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A REMARK ON ALMOST CONTINUOUS MULTIFUNCTIONS

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The term "almost continuity" is used here in the sense of Husain. The notion of almost continuity of a function was studied by Blumberg, Banach, Pták and by several other authors ([1], [6], [10]). We investigate "upper almost continuity" of multifunctions. In this paper we give a characterization of upper almost continuity. We show that under some assumptions on spaces for each compact-valued multifunction F there is a dense set A in domain, such that F/A is upper semicontinuous.

We introduce some definitions which we shall use. By a multifunction F of X to Y ($F: X \to Y$) we mean a function which to every point $x \in X$ assigns a nonempty subset $F(x)$ of Y. For any $A \subset Y$ we denote $F^-(A) = \{x \in X: F(x) \cap A \neq \emptyset\}$ and $F^+(A) = \{x \in X: F(x) \subset A\}$.

All topological spaces considered in this paper are supposed to be Hausdorff. For a subset A of a topological space X, $\bar{A}$ and $\operatorname{Int} A$ denote the closure or the interior of A respectively.

A multifunction $F: X \to Y$ is called upper (lower) semicontinuous at a point x if for any open set $V \subset Y$ such that $x \in F^+(V)(x \in F^-(V))$ there exists a neighbourhood U of x such that $U \subset F^+(V)(U \subset F^-(V))$.

A multifunction $F: X \to Y$ is upper (lower) almost continuous at a point $x \in X$ if for every open set V in Y, $x \in F^+(V)(F^-(V))$ implies $x \in \operatorname{Int}\overline{F^+(V)}(x \in \operatorname{Int}\overline{F^-(V)})$.

By a graph of a multifunction $F: X \to Y$ we mean the set $\operatorname{Gr} F = \{(x, y): x \in X, y \in F(x)\}$.

If a single-valued function $f: X \to Y$ is given, then it is considered as a multifunction which associates $\{f(x)\}$ to any $x \in X$. Thus f is upper (lower) almost continuous exactly if it is almost continuous in the sense as introduced in [1].

A subset A of a topological space X is called almost open (or nearly open [8]) if $A \subset \operatorname{Int}\bar{A}$ and almost closed if $X \setminus A$ is almost open. If for some $x \in X$ and an almost open set $A \subset X$ we have $x \in A$, we say that A is an almost-neighbourhood of x.

Remark 1. The following properties of almost open sets are evident:

(a) A set $A \subset X$ is almost open if and only if there is an open set U such that $A \subset U$ and A is dense in U.

(b) The intersection of an open set and an almost open set is almost open.

(c) If $A \subset X$ is almost open in X and $B \subset A$ is almost open in A (with the induced topology), then B is almost open in X.

(d) The union of almost open sets is almost open.

The following remark is a trivial exercise. We will frequently use it without a specific reference.

Remark 2. The following conditions on a multifunction $F\colon X \to Y$ are equivalent:

(a) F is upper (lower) almost continuous at $x \in X$;

(b) for any open set $V \subset Y$ such that $x \in F^+(V)(x \in F^-(V))$ there exists an almost neighbourhood G of x such that $G \subset F^+(V)(G \subset F^-(V))$;

(c) for any open set V such that $x \in F^+(V)(x \in F^-(V))$ there exists an open neighbourhood U of x such that $F^+(V)(F^-(V))$ is dense in U.

The proofs of the following two propositions are based only on the topological properties of the domain of multifunctions. We give proofs only for single-valued functions since their generalization for multifunctions is evident.

Proposition 1. Let $f\colon X \to Y$ be a function. Let A be an almost open set and f/A be almost continuous. Then f is almost continuous at every $x \in A$.

Proof. The proof is clear from Remark 1 (c).

For A dense in X and f such that f/A is continuous, Proposition 1 is proved in [1].

If $f\colon X \to Y$ is almost continuous and A is an almost open set, then f/A need not be almost continuous. (See Example 3 in [1]) But the following proposition is true.

Proposition 2. Let $f\colon X \to Y$ be a function. Let $M = G \setminus R$, where G is a nonempty open set in X and R is a nowhere dense set in X. Then f is almost continuous at $x \in M$ if and only if f/M is almost continuous at x.

