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INTEGRATION OF REAL FUNCTIONS WITH RESPECT TO A $\oplus$-MEASURE

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ABSTRACT. In the paper, a new definition of the integral with respect to $\oplus$-measures in the case of real functions is suggested, and properties of this integral are studied. The reasons explaining necessity of changing the definition introduced in [5] are given.

1. Introduction

The integral with respect to $\oplus$-measures introduced by Marinová [5] is one of the integrals based on non-additive set functions (see, e.g., [2], [8], [11], [12], [13]). This integral is based on a special type of a pseudo-addition $\oplus$ on $[0,\infty]$, on ordinary multiplication of real numbers, and on $\oplus$-measures. If the operation $\oplus$ is ordinary addition $+$ of real numbers, then the $\oplus$-integral of non-negative measurable functions is the Lebesgue integral. The case $\oplus = \max$ leads to the integral introduced by Shilkret [11].

In [4], the structure of the operation $\oplus$ considered in [5] was explained, and all operations satisfying conditions given in [5] were described. Due to these results, the connection between the $\oplus$-integral of non-negative functions and the Lebesgue integral was discovered (in the case of $\oplus \neq \max$).

The aim of the present paper is to give another definition of the $\oplus$-integral in the case of real functions which would be more appropriate than that of [5]. The reasons for this change will be explained. We are not able to extend the $\oplus$-integral in the case of $\oplus = \max$. As it will be shown, the integral obtained in this case has not satisfactory properties.

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2. Basic notions

Let us recall the basic notions as they were introduced in [5].

Let $(X, \mathcal{S})$ be a measurable space, i.e., let X be an arbitrary non-empty set, and let $\mathcal{S}$ be a σ-algebra of its subsets.

$\oplus$-measure is a set function $m\colon \mathcal{S} \to [0,\infty]$ such that:

(i) $m(\emptyset) = 0$,

(ii) if $\{A_n\}_{n\in\mathrm{N}} \subset \mathcal{S}$, $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup_{n\in\mathrm{N}}\{m(A_1) \oplus \cdots \oplus m(A_n)\},$$

where $\oplus$ is a binary operation defined on $[0,\infty]$ with properties:

(A1) $a \oplus b = b \oplus a$,

(A2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,

(A3) $k \cdot (a \oplus b) = (k \cdot a) \oplus (k \cdot b)$,

(A4) $a \oplus 0 = a$, $a \oplus \infty = \infty$,

(A5) $a \le b \implies a \oplus c \le b \oplus c$,

(A6) $(a+b) \oplus (c+d) \le (a \oplus c) + (b \oplus d)$,

(A7) $a_n \to a$ and $b_n \to b \implies a_n \oplus b_n \to a \oplus b$

for each $a, b, c, d, a_n, b_n \in [0,\infty]$, $n = 1, 2, \dots$, and for each $k > 0$.

Note that $\le$ means usual order of real numbers, and the symbol $\cdot$ in (A3) is used for ordinary multiplication. We will omit it if there can be no confusion. The symbol $+$ in (A6) denotes ordinary addition of real numbers.

The integral with respect to a $\oplus$-measure for non-negative functions was defined in the following way:

[A] If s is a simple non-negative measurable function defined on X, $s = \sum_{i=1}^{n} a_i \cdot \mathbf{1}_{A_i}$ ($a_i \ge 0$, $A_i \in \mathcal{S}$, $A_i \cap A_j = \emptyset$, for $i \neq j$; $i, j = 1, 2, \dots, n$), then

$$\int_X^{\oplus} s \,\mathrm{d}m = a_1 m(A_1) \oplus a_2 m(A_2) \oplus \cdots \oplus a_n m(A_n). \tag{1}$$

[B] If f is a non-negative measurable function defined on X, then

$$\int_X^{\oplus} f \,\mathrm{d}m = \sup\left\{ \int_X^{\oplus} s \,\mathrm{d}m;\ s \le f,\ s \text{ is simple, non-negative} \right\}. \tag{2}$$

A function f is called integrable if $\int_X^{\oplus} f \,\mathrm{d}m < \infty$.

