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AN IDEAL THEORETIC CHARACTERIZATION OF FINITE SETS, FINITE ALGEBRAS, AND σ-ALGEBRAS OF COUNTABLY GENERATED TYPE

DETLEF PLACHKY

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ABSTRACT. It is shown that a non-empty set Ω is finite if and only if any ideal of the set $\mathcal{P}(\Omega)$ of all subsets of Ω is of the type $\{A \subset \Omega : \mu(A) = 0\}$ for some probability charge μ on $\mathcal{P}(\Omega)$. Moreover, it is proved that an algebra $\mathcal{A}$ of subsets of an arbitrary set Ω is finite if and only if any maximal ideal of $\mathcal{A}$ is finitely generated. Finally, it is shown that a σ-algebra $\mathcal{A}$ of subsets of an arbitrary set Ω is of countably generated type, i.e. there does not exist a $\{0,1\}$-valued probability measure P defined on $\mathcal{A}$ vanishing for all atoms of $\mathcal{A}$, if and only if every maximal σ-ideal of $\mathcal{A}$ is countably generated.

1. Introduction and main result

It is well-known that $\mathcal{P}(\Omega)$ and, therefore, also any algebra $\mathcal{A}$ of subsets of Ω is a ring by introducing addition as Δ (symmetric difference) and multiplication as $\cap$ (intersection). An ideal $I \subset \mathcal{A}$ might be described by the property of stability under unions (i.e. $A_j \in I$, $j = 1,2$, implies $A_1 \cup A_2 \in I$) and stability under inclusion (i.e. $B \subset A \in I$, $B \in \mathcal{A}$, yields $B \in I$). Moreover, $\emptyset \in I$ and $\Omega \notin I$ should be valid. In particular, $I(\mathcal{G}) = \Big\{ \bigcup\limits_{k=1}^{n} A_k \cap G_k : A_k \in \mathcal{A},\ G_k \in \mathcal{G},\ k = 1,\dots,n,\ n \in \mathbb{N}\Big\}$ is the ideal generated by $\mathcal{G} \subset \mathcal{A}$ provided $\bigcup\limits_{k=1}^{n} G_k = \Omega$ is impossible for a finite number G_k, $k = 1,\dots,n$, $n \in \mathbb{N}$, of elements of $\mathcal{G}$. Therefore, $I(\mathcal{G}) = \mathcal{A} \cap \Big(\bigcup\limits_{k=1}^{n} G_k\Big)$ holds true in the case $\mathcal{G} = \{G_1,\dots,G_n\} \subset \mathcal{A}$, where $\bigcup\limits_{k=1}^{n} G_k \neq \Omega$ is fulfilled. Hence, a maximal ideal $I \subset \mathcal{A}$ of $\mathcal{A}$ being finitely

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generated, i.e. $I = I(\mathcal{G})$ is valid with $\mathcal{G} = \{G_1, \ldots, G_n\} \subset \mathcal{A}$, $\bigcup_{k=1}^{n} G_k \neq \Omega$, if and only if $I = \mathcal{A} \cap A_I^c$ holds true, where A_I $(\neq \Omega)$ is an atom of $\mathcal{A}$.

Moreover, a maximal ideal $I \subset \mathcal{A}$ of $\mathcal{A}$ might be characterized by $I = \{A \in \mathcal{A} : \mu(A) = 0\}$, where μ is a $\{0,1\}$-valued probability charge on $\mathcal{A}$ (cf. [2; p. 38]), where some non-negative, finitely additive set function $\mu \colon \mathcal{A} \to \mathbb{R}$ with $\mu(\Omega) = 1$ is called in the sequel probability charge. Therefore, the property of $\mathcal{A}$ that any maximal ideal of $\mathcal{A}$ is finitely generated yields for the ideal I_ω defined by $\{A \subset \Omega : \omega \notin A\}$, $\omega \in \Omega$ fixed, the representation $I_\omega = \mathcal{A} \cap A_\omega^c$, where A_ω $(\neq \Omega)$ is an atom of $\mathcal{A}$. Hence $\mathcal{A}$ is in the case under consideration already atomic, i.e. Ω is equal to the union of the atoms of $\mathcal{A}$. Finally, the number of atoms of $\mathcal{A}$ must be finite in the case where all maximal ideals of $\mathcal{A}$ are already finitely generated, since otherwise there would exist some $\{0,1\}$-valued probability charge μ on $\mathcal{A}$ vanishing for all atoms of $\mathcal{A}$ on account of the fact that any $\{0,1\}$-valued probability charge on an algebra of subsets of Ω might be extended to any larger algebra of subsets of Ω as a $\{0,1\}$-valued probability charge (cf. [2; p. 75]). Now the maximal ideal $I \subset \mathcal{A}$ of $\mathcal{A}$ defined by $\{A \in \mathcal{A} : \mu(A) = 0\}$ is of the type $\mathcal{A} \cap A_I^c$ with A_I $(\neq \Omega)$ as an atom of $\mathcal{A}$. Hence one arrives at the contradiction $A_I \notin I$ and $\mu(A_I) = 0$.