Proof. Let f/M be almost continuous at $x \in M$. Since R is nowhere dense in X, $G \setminus R$ is almost open. By Proposition 1 f is almost continuous at x.

Now let f be almost continuous at $x \in M$. Let V be an open set in Y such that $f(x) \in V$. There is an open set U in X such that $x \in U$ and $f^{-1}(V)$ is dense in U. Put $H = U \cap M$. H is open in M. We show that $(f/M)^{-1}(V)$ is dense in H. Let H_1 be a nonempty open set in M such that $H_1 \subset H$. Then $H_1 = V_1 \cap M$ for some open set V_1 in X. $V_1 \cap U \cap G$ is a nonempty open set in X. Since R is nowhere dense in X there exists a nonempty open set G_1 in X such that $G_1 \subset V_1 \cap U \cap G$ and $G_1 \subset X \setminus R$. The density $f^{-1}(V)$ in U implies that $f^{-1}(V) \cap G_1 \neq 0$, i.e. $f^{-1}(V) \cap H_1 \neq 0$.

Remark 3. Let $F\colon X \to Y$ be a multifunction. Denote the set of points of upper (lower) almost continuity by $A_U(F)(A_L(F))$. In the paper [2] it is proved that if Y is a second countable space, then for any multifunction $F: X \to Y$, $A_L(F)$ is a complement of a set of the first category. If F is a compact-valued multifunction of X to a second countable space Y, then the same is true for the set $A_U(F)$. Thus if X is a second category space and Y a second countable space, the sets $A_L(F)$, $A_U(F)$ are nonempty and in spaces in which any set of the first category is nowhere dense the restrictions $F/A_L(F)$ and $F/A_U(F)$ are lower or upper almost continuous respectively. (See Proposition 2) But in general the restriction $F/A_{L(F)}$ $(F/A_U(F))$ need not be lower (upper) almost continuous.

Example 1. Let X be the unit interval with the usual topology and Y be the set of real numbers with the usual topology. Let $\{x_n\}$ be a sequence of different real numbers convergent to 2. For any $n \in N$ let A_n be the set of rational numbers in the open interval $(1/(n+1), 1/n)$ and f_n be a bijection from A_n onto the set $\{x_m : m \geqslant n\}$. Define the function f as follows: $f(0) = 2$, $f(x) = f_n(x)$ for $x \in A_n$ and $f(x) = x$ otherwise. It is easy to verify that $A_L(f) = X \setminus \bigcup\limits_{n=1}^{\infty} A_n$ and $f/A_L(f)$ is not almost continuous at 0.

Proposition 3. Let X be a Baire space and Y be a second countable space. Let $F: X \to Y$ be a multifunction. There is a dense set D in $A_L(F)$ such that F/D is lower almost continuous. If F is a compact-valued multifunction, then there exists a dense set T in $A_U(F)$ such that F/T is upper almost continuous.

Proposition 3 is stated here for reference. The case of lower almost continuity is proved in [10] and the proof of upper almost continuity is similar.

The following theorem gives a characterization of upper almost continuity.

Theorem 1. *Let X, Y be topological spaces, $F: X \to Y$, $x \in X$. Let there exist a countable base of neighbourhoods of $F(x)$ and a countable family of closed neighbourhoods of x the intersection of which is the set $\{x\}$. Then F is upper almost continuous at x if and only if there exists an almost neighbourhood A of x such that F/A is upper semicontinuous at x.*

Proof. Let A be an almost-neighbourhood of x such that F/A is upper semicontinuous at x. By Proposition 1, F is upper almost continuous at x.

Now let F be upper almost continuous at x. If $\{x\}$ is open, then the theorem is proved. Suppose $\{x\}$ is not open. Let $\{G_n\}$ be a non-increasing base of open neighbourhoods of $F(x)$ and $\{V_n\}$ be a sequence of closed neighbourhoods of x such that $\bigcap\limits_{n=1}^{\infty} V_n = \{x\}$.