In [4], the structure of $\oplus$ was explained. There was shown that a binary operation $\oplus$ defined on $[0,\infty]$ with the properties (A1) – (A7) is either $\vee$ (max) or the operation of the type $\oplus_r$, where

$$x \oplus_r y = \sqrt[r]{x^r + y^r} \qquad \text{for some} \quad r \geq 1 . \tag{3}$$

Each operation $\oplus_r$, $r \geq 1$, is generated on the whole interval $[0,\infty]$ by any of the functions $g_{r,a}(x) = ax^r$, $a > 0$. It means that

$$x \oplus_r y = g_{r,a}^{-1}\left[g_{r,a}(x) + g_{r,a}(y)\right] .$$

In what follows, we only will use the normed generator $g_{r,1}$, $g_{r,1}(x) = x^r$, which, for brevity's sake, will always be denoted by g.

Note that $\vee$ has no generator. More facts can be found in [4]. There was also proved that, if $\oplus \neq \vee$, the integral of a non-negative function f with respect to a $\oplus$-measure m is given by

$$\int_X^{\oplus} f \, \mathrm{d}m = g^{-1}\left[\int_X (g \circ f) \, \mathrm{d}(g \circ m)\right] , \tag{4}$$

where the integral on the right-hand side is Lebesgue, and g is the normed generator of $\oplus$.

It should be noted that Marinová's $\oplus$-integral, which is based on a binary operation $\oplus$ with properties (A1) – (A7), on ordinary multiplication and a $\oplus$-measure, is a special type of Pap's integral on $[0,\infty]$ ([8]).

The $\oplus$-integral for real functions was defined in [5] as follows:

[C] If $f\colon X \to (-\infty,\infty)$ is a measurable function and at least one of the functions $f^+ = \max(f,0)$, $f^- = \max(-f,0)$ is integrable, then

$$\int_X^{\oplus} f \, \mathrm{d}m = \int_X^{\oplus} f^+ \, \mathrm{d}m - \int_X^{\oplus} f^- \, \mathrm{d}m . \tag{5}$$

A function f is called integrable if $-\infty < \int_X^{\oplus} f \, \mathrm{d}m < \infty$.

It is desired that certain properties of the $\oplus$-integral of non-negative functions remain preserved (or can be generalized) for real functions.

Given a measurable space $(X, \mathcal{S})$ with a $\oplus$-measure m and a non-negative integrable function f, then according to Theorem 2 in [5], a set function ν_f defined on $\mathcal{S}$ by

$$\nu_f(A) = \int_A^{\oplus} f \, \mathrm{d}m , \qquad A \in \mathcal{S} , \tag{6}$$

is a finite $\oplus$-measure on $(X, \mathcal{S})$ (note that $\int\limits_A^{\oplus} f \, dm = \int\limits_X^{\oplus} f \cdot \underline{1}_A \, dm$).

The following two examples show that this property cannot be generalized for real functions if the integral is defined by (5).

EXAMPLE 1. Let $X = [0, \infty]$, $\mathcal{S} = \mathcal{B}(X)$, $m = \sqrt{\lambda}$, where $\mathcal{B}(X)$ is the system of Borel subsets of X, and λ is the Lebesgue measure on $(X, \mathcal{S})$. Then m is a $\oplus_2$-measure, where $\oplus_2$ is the operation defined by (3), i.e.,

$$x \oplus_2 y = \sqrt{x^2 + y^2}\,, \qquad x, y \in [0, \infty]\,.$$

Let $A = [0, 1) \cup [2, 11)$, $B = [1, 2) \cup [11, 27]$, and let $f = 15 \cdot \underline{1}_{[0,2)} - 1 \cdot \underline{1}_{[2,27]}$. Then

$$\int\limits_{A \cup B}^{\oplus_2} f \, dm \neq \int\limits_A^{\oplus_2} f \, dm \oplus_2 \int\limits_B^{\oplus_2} f \, dm\,.$$