Obviously, any ideal of $\mathcal{A}$ is finitely generated in the case where $\mathcal{A}$ is already finite. Therefore, the first part of the following result has been proved:

THEOREM 1. *Let $\mathcal{A}$ stand for an algebra of subsets of a non-empty set Ω. Then $\mathcal{A}$ is finite if and only if any maximal ideal of $\mathcal{A}$ is finitely generated. Moreover, Ω is finite if and only if any ideal of $\mathcal{P}(\Omega)$ is of the type $\{A \subset \Omega : \mu(A) = 0\}$ for some probability charge μ on $\mathcal{P}(\Omega)$.*

Proof. It remains to show that the property of Ω to be finite is equivalent to the property of $\mathcal{P}(\Omega)$ that any ideal of $\mathcal{P}(\Omega)$ is of the type $\{A \subset \Omega : \mu(A) = 0\}$ for some probability charge μ on $\mathcal{P}(\Omega)$. Obviously, any ideal of $\mathcal{P}(\Omega)$ is of the type $\{A \subset \Omega : \mu(A) = 0\}$ for some probability charge μ on $\mathcal{P}(\Omega)$ in the case where Ω is already finite, since any ideal is then a principal ideal, i.e. of the type $\mathcal{P}(\Omega) \cap A$ for some $A \subset \Omega$, $A \neq \Omega$.

In the case where any ideal of $\mathcal{P}(\Omega)$ is of the type $\{A \subset \Omega : \mu(A) = 0\}$ the underlying set Ω must be countable. Otherwise, according to [5; p. 45], there would exist pairwise disjoint subsets Ω_i of Ω, $i \in I$, satisfying $\mathrm{card}(\Omega_i) = \mathrm{card}(\Omega)$, $i \in I$, and $\mathrm{card}(I) = \mathrm{card}(\Omega)$. Therefore, the ideal I of $\mathcal{P}(\Omega)$ defined by $\{A \subset \Omega : \mathrm{card}(A) \leq \mathrm{card}(\mathbb{N})\}$ is not of the type $\{A \subset \Omega : \mu(A) = 0\}$ for some probability charge μ on $\mathcal{P}(\Omega)$, since otherwise $\mu(A_i) > 0$, $i \in I$, with $\mathrm{card}(I) > \mathrm{card}(\mathbb{N})$ would be valid, which is impossible.

Now it is shown that the ideal of $\mathcal{P}(\mathbb{N})$ defined by $\{A \subset \mathbb{N} : A \text{ finite}\}$ is not of the type $\{A \subset \mathbb{N} : \mu(A) = 0\}$ for some probability charge μ on $\mathcal{P}(\mathbb{N})$, which proves that Ω must be already finite if any ideal of $\mathcal{P}(\Omega)$ is of the type $\{A \subset \Omega : \mu(A) = 0\}$ for some probability charge μ on $\mathcal{P}(\Omega)$.