$\{F^+(G_n)\}$ is a non-increasing sequence such that $x \in F^+(G_n)$ and $\overline{F^+(G_n)}$ is a neighbourhood of x for $n = 1, 2, \ldots$. There exist open neighbourhoods U_1, H_1 of x such that $U_1 \subset \overline{F^+(G_1)} \cap V_1$, $H_1 \subset U_1$ and $U_1 \setminus \bar{H}_1 \neq \emptyset$. By induction we can construct sequences $\{U_n\}$, $\{H_n\}$ of open neighbourhoods of x such that for any $n \in N$ $H_n \subset U_n$, $U_n \subset \overline{F^+(G_n)} \cap V_n$, $U_{n+1} \subset H_n$, and $U_n \setminus \bar{H}_n \neq \emptyset$.

Put $A = \bigcup_{n=1}^{\infty} F^+(G_n) \cap (U_n \setminus \bar{U}_{n+1}) \cup \{x\}$. Then A is the searched set.

Notice that if F in Theorem 1 is upper almost continuous at every $x \in X$, then F/A is upper almost continuous.

Let $z \in A \setminus \{x\}$ and U be an open set in Y such that $z \in F^+(U)$. There is $n \in N$ such that $z \in F^+(G_n) \cap (U_n \setminus \bar{U}_{n+1})$. The upper almost continuity of F at z implies that there is an almost-neighbourhood H of z such that $H \subset F^+(G_n \cap U)$, i.e. $H \cap (U_n \setminus \bar{U}_{n+1})$ is a subset of A. By Remark 1 the set $H \cap (U_n \setminus \bar{U}_{n+1})$ is almost open in X and thus in A.

For a single-valued function and X, Y metric spaces, Theorem 1 is proved in [8].

The following examples show that the assumptions in Theorem 1 are essential.

Example 2. Let X be the set of all ordinal numbers less than or equal to ω_1 with the topology $\{\{\lambda \in X : \lambda > \gamma\} : \gamma \in X\} \cup \{X, \emptyset\} \cup \{\{\lambda \in X : \lambda \neq \omega_1, \lambda > \gamma\} : \gamma \in X\}$ and $Y = R$ with the usual topology. Then for any sequence $\{V_n\}$ of neighbourhoods of ω_1 $\bigcap_{n=1}^{\infty} V_n \neq \{\omega_1\}$. If λ is an ordinal number, there are a unique non-negative integer n and a limit number β such that $\lambda = \beta + n$. Define the single-valued function $f: X \to Y$ by $f(\lambda) = 1/n$ if λ is a non-limit ordinal number, $f(\lambda) = 1$ if $\lambda < \omega_1$ is a limit ordinal number and $f(\omega_1) = 0$.

It is easy to verify that f is almost continuous at ω_1.

Suppose that A is an almost-neighbourhood of ω_1 and f/A is continuous at ω_1. For any $n \in N$ there is a neighbourhood U_n of ω_1 such that $f(U_n \cap A) \subset \{y \in Y : y < 1/n\}$. Put $U = \bigcap_{n=1}^{\infty} U_n$. Then U is a neighbourhood of ω_1 and $f(U \cap A) = \{0\}$, hence $U \cap A = \{\omega_1\}$, thus $\omega_1 \notin \operatorname{Int} \bar{A}$, which is a contradiction.

Example 3. Let $\{B_n\}$ be a sequence of mutually disjoint countable dense sets in $[0, 1] \setminus \{1, 1/2, \ldots, 1/n, \ldots\}$. Put $X = \left(\bigcup_{n=1}^{\infty} B_n\right) \cup \{0\}$ with the induced topology. Let Y be the set of real numbers with the usual topology.

Let for every $k \in N$, $\{x_n^k\}_n$ be a sequence of different real numbers in the open interval $(k, k+1)$ convergent to k and $\{f_j^k\}_j$ be a sequence of bijections from $B_j \cap (1/(k+1), 1/k)$ to the set $\{x_n^k : n \geqslant j\}$. Define F by $F(0) = \{1, 2, \ldots, n, \ldots\}$ and $F(x) = \{1, 2, \ldots, (k-1), f_j^k(x), (k+1), \ldots\}$ for $x \in B_j \cap (1/(k+1), 1/k)$.