Proof. It is clear that $A \cap B = \emptyset$. Let us denote $f_1 = f \cdot \underline{1}_A$, $f_2 = f \cdot \underline{1}_B$. It holds:

$$f_1^+ = 15 \cdot \underline{1}_{[0,1)}, \quad f_1^- = 1 \cdot \underline{1}_{[2,11]} \quad \text{and} \quad f_2^+ = 15 \cdot \underline{1}_{[1,2)}, \quad f_2^- = 1 \cdot \underline{1}_{[11,27]}\,.$$

Therefore, by (5),

$$\int\limits_A^{\oplus_2} f \, dm = \int\limits_X^{\oplus_2} f_1 \, dm = \int\limits_X^{\oplus_2} f_1^+ \, dm - \int\limits_X^{\oplus_2} f_1^- \, dm = 15 \cdot 1 - 1 \cdot \sqrt{9} = 12\,.$$

Analogously,

$$\int\limits_B^{\oplus_2} f \, dm = 15 \cdot 1 - 1 \cdot \sqrt{16} = 11\,.$$

The $\oplus_2$-sum of these integrals is $\int\limits_A^{\oplus_2} f \, dm \oplus_2 \int\limits_B^{\oplus_2} f \, dm = \sqrt{144 + 121} = \sqrt{265}$.

If we compare this number with the value of the integral $\int\limits_{A \cup B}^{\oplus_2} f \, dm$, where $\int\limits_{A \cup B}^{\oplus_2} f \, dm = 15 \cdot \sqrt{2} - 1 \cdot \sqrt{25} = 15 \cdot \sqrt{2} - 5$, we see that

$$\int\limits_{A \cup B}^{\oplus_2} f \, dm \neq \int\limits_A^{\oplus_2} f \, dm \oplus_2 \int\limits_B^{\oplus_2} f \, dm\,.$$

□

EXAMPLE 2. Let again $X = [0,\infty]$, $\mathcal{S} = \mathcal{B}(X)$. Let $m(E) = \sup_{x\in E} x$, $E \in \mathcal{S}$. Then m is a $\vee$-measure on $(X,\mathcal{S})$. Let a function f and sets A, B be as in Example 1. Then

$$\overset{\vee}{\int_A} f\ dm = 15\cdot 1 - 1\cdot 11 = 4\,,\qquad \overset{\vee}{\int_B} f\ dm = 15\cdot 2 - 1\cdot 27 = 3 \qquad \text{and}$$

$$\overset{\vee}{\int_{A\cup B}} f\ dm = 15\cdot 2 - 1\cdot 27 = 3\,.$$

It means that

$$\overset{\vee}{\int_{A\cup B}} f\ dm \neq \overset{\vee}{\int_A} f\ dm \vee \overset{\vee}{\int_B} f\ dm\,.$$

3. New definition of the $\oplus$-integral for real functions

In order to remove shortcomings of the $\oplus$-integral, we have to change its definition for real functions given in [C].

As it was mentioned above, the operation $\oplus$ with properties (A1)–(A7) is either $\vee$ or an operation $\oplus_r$, $r \geq 1$, which is generated on $[0,\infty]$ by the normed generator g, $g(x) = x^r$.

Let $\oplus \neq \vee$. Let us extend the generator g of the operation $\oplus$ into the odd function $\bar{g}$ putting

$$\bar{g}(x) = \begin{cases} g(x) & \text{for } x \in [0,\infty]\,, \\ -g(-x) & \text{for } x \in [-\infty, 0) \end{cases} \tag{7}$$

(or briefly $\bar{g}(x) = \operatorname{sgn} x \cdot g(|x|)$, $x \in [-\infty,\infty]$).

Then we can define a binary operation $\bar{\oplus}$ on the interval $[-\infty,\infty]$ by:

$$x \mathbin{\bar{\oplus}} y = \bar{g}^{-1}\big[\bar{g}(x) + \bar{g}(y)\big]\,. \tag{8}$$

One has $\bar{\oplus}\big|_{[0,\infty]} = \oplus$, and it can easily be shown that the operation $\bar{\oplus}$ is also commutative, associative and continuous. The expression $\infty \mathbin{\bar{\oplus}} (-\infty)$ is not defined.