For the proof that there does not exist any probability charge μ on $\mathcal{P}(\mathbb{N})$ satisfying $\mu(\{n\}) = 0$, $n \in \mathbb{N}$, and $\mu(A) > 0$ for all infinite subsets A of $\mathbb{N}$, one starts from a system of subsets A_i of $\mathbb{N}$, $i \in I$, such that $A_i \cap A_j$ is finite for $i, j \in I$, $i \neq j$, and being maximal with respect to inclusion among all systems of subsets of $\mathbb{N}$ of the type which has just been introduced. The lemma of Zorn yields the existence of such a maximal system of subsets where the property of maximality is equivalent to the effect that for any infinite subset B of $\mathbb{N}$ satisfying $B \notin \{A_i : i \in I\}$ there exists some $i_0 \in I$ such that $B \cap A_{i_0}$ is infinite. Moreover, one might assume in the sequel that I is not finite, since one might start from a system of infinite subsets A_i^0 of $\mathbb{N}$, $i = 1, 2, \ldots$, being pairwise disjoint and might consider a maximal system $\{A_i : i \in I\}$ of infinite subsets of $\mathbb{N}$ with the property that $A_i \cap A_j$ is finite for $i, j \in I$, $i \neq j$, containing already $\{A_i^0 : i = 1, 2, \ldots\}$. Here A_i^0 might be chosen as $\{2^n p_i : n = 1, 2, \ldots\}$, $i = 1, 2, \ldots$, where $p_1 < p_2 < \ldots$ denotes all prime numbers. It will now be shown that the corresponding set I is not countable. Otherwise, $\operatorname{card}(I) = \operatorname{card}(\mathbb{N})$ admits the introduction of pairwise disjoint subsets A_i' defined by $A_i \cap (A_i \cup \cdots \cup A_{i-1})^c$, $i = 1, 2, \ldots$, (A_0 stands for $\mathbb{N}$). Now one arrives by $A_i = A_i' \cup (A_i \cap (A_1 \cup \cdots \cup A_{i-1}))$, $i = 1, 2, \ldots$, that A_i' are infinite subsets of Ω, $i = 1, 2, \ldots$, since $A_i \cap (A_1 \cup \cdots \cup A_{i-1})$, $i = 2, 3, \ldots$, is finite. Moreover, there does not exist any infinite subset B of Ω satisfying $B \notin \{A_i' : i \in i\}$ and $B \cap A_i'$ being finite for all $i \in I$ on account of the maximality of $\{A_i : i \in I\}$. In the case $B \notin \{A_i : i \in I\}$ there exists some $i_0 \in I$ such that $B \cap A_{i_0}$ is infinite. Therefore, $B \cap A_{i_0} = (B \cap A_{i_0}') \cup (B \cap A_{i_0} \cap (A_1 \cup \cdots \cup A_{i_0 - 1}))$ yields that $B \cap A_{i_0}'$ is infinite, too. Moreover, the case $B = A_{k_0}$ for some $k_0 \in I$ results in $B \cap A_{k_0}' = A_{k_0}'$, where A_{k_0}' is infinite. However, in the case $\operatorname{card}(I) = \operatorname{card}(\mathbb{N})$ there exists obviously some infinite subset B of $\mathbb{N}$ satisfying $B \notin \{A_i' : i \in I\}$ such that $B \cap A_i'$ is finite for all $i \in I$. One might choose for any $i \in I$ some $n_i \in A_i$ and introduce B as $\{n_i : i \in I\}$. Hence, $\{A_i : i \in I\}$ is an uncountable system of infinite subsets A_i of $\mathbb{N}$ such that $A_i \cap A_j$ is finite for $i, j \in I$, $i \neq j$. In particular, it has been shown that any maximal system of infinite subsets A_i of $\mathbb{N}$, $i \in \mathbb{N}$, being almost disjoint, i.e. $A_i \cap A_j$ is finite, $i, j \in I$, $i \neq j$, is finite or uncountable (cf. [5; p. 242, Lemma 23.9], concerning an explicit description of an uncountable system of subsets of $\mathbb{N}$ being almost disjoint). Now the equation $\mu\Big(\bigcup_{j=1}^{n} B_j\Big) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le j_1 < \cdots < j_k \le n} \mu(B_{j_1} \cap \cdots \cap B_{j_k})$ for all $B_j \subset \Omega$, $j = 1, \ldots, n$, and any probability charge μ on $\mathcal{P}(\Omega)$ (cf. [2; p. 36]) implies for some probability charge μ on $\mathcal{P}(\mathbb{N})$ satisfying $\mu(\{n\}) = 0$ and $\mu(A) > 0$ for all infinite subsets A of $\mathbb{N}$, that $\mu\Big(\bigcup_{i \in J} A_i\Big) = \sum_{i \in J} \mu(A_i)$ is valid for any finite subset J of I, where A_i, $i \in I$, are infinite subsets of $\mathbb{N}$ being almost disjoint with $\operatorname{card}(I) > \operatorname{card}(\mathbb{N})$. In particular, $\mu(\mathbb{N}) = 1$ yields that there exists only finitely many sets $A_i \in \{A_i : i \in I\}$ satisfying $\frac{1}{n+1} < \mu(A_i) \leq \frac{1}{n}$ for some fixed $n \in \mathbb{N}$. Thus one arrives at the contradiction that I is countable. □