It is easy to verify that F is upper almost continuous at 0.

Suppose A is an almost-neighbourhood of 0 and F/A is upper semicontinuous at 0. There exists $r \in N$ such that A is dense in $X \cap (0, 1/r)$. For any $l \geqslant r$ choose $x_l \in A \cap (1/(l+1), 1/l)$. Let j_l be such that $x_l \in B_{j_l}$. Put $V = Y \setminus \{f_{j_l}^l(x_l) : l \geqslant r\}$. Then $F^-(V)$ is not a neighbourhood of 0 in A and that is a contradiction.

The following simple example shows that for $F\colon X \to Y$ the lower almost continuity does not imply the existence of an almost-neighbourhood A of x such that F/A is lower continuous at x.

Example 4. Let $X = Y = R$, where R is the set of real numbers with the usual topology. Let F be defined as $F(0) = \{1, 2\}$, $F(x) = \{1\}$ for x rational and $F(x) = \{2\}$ for x irrational. Then F is lower almost continuous at 0 and there is no almost open set A containing 0 for which F/A is lower semicontinuous at 0.

Theorem 2. *Let X be a topological space with a σ-discrete base. Let $F\colon X \to Y$ be upper almost continuous. Let there exist for any $x \in X$ a countable base of neighbourhoods of $F(x)$. Then there exists a dense set D in X such that F/D is upper semicontinuous.*

Proof. Let $\{\mathscr{V}_n : n \in N\}$ be discrete systems of nonempty open sets such that $\mathscr{V} = \cup \{\mathscr{V}_n : n \in N\}$ is a base for X. For any $V \in \mathscr{V}_1$ choose $x_V \in V$ and put $D_1 = \{x_V : V \in \mathscr{V}_1\}$. For any $x_V \in D_1$ denote A_{x_V} an almost-neighbourhood of x_V such that $A_{x_V} \subset V$ and F/A_{x_1} is upper almost continuous and upper semicontinuous at x_V. (See Theorem 1) Put $A_1 = \cup \{A_x : x \in D_1\}$ and $X_1 = (X \setminus \bar{A}_1) \cup A_1$. Then X_1 is dense in X. Since $\mathscr{V}_1$ is a discrete family, F/X_1 is upper almost continuous and upper semicontinuous at every $x \in D_1$.

By induction we will construct sequences $\{D_n\}$, $\{X_n\}$ with the following properties: (a) X_n is a dense subset of X_{n-1}, (b) $D_n \subset X_n$, (c) $D_{n-1} \subset D_n$, (d) for any $V \in \{\mathscr{V}_i : i = 1, 2, \ldots, n\}$ $V \cap D_n \neq \emptyset$, (e) there exists a pairwise disjoint locally finite family of open neighbourhoods of points of D_n, (f) F/X_n is upper almost continuous and upper semicontinuous at every $x \in D_n$.

Suppose $D_1, D_2, \ldots, D_{n-1}, X_1, X_2, \ldots, X_{n-1}$ were constructed. Put $\mathscr{B}_n = \mathscr{V}_n \setminus \{V \in \mathscr{V}_n : V \cap D_{n-1} \neq \emptyset\}$. For any $V \in \mathscr{B}_n$ choose $x_V \in V \cap X_{n-1}$ and put $C_n = \{x_V : V \in \mathscr{B}_n\}$. For any $x \in D_{n-1}$ there exists an open neighbourhood U_x such that $\bar{U}_x \cap C_n = \emptyset$ and such that the family $\{U_x : x \in D_{n-1}\}$ is pairwise disjoint. By assumption there exists a pairwise disjoint locally finite family $\{V_x : x \in D_{n-1}\}$ of open neighbourhoods of points of D_{n-1}. Let $x \in D_{n-1}$. Since $\mathscr{B}_n$ is a discrete family in X there exists an open neighbourhood O of x such that $O \cap V \neq \emptyset$ for at most one member V from $\mathscr{B}_n$. Since X is Hausdorff, there exists an open set O_1 such that $x \in O_1 \subset O$ and $x_V \notin \bar{O}_1$. Put $U_x = V_x \cap O_1$.