Using the extended operation $\bar{\oplus}$, a pseudo-subtraction $\ominus$ can be introduced. Let us put

$$x \ominus y = x \mathbin{\bar{\oplus}} (-y) \qquad \text{for all } x, y \in [-\infty,\infty] \tag{9}$$

except expressions $\infty \ominus \infty$ and $(-\infty)\ominus(-\infty)$, which are not defined.

Then, using (8) and (7) we get

$$x \ominus y = \bar{g}^{-1}\big[\bar{g}(x) - \bar{g}(y)\big]\,. \tag{10}$$

Note that this way of extending pseudo-additions was proposed in [7].

Instead of the definition given in [C], we suggest using the next one.

DEFINITION 1. Let $(X, \mathcal{S})$ be a measurable space with a $\oplus$-measure m, $\oplus \neq \vee$, and let $f\colon X \to (-\infty, \infty)$ be a measurable function. Then

$$\int_X^{\oplus} f \;\mathrm{d}m = \int_X^{\oplus} f^+ \;\mathrm{d}m \ominus \int_X^{\oplus} f^- \;\mathrm{d}m \tag{11}$$

if at least one of the functions f^+, f^- is integrable.

PROPOSITION 1. *Let $(X, \mathcal{S})$ be a measurable space with a $\oplus$-measure m, $\oplus \neq \vee$. If $f\colon X \to (-\infty, \infty)$ is a measurable function (for which $\int_X^{\oplus} f\;\mathrm{d}m$ is defined), then*

$$\int_X^{\oplus} f \;\mathrm{d}m = \bar{g}^{-1}\left[\int_X (\bar{g} \circ f)\;\mathrm{d}(\bar{g} \circ m)\right], \tag{12}$$

where $\bar{g}$ is the extension of the normed generator of the operation $\oplus$, and the integral on the right-hand side is Lebesgue.

Proof. To prove this proposition it is enough to use Definition 1, formula (4), the fact that $\bar{\oplus}|_{[0,\infty]} = \oplus$, and additivity of the Lebesgue integral. Concretely:

$$\begin{aligned}
\int_X^{\oplus} f \;\mathrm{d}m &= \int_X^{\oplus} f^+ \;\mathrm{d}m \ominus \int_X^{\oplus} f^- \;\mathrm{d}m \\
&= \bar{g}^{-1}\left[\bar{g}\left(\int_X^{\oplus} f^+ \;\mathrm{d}m\right) - \bar{g}\left(\int_X^{\oplus} f^- \;\mathrm{d}m\right)\right] \\
&= \bar{g}^{-1}\left\{\bar{g}\left[g^{-1}\left(\int_X (g \circ f^+)\;\mathrm{d}(g \circ m)\right)\right] - \bar{g}\left[g^{-1}\left(\int_X (g \circ f^-)\;\mathrm{d}(g \circ m)\right)\right]\right\} \\
&= \bar{g}^{-1}\left\{\int_X (\bar{g} \circ f^+)\;\mathrm{d}(\bar{g} \circ m) - \int_X (\bar{g} \circ f^-)\;\mathrm{d}(\bar{g} \circ m)\right\} \\
&= \bar{g}^{-1}\left[\int_X (\bar{g} \circ f)\;\mathrm{d}(\bar{g} \circ m)\right].
\end{aligned}$$

□

EXAMPLE 3. Let X, $\mathcal{S}$, f be as in Example 1. The operation $\oplus_2$ is generated on the interval $[0,\infty]$ by the normed generator g, $g(x) = x^2$. So the extended generator $\bar{g}$ of $\bar{\oplus}$ is given by $\bar{g}(x) = (\operatorname{sgn} x) \cdot x^2$, $x \in [-\infty, \infty]$. Therefore

$$\bar{g} \circ f = 225 \cdot \underline{1}_{[0,2)} - 1 \cdot \underline{1}_{[2,27]},$$

and so, by (12), we obtain

$$\int_X^{\oplus_2} f \ \mathrm{d}m = \bar{g}^{-1}[225 \cdot 2 - 1 \cdot 25] = \bar{g}^{-1}(425) = \sqrt{425}.$$

