph G_1 obtained in this way is not geodetic.*

Proof. We shall suppose that $\varrho_G(u, v) = k$ and $|P| = n$. Three cases are distinguished.

Case 1. If $n \geq k$, then $\varrho_{G_1}(u, v) = k \leq n$ and G_1 cannot be geodetic because of Theorem A.

Case 2. If $n < k$ and $n - k \equiv 0 \pmod 2$, then an even cycle C consisting of $\Gamma_G[u, v]$ and P contradicts the geodeticity of G_1 by Theorem B.

Case 3. Let $n < k$ and $n - k \equiv 1 \pmod 2$. a) If $k = d$, then there is a vertex $z \in N_d(u)$, $vz \in E(G)$, because G is a block. Then P together with $\Gamma_G[u, z]$ and zv form an even cycle C of length $n + d + 1$. Clearly, from the choice of the vertex

$z \in V(G)$ there follows the existence of a pair z, z' of C-opposite vertices with $\varrho_{G_1}(z, z') = (n + d + 1)/2 = |C|/2$. This contradicts the geodetic nature of G. b) Let $k < d$. Now we shall draw the graph G using the levels $N_0(u)$, $N_1(u)$, ..., $N_k(u)$, ..., $N_d(u)$. Let $t \geq k$ be the smallest number such that there are the vertices $y, z \in N_t(u)$, $yz \in E(G)$ and y is the $(t-k)$-th successor of v, but z is not. If there is for no $t \geq k$ such a pair of vertices, then G cannot be a block (a predecessor $w \in N_{k-1}(u)$ of v is a cutvertex). The same reason ensures paths $\Gamma_G[u, y]$, $\Gamma_G[u, z]$ such that $v \in \Gamma_G[u, y]$, $v \notin \Gamma_G[u, z]$ (see Figure 2) and no edge excepting yz joins a vertex of $\Gamma_G[u, z]$ with a vertex of $\Gamma_G[u, y]$ (see Theorems C and D). These paths can have one common section at most ($\Gamma_G[u, x]$ for $x \in N_j(u)$, $j < k$). Then P together with $\Gamma_G[u, z]$, zy and $\Gamma_G[v, y] \subset \Gamma_G[u, y]$ form an even cycle C of length $2t - k + n + 1$ (see Figure 2). Clearly, from the choice of the vertex $z \in V(G)$ there follows the existence of a pair z, z' of C-opposite vertices with $\varrho_{G_1}(z, z') = (2t - k + n + 1)/2 = |C|/2$. This contradicts the geodetic nature of G. Hence the theorem. Q. E. D.

In particular, if P in Theorem 1 is just an edge we have:

Corollary 1. *Every geodetic block is upper geodetic oritical.*

Theorem 2. *Let G be a geodetic block other than an odd cycle. Let P be a suspended path of G. Then the graph $G - P$ is not geodetic.*

Proof. We distinguish two cases.

Case 1. P is a maximal suspended path. Then the graph $G_1 = G - P$ is a block by Lemma 1. (clearly, G_1 is not an edge). If the block G_1 is geodetic, then adding P to G_1 gives the geodetic block G, which is impossible by Theorem 1.

Case 2. P is a proper subpath of a maximal suspended path Q. Then by the assumption that $G - P$ is a geodetic graph and using Lemma 1 it follows that $G_1' = G - Q$ is a geodetic block. As adding Q to G_1' gives G and it is geodetic, we have a contradiction to Theorem 1. Hence the theorem. Q. E. D.

If P in Theorem 2 is just an edge we have:

Corollary 2. *Every geodetic block other than an add cycle is lower geodetic critical.*

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КРИТИЧЕСКИЕ ПОНЯТИЯ В ГЕОДЕЗИЧЕСКИХ ГРАФАХ

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Резюме

Геодезический граф G называется сверху (снизу) геодезически критическим, если $G+e$ $(G-e)$ уже не является геодезическим графом. Показано, что всякий неориентированный геодезический граф является сверху и снизу геодезически критическим графом.