For any $x_V \in C_n$ put $U_{x_V} = V \cap (X \setminus \cup \{\bar{U}_x : x \in D_{n-1}\})$. Since $\{U_x\}$ is a locally finite family, we have $\cup \{\bar{U}_x : x \in D_{n-1}\} = \overline{\cup \{U_x : x \in D_{n-1}\}}$ and thus U_{x_V} is an open neighbourhood of every x_V from C_n. Put $D_n = D_{n-1} \cup C_n$. The family $\{U_x : x \in D_n\}$ is pairwise disjoint and locally finite. $D_n \subset X_{n-1}$ and F/X_{n-1} is upper almost continuous. For any $x \in D_n$ denote A_x an almost open set in X_{n-1} such that $x \in A_x$, $A_x \subset U_x$ and F/A_x is upper almost continuous and upper semicontinuous at x.

Put $A_n = \cup \{A_x : x \in D_n\}$ and $X_n = (X_{n-1} \setminus \bar{A}_n) \cup A_n$. It is evident that X_n is dense in X_{n-1} and F/X_n is upper almost continuous and upper semicontinuous at every $x \in D_n$. It follows from the construction that $D = \bigcup_{i=1}^{\infty} D_i$ is dense in X and F/D is upper semicontinuous.

Remark 4. Notice that the set D constructed in the proof of Theorem 2 is an F_σ-set and in spaces without isolated points, D is a set of the first category. The following example shows that this result is the best possible.

Example 5. Let X be the set of real numbers with the usual topology and Y be the set of real numbers with the discrete topology. For any irrational number p put $C_p = p + Q$, where Q is the set of rational numbers. Choose c_p from C_p for any irrational p. ($c_p = c_q$ for any $q \in p + Q$)

Define $f: X \to Y$ as $f(x) = 0$ for $x \in Q$ and $f(x) = c_p$ for $x \in C_p$. It is easy to see that f is almost continuous. If D is a set in X such that f/D is continuous, then D is countable. Suppose that D is uncountable. Then there exists $x \in D$ such that for every neighbourhood V of x, $V \cap D$ is an uncountable set. It is clear that f/D is not continuous at x.

Theorem 3. *Let X be a space with a σ-discrete base and Y be a second countable space with infinitely many points. The following statements are equivalent.*

(1) X is a Baire space,

(2) for every compact-valued multifunction $F: X \to Y$ there is a dense set D in X such that F/D is upper semicontinuous.

Proof. Suppose that X is a Baire space Then the assertion is clear from Remark 3, Proposition 3 and Theorem 2.

Now assume that X is not Baire and choose a nonempty open set U which is of the first category. Let $C_1, C_2, \ldots$ be a sequence of mutually disjoint nowhere dense sets with $\cup\{C_n : n \in N\} = U$. Let L be an infinite discrete subset of Y and let $(c_n : n \geqslant 0)$ be an enumeration of L. Define $f: X \to Y$ by $f(C_n) = c_n$, $n \geqslant 1$ and $f(X \setminus U) = c_0$. There is no set D dense in X for which the restriction f/D is continuous. Suppose that there is a dense set D in X such that the restriction f/D is continuous. Choose $x \in D \cap U$. There is $n \geqslant 1$ such that $f(x) = c_n$. Since L is a discrete set and f/D is continuous at x there is an open neighbourhood V of x in X such that $f(V \cap U \cap D) = c_n$. Thus $C_n = f^{-1}(c_n) \supset V \cap U \cap D$, i.e. $\bar{C}_n \supset V \cap U$ and that is a contradiction since C_n is nowhere dense.

Remark 4. The question is, whether the assumption on X in Theorem 2 is essential?

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ЗАМЕЧАНИЕ К ПОЧТИ НЕПРЕРЫВНЫМ ОТНОШЕНИЯМ

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Резюме

В этой статье изучается почти непрерывность отношений, дана характеристика сверху почти непрерывных отношений.