In addition, if we consider sets A, B as in Example 1, we obtain

$$\int_A^{\oplus_2} f \ \mathrm{d}m = \sqrt{216}, \qquad \int_B^{\oplus_2} f \ \mathrm{d}m = \sqrt{209} \quad \text{and} \quad \int_{A\cup B}^{\oplus_2} f \ \mathrm{d}m = \sqrt{425}.$$

So, it holds $\int_{A\cup B}^{\oplus_2} f \ \mathrm{d}m = \int_A^{\oplus_2} f \ \mathrm{d}m \ \bar{\oplus}_2 \int_B^{\oplus_2} f \ \mathrm{d}m$.

The last property can be proved generally.

LEMMA 1. *Let $(X,\mathcal{S})$ be a measurable space with a $\oplus$-measure m, $\oplus \neq \vee$. Let $f\colon X \to (-\infty,\infty)$ be an integrable function. Then the function ν_f defined on $\mathcal{S}$ by*

$$\nu_f(A) = \int_A^{\oplus} f \ \mathrm{d}m, \qquad A \in \mathcal{S},$$

where the integral is given by (11), is a $\bar{\oplus}$-additive function on $\mathcal{S}$.

Proof. Since $\oplus \neq \vee$, the operation $\oplus$ is generated by the normed generator g, and for $A, B \in \mathcal{S}$, $A \cap B = \emptyset$, we have

$$\nu_f(A) \mathbin{\bar{\oplus}} \nu_f(B) = \int_A^{\oplus} f \ \mathrm{d}m \mathbin{\bar{\oplus}} \int_B^{\oplus} f \ \mathrm{d}m$$

$$= \bar{g}^{-1}\left[\bar{g}\left(\int_A^{\oplus} f \ \mathrm{d}m\right) + \bar{g}\left(\int_B^{\oplus} f \ \mathrm{d}m\right)\right]$$

$$= \bar{g}^{-1}\left\{\bar{g}\left[\bar{g}^{-1}\left(\int_A (\bar{g}\circ f) \ \mathrm{d}(\bar{g}\circ m)\right)\right] + \bar{g}\left[\bar{g}^{-1}\left(\int_B (\bar{g}\circ f) \ \mathrm{d}(\bar{g}\circ m)\right)\right]\right\}$$

$$= \bar{g}^{-1}\left\{\int_{A\cup B} (\bar{g}\circ f) \ \mathrm{d}(\bar{g}\circ m)\right\} = \int_{A\cup B}^{\oplus} f \ \mathrm{d}m = \nu_f(A\cup B).$$

□

PROPOSITION 2. *Let $(X,\mathcal{S})$ be a measurable space with a $\oplus$-measure m, $\oplus \neq \vee$, and let $f\colon X \to (-\infty,\infty)$ be an integrable function. Then the function ν_f defined in Lemma 1 is a finite σ-$\bar{\oplus}$-additive function on $\mathcal{S}$. If f is non-negative, then ν_f is a $\oplus$-measure.*

P r o o f . By Lemma 1, the set function ν_f is a $\bar{\oplus}$-additive function on $\mathcal{S}$. To prove the σ-$\bar{\oplus}$-additivity of ν_f, it is enough to prove its continuity from bellow.

Let $A_n \in \mathcal{S}$, $n = 1,2,\dots$, and let $A_1 \subset A_2 \subset \cdots \subset A_n \dots$, $A_n \nearrow A$, $A \in \mathcal{S}$. Then, from continuity of $\bar{g}$ and properties of the Lebesgue integral, we get

$$\begin{aligned}\lim_{n\to\infty} \nu_f(A_n) &= \lim_{n\to\infty} \overset{\oplus}{\int\limits_{A_n}} f \,\mathrm{d}m = \lim_{n\to\infty} \bar{g}^{-1}\Bigg[\int\limits_{A_n} (\bar{g}\circ f)\,\mathrm{d}(\bar{g}\circ m)\Bigg] \\ &= \bar{g}^{-1}\Bigg[\lim_{n\to\infty}\int\limits_{A_n} (\bar{g}\circ f)\,\mathrm{d}(\bar{g}\circ m)\Bigg] = \bar{g}^{-1}\Bigg[\int\limits_{A} (\bar{g}\circ f)\,\mathrm{d}(\bar{g}\circ m)\Bigg] \\ &= \bar{g}^{-1}\Bigg[\bar{g}\Bigg(\overset{\oplus}{\int\limits_{A}} f\,\mathrm{d}m\Bigg)\Bigg] = \overset{\oplus}{\int\limits_{A}} f\,\mathrm{d}m\,.\end{aligned}$$

Finitness of the function ν_f follows from integrability of f.