Remark (Related topics of the theory of Boolean algebras). M. Erné has kindly drawn my attention to the following facts related to the first part of Theorem 1, which might be found in [3] on:

page 181: Show that every Noetherian Boolean lattice is finite (Exercise 5);
page 184: A lattice L is Noetherian if and only if every ideal of L is principal;
page 184: Let A be an algebra whose congruence relations form a Noetherian distributive lattice. Then A has a unique representation as irredundant finite subdirected product of subdirectly irreducible factors (Theorem 5).

However, this references do not meet the case of maximal ideals. This is the case concerning [1; p. 199, Theorem 9]: Let A be an infinite Boolean algebra. Then there exists a prime ideal P of A which is in its own right a Boolean ring without unit.

Taking into consideration that a finitely generated algebra $\mathcal{A}$ of subsets of a non-empty set Ω is already finite, one might expect that a countably generated σ-algebra $\mathcal{A}$ of subsets of Ω might be characterized by the property that any maximal ideal of $\mathcal{A}$ being a σ-ideal ("maximal σ-ideal" for brevity) is countably generated. Here an ideal I of the σ-algebra $\mathcal{A}$ is called *σ-ideal* if I is stable with respect to unions of countably many sets belonging to I. The notion of a *countably generated σ-ideal* I of a σ-algebra $\mathcal{A}$ of subsets of a non-empty set Ω might be introduced as follows: Let $\mathcal{G} \subset \mathcal{A}$ satisfy $\bigcup\limits_{k=1}^{\infty} G_k \neq \Omega$ for any countable subset $\{G_1, G_2, \dots\}$ of $\mathcal{G}$. Then the σ-ideal $I(\mathcal{G})$ of the σ-algebra $\mathcal{A}$ generated by $\mathcal{G}$ is defined by $\left\{ \bigcup\limits_{k=1}^{\infty} A_k \cap G_k : A_k \in \mathcal{A},\ G_k \in \mathcal{G},\ k = 1, 2, \dots \right\}$ and a σ-ideal I of the σ-algebra $\mathcal{A}$ is called *countably generated* if $I = I(\mathcal{G})$ for some countable subset $\mathcal{G} = \{G_1, G_2, \dots\}$ of $\mathcal{A}$ is valid, i.e. $I = \mathcal{A} \cap \left(\bigcup\limits_{k=1}^{\infty} G_k \right)$ holds true with $\bigcup\limits_{k=1}^{\infty} G_k \neq \Omega$.

It might be surprising that the preceding ideal theoretic property of a σ-algebra $\mathcal{A}$ of subsets of $\Omega \neq \emptyset$ is characterized by the property of $\mathcal{A}$ to be of countably generated type, i.e. there does not exist any probability measure P on $\mathcal{A}$ being $\{0, 1\}$-valued and vanishing for all atoms of $\mathcal{A}$.

THEOREM 2. *Let $\mathcal{A}$ stand for a σ-algebra of subsets of a non-empty set Ω. Then $\mathcal{A}$ is of countably generated type if and only if every maximal σ-ideal of $\mathcal{A}$ is countably generated.*