Finally, as $\oplus$ has already been extended on the interval $[-\infty,\infty]$, it makes sense for functions f, $h\colon X \to (-\infty,\infty)$ to put:

$$(f \mathbin{\bar{\oplus}} h)(x) = f(x) \mathbin{\bar{\oplus}} h(x) = \bar{g}^{-1}\big[\bar{g}\big(f(x)\big) + \bar{g}\big(h(x)\big)\big]\,.$$

Then it can be proved (technically in the same way as in Proposition 1 or Lemma 1) that

$$\overset{\oplus}{\int\limits_{X}} (f \mathbin{\bar{\oplus}} h)\,\mathrm{d}m = \overset{\oplus}{\int\limits_{X}} f\,\mathrm{d}m \mathbin{\bar{\oplus}} \overset{\oplus}{\int\limits_{X}} h\,\mathrm{d}m \tag{13}$$

for all functions for which the expressions on both sides make sense.

It means that, in case $\oplus \neq \vee$, the suggested extended integral for real function is $\bar{\oplus}$-additive.

Similarly, we can show that

$$\overset{\oplus}{\int\limits_{X}} cf\,\mathrm{d}m = c\overset{\oplus}{\int\limits_{X}} f\,\mathrm{d}m$$

for each measurable function f, for which the integral is defined, and each constant $c \in (-\infty, \infty)$. It means that the proposed integral is a homogeneous functional. □

Due to the obtained results, we can make the conclusion that, if $\oplus$ differs from $\vee$, Definition 1 is an appropriate definition for the $\oplus$-integral of real-valued functions.

Remark 1. E. Pap [8] has introduced an integral using a pseudo-addition $\oplus$ on the interval $[a,b] \subseteq [-\infty, \infty]$, a pseudo-multiplication $\otimes$, and a $\oplus$-measure m. According to [8], the integral is $\oplus$-additive and $\otimes$-homogeneous.

If $\oplus$ is a pseudo-addition with a strictly increasing generator φ, then the pseudo-multiplication is given by $u \otimes v = \varphi^{-1}\big[\varphi(u) \cdot \varphi(v)\big]$. For a measurable function $f\colon X \to [a,b]$ the integral can be expressed in the form

$$\int_X^{\oplus} f \otimes \mathrm{d}m = \varphi^{-1}\Bigg[\int_X (\varphi \circ f)\ \mathrm{d}(\varphi \circ m)\Bigg],$$

where $\varphi \circ m$ is the Lebesgue measure.

Our $\oplus$-integral for real-valued functions, in case $\oplus \neq \vee$, is based on the pseudo-addition $\bar{\oplus}$ generated on the interval $[-\infty, \infty]$ by the function $\bar{g}$, and on ordinary multiplication of real numbers (and on the $\oplus$-measure), what means that it is of Pap's integral type on $[-\infty, \infty]$.

Any Pap's integral based on a pseudo-addition $\bar{\oplus}$ with a generator φ for which $\varphi|_{[0,\infty]} = g$ (we continue in the above used notation) is a possible extension of Marinová's integral for non-negative functions. But the obtained integral is homogeneous (with respect to ordinary multiplication) only in the case of $\varphi = \bar{g}$.

In fact, let $\bar{\oplus}$ be a pseudo-addition on the interval $[-\infty, \infty]$ with the additive generator φ for which $\varphi|_{[0,\infty]} = g$. Let ordinary multiplication be taken as the pseudo-multiplication $\otimes$. Then $u \otimes v = u \cdot v = \varphi^{-1}\big[\varphi(u) \cdot \varphi(v)\big]$, and the integral is $\cdot$-homogeneous.