Proof. At first it will be shown that a maximal σ-ideal of $\mathcal{A}$ being countably generated might be characterized by the property to be of the type $\{A \in \mathcal{A} : P(A) = 0\}$, where P stands for some $\{0,1\}$-valued probability measure on $\mathcal{A}$ satisfying $P(A_0) = 1$ for some atom $A_0 \in \mathcal{A}$ of $\mathcal{A}$ with $A_0 \neq \Omega$. In the case where I is a maximal σ-ideal of $\mathcal{A}$ being countably generated, the property of I to be a maximal σ-ideal of $\mathcal{A}$ implies $I = \{A \in \mathcal{A} : P(A) = 0\}$ for some $\{0,1\}$-valued probability measure P on $\mathcal{A}$. Moreover, one arrives by the property of I to be countably generated at $I = \mathcal{A} \cap A_0^c$ with $A_0 \in \mathcal{A}$ being some atom $A_0 \in \mathcal{A}$ satisfying $A_0 \neq \Omega$, i.e. $P(A_0) = 1$ is valid. For the proof of the converse implication one might start from some ideal I of $\mathcal{A}$ being of the type $\{A \in \mathcal{A} : P(A) = 0\}$ for some $\{0,1\}$-valued probability measure P on $\mathcal{A}$ satisfying $P(A_0) = 1$ for some atom $A_0 \in \mathcal{A}$ of $\mathcal{A}$ with $A_0 \neq \Omega$. Hence $I = \mathcal{A} \cap A_0^c$ holds true, i.e. I is even a principal ideal on account of $I = I(\{A_0^c\})$. Now everything is prepared for the proof of Theorem 2, since in the case where any maximal σ-ideal is countably generated, one arrives by means of the maximal σ-ideal defined by $\{A \in \mathcal{A} : P(A) = 0\}$, where P is some $\{0,1\}$-valued probability measure on $\mathcal{A}$ at $\{A \in \mathcal{A} : P(A) = 0\} = \mathcal{A} \cap A_0^c$ for some atom $A_0 \neq \Omega$ of $\mathcal{A}$, from which $P(A_0) = 1$ follows. For the proof of the converse implication one concludes that any maximal σ-ideal I of $\mathcal{A}$ is countably generated if every $\{0,1\}$-valued probability measure is already concentrated with probability one on some atom of $\mathcal{A}$ different from Ω. This follows from $I = \{A \in \mathcal{A} : P(A) = 0\}$ for some $\{0,1\}$-valued probability measure P on $\mathcal{A}$, i.e. $P(A_0) = 1$ for some atom $A_0 \neq \Omega$ of $\mathcal{A}$ is valid. Therefore, $I = \mathcal{A} \cap A_0^c$, i.e. $I = I(\{A_0^c\})$ holds true. □

Remark (Generalization of Theorem 1 and Theorem 2). The preceding Theorem 2 might be extended as follows: A system $\mathcal{A}$ of subsets of a set $\Omega \neq \emptyset$ will be called *κ-algebra*, where κ denotes some cardinal number, if $\Omega \in \mathcal{A}$ is valid and $\mathcal{A}$ is stable with respect to complements and unions of sets $A_k \in \mathcal{A}$, $k \in K$, of the type $\bigcup_{k \in K} A_k$, $\operatorname{card}(K) \leq \kappa$. Moreover, an ideal $I \subset \mathcal{A}$ of $\mathcal{A}$ being stable under unions $\bigcup_{k \in K} A_k$, $A_k \in I$, $k \in K$, $\operatorname{card}(K) \leq \kappa$, will be called *$\kappa$-ideal*, and a maximal ideal of $\mathcal{A}$ being also a κ-ideal will be called *maximal κ-ideal* for brevity. Finally, a set function $P \colon \mathcal{A} \to \mathbb{R}$ with $\mathcal{A}$ as a κ-algebra satisfying $P(A) \geq 0$, $A \in \mathcal{A}$, $P(\Omega) = 1$, and being κ-additive, i.e. $P\Big(\bigcup_{k \in K} A_k\Big) = \sum_{k \in K} P(A_k)$ is valid for pairwise disjoint sets $A_k \in \mathcal{A}$, $k \in K$, $\operatorname{card}(K) \leq \kappa$, will be called *$\kappa$-additive probability measure*. Here $\sum_{k \in K} P(A_k)$ is defined by $\sup_{F \in \mathcal{K}} \sum_{k \in F} P(A_k)$, where $\mathcal{K}$ stands for all finite subsets of K. In particular, $P\Big(\bigcup_{k \in K} A_k\Big) = 0$ is valid in the case $A_k \in \mathcal{A}$, $P(A_k) = 0$, $k \in K$,

$\operatorname{card}(K) \leq \kappa$, if P denotes some *κ-additive probability measure* defined on the κ-algebra $\mathcal{A}$. Moreover, one arrives in the case of a finite cardinal number $\kappa \geq 2$ by means of κ-algebras and κ-probability measures at algebras and probability charges, respectively.