Let $a > 0$. Then $\varphi(a) = g(a) > 0$, and

$$a^2 = a \cdot a = \varphi^{-1}\big[\varphi(a) \cdot \varphi(a)\big] \quad \text{or} \quad \varphi(a^2) = \big[\varphi(a)\big]^2.$$

Hence, $\varphi(a) = \sqrt{\varphi(a^2)} = \sqrt{g(a^2)}$. Simultaneously, we have

$$a^2 = (-a) \cdot (-a) = \varphi^{-1}\big[\varphi(-a) \cdot \varphi(-a)\big] \quad \textit{or} \quad \varphi(a^2) = \big[\varphi(-a)\big]^2,$$

what is the same as

$$|\varphi(-a)| = \sqrt{\varphi(a^2)} = \sqrt{g(a^2)}.$$

As the generator φ is strictly increasing, from $-a < 0$, we get $\varphi(-a) < \varphi(0) = 0$.

Hence, $\varphi(a) = \sqrt{g(a^2)}$ and $\varphi(-a) = -\sqrt{g(a^2)}$, $a > 0$.

We have proved $\varphi(-a) = -\varphi(a)$, $a > 0$, what means that φ is an odd function. Since both functions φ and $\bar{g}$ are odd and $\varphi\big|_{[0,\infty]} = g = \bar{g}\big|_{[0,\infty]}$, we have $\varphi = \bar{g}$.

So the extension of Marinová's integral suggested in this paper is the only possible homogeneous extension.

So far we have dealt only with operations $\oplus$ different from $\vee$. The latter has been excluded from our considerations as it has no generator. The question arises how $\vee$ could be extended on the interval $[-\infty,\infty]$. Considering that $\vee$ on $[0,\infty]$ is the limit of the operations $\oplus_r$, $r \geq 1$, i.e.,

$$x \vee y = \lim_{r\to\infty} x \oplus_r y = \lim_{r\to\infty} \sqrt[r]{x^r + y^r}\,,$$

it is natural to suggest extending $\vee$ on the interval $[-\infty,\infty]$ in the same way as the limit of the extended operations $\bar{\oplus}_r$. Using this procedure we get:

$$x \mathbin{\bar{\vee}} y = \lim_{r\to\infty} x \mathbin{\bar{\oplus}_r} y = \operatorname{sgn}(x+y)\cdot\big(|x| \vee |y|\big)\,.$$

If we again put $x \ominus y = x \mathbin{\bar{\vee}} (-y)$, the integral of real measurable functions can again be defined by (11) (in Definition 1).

EXAMPLE 4. Let $X = [0,1]$, $\mathcal{S} = \mathcal{B}(X)$, $m(E) = \sup\limits_{x\in E} x$, $E \in \mathcal{S}$. Then m is a $\vee$-measure on $(X,\mathcal{S})$. Let us consider the functions f, h: $f(x) = x$ and $h(x) = -x^2$, $x \in X$. Then

$$(f \mathbin{\bar{\vee}} h)(x) = \begin{cases} x & \text{for } x \in [0,1)\,, \\ 0 & \text{for } x = 1\,. \end{cases}$$

Using the fact that $\overset{\vee}{\int\limits_E} f \,\mathrm{d}m = \sup\limits_{x\in E} x\cdot f(x)$, (see, e.g., [1]), we get:

$$\overset{\vee}{\int\limits_X} f \,\mathrm{d}m = \sup_{x\in X} x^2 = 1 \quad\text{and}\quad \overset{\vee}{\int\limits_X} h \,\mathrm{d}m = 0 \mathbin{\bar{\vee}} \Bigg(-\overset{\vee}{\int\limits_X} h^- \,\mathrm{d}m\Bigg) = -\sup_{x\in X} x^3 = -1\,,$$

and, analogously, $\overset{\vee}{\int\limits_X} (f \mathbin{\bar{\vee}} h) \,\mathrm{d}m = 1$.

From

$$\overset{\vee}{\int\limits_X} f \,\mathrm{d}m \mathbin{\bar{\vee}} \overset{\vee}{\int\limits_X} h \,\mathrm{d}m = 1 \mathbin{\bar{\vee}} (-1) = 0 \quad\text{and}\quad \overset{\vee}{\int\limits_X} (f \mathbin{\bar{\vee}} h) \,\mathrm{d}m = 1$$

we conclude that $\overset{\vee}{\underset{X}{\int}} (f \mathbin{\bar{\vee}} h)\, \mathrm{d}m \neq \overset{\vee}{\underset{X}{\int}} f\, \mathrm{d}m \mathbin{\bar{\vee}} \overset{\vee}{\underset{X}{\int}} h\, \mathrm{d}m$.

The previous example has shown that the suggested extension of $\vee$ and, consequently, the definition of the integral for real functions are not appropriate.

Different properties of integrals with respect to $\oplus_r$- and $\vee$-measures in the case of real functions are caused by an essential difference between $\oplus_r$ and $\vee$. Their common properties are expressed by axioms (A1) – (A7). As we can see, both types have such important properties as associativity (A2) and continuity (A7). But while all operations $\oplus_r$ are Archimedean, the operation $\vee$ has not this property (a binary operation $\oplus$ on $[0,\infty]$ is said to be Archimedean if for each $x, y \in (0,\infty)$ there exists $n \in \mathbb{N}$ such that $\underbrace{x \oplus \cdots \oplus x}_{n\text{-times}} \geq y$). Contrary to the Archimedean operations $\oplus_r$ which extensions $\bar{\oplus}_r$ remain continuous and associative, the extended operation $\bar{\vee}$ is neither associative nor continuous.

Indeed, if $a \in (0,\infty]$, then

$$(a \mathbin{\bar{\vee}} a) \mathbin{\bar{\vee}} (-a) = 0\,, \qquad \text{but} \qquad a \mathbin{\bar{\vee}} \big[a \mathbin{\bar{\vee}} (-a)\big] = a \mathbin{\bar{\vee}} 0 = a\,,$$

and further, if $0 \leq a_n \nearrow a$, then

$$\lim_{n\to\infty} \big[a \mathbin{\bar{\vee}} (-a_n)\big] = a\,, \qquad \text{but} \qquad a \mathbin{\bar{\vee}} (-a) = 0\,.$$

Loss of continuity and associativity of the operation $\bar{\vee}$ is the reason why it is impossible to introduce for real measurable functions a reasonable integral based on the operation $\bar{\vee}$.

In the remark that follows, we turn briefly to the question of defining pseudo-subtraction.

Remark 2. Weber [13] has introduced a subtraction $\boxminus$ on the interval $[0,1]$ based on a t-conorm $\perp$ (i.e., on a binary operation from the unit square into the interval $[0,1]$ which is commutative, associative, non-decreasing in each argument, and with 0 as a neutral element) in the following way

$$a \boxminus b = \inf\big\{c \in [0,1]\,;\ b \perp c \geq a\big\}\,. \tag{14}$$

The operations $\oplus$ considered in this paper are generalized t-conorms on the interval $[0,\infty]$. We could modify (14) and define the pseudo-subtraction by

$$a \boxminus b = \inf\big\{c \in [0,\infty]\,;\ b \oplus c \geq a\big\}\,. \tag{15}$$

For $0 \leq b < a \leq \infty$ it holds $a \boxminus b = a \ominus b$, but for $a \leq b$ we have $a \boxminus b = 0$, and so this way of defining pseudo-subtraction is not appropriate for us.

But, if we used the extended operation $\bar{\oplus}$ and defined pseudo-subtraction by

$$a \mathbin{\bar{\boxminus}} b = \inf\big\{c \in [-\infty,\infty]\,;\ b \mathbin{\bar{\oplus}} c \geq a\big\}\,,$$

we would come to the same results as by means of $\ominus$ given by $a \ominus b = a \mathbin{\bar{\oplus}} (-b)$. This remark is valid for both types of the operation $\bar{\oplus}$ which have been introduced in this paper, i.e., for $\bar{\oplus}_r$ and also for $\bar{\vee}$.

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