In the case where $\mathcal{A}$ is a κ-algebra of subsets of a set $\Omega \neq \emptyset$, the ideal $I(\mathcal{G})$ generated by $\mathcal{G} \subset \mathcal{A}$ satisfying $\bigcup_{k \in K} G_k \neq \Omega$ for any collection $G_k \in \mathcal{G}$, $k \in K$, $\operatorname{card}(K) \leq \kappa$, is defined by $\Big\{ \bigcup_{k \in K} A_k \cap G_k : \ A_k \in \mathcal{A}, \ G_k \in G, \ k \in K, \operatorname{card}(K) = \kappa \Big\}$. In particular, an ideal $I \subset \mathcal{A}$ of the κ-algebra $\mathcal{A}$ is called *κ-generated* if $I = I(\mathcal{G})$ holds true for some $\mathcal{G} = \{G_k \in A : \ k \in K\}$, $\operatorname{card}(K) = \kappa$, satisfying $\bigcup_{j \in J} G_j \neq \Omega$, $G_j \in \mathcal{G}$, $j \in J$, $J \subset K$. Now the proof of the preceding Theorem 2 yields in the case where $\kappa \geq 2$ is some cardinal number that any maximal κ-ideal of the κ-algebra is κ-generated if and only if any κ-probability measure on $\mathcal{A}$ being $\{0,1\}$-valued is equal to 1 for some atom of $\mathcal{A}$. In particular, the special case of a finite cardinal number $\kappa \geq 2$ yields the preceding Theorem 1, since an $\mathcal{A}$ algebra of subsets of a set Ω is finite if and only if any probability charge on $\mathcal{A}$ being $\{0,1\}$-valued is equal to 1 for some atom of $\mathcal{A}$. This might be seen by the proof of the preceding Theorem 1 yielding in the case where $\mathcal{A}$ has the property that every $\{0,1\}$-valued probability charge on $\mathcal{A}$ is already equal to 1 for some atom of $\mathcal{A}$ together with the assumption that $\mathcal{A}$ is infinite the contradiction that there exists some $\{0,1\}$-valued probability charge on $\mathcal{A}$ vanishing for all atoms of $\mathcal{A}$.

2. Examples and applications

The notion of a σ-algebra of countably generated type occurring in connection with the preceding Theorem 2 might be illustrated by the following example.

EXAMPLE (COUNTABLY GENERATED σ-ALGEBRAS). Every countably generated σ-algebra $\mathcal{A}$ of subsets of a set Ω is of countably generated type, since the atoms of $\mathcal{A}$ might easily be described by the non-empty sets of the type $\bigcap_{i=1}^{\infty} A_i$ with $A_i \in \{G_i, G_i^c\}$, where $G_i \in \mathcal{A}$, $i = 1, 2, \ldots$, generate $\mathcal{A}$. In particular, any $\{0,1\}$-valued probability measure P on $\mathcal{A}$ has the property $P(A_0) = 1$ for some atom $A_0 \in \mathcal{A}$ of $\mathcal{A}$, namely $A_0 = \bigcap_{i=1}^{\infty} A_i$, where A_i is equal to G_i or G_i^c in the case $P(G_i) = 1$ or $P(G_i^c) = 1$ respectively, $i = 1, 2, \ldots$. However, the special case where $\mathcal{A}$ is equal to the set $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ shows that $\mathcal{A}$ is of countably generated type without being already countably generated. This follows from the observation that any $\{0,1\}$-valued probability measure P

on $\mathcal{P}(\mathbb{R})$ is equal to some one-point mass at some $x \in \mathbb{R}$, since the restriction $P|_{\mathcal{B}(\mathbb{R})}$ with $\mathcal{B}(\mathbb{R})$ as the Borel σ-algebra of $\mathbb{R}$ is of this type according to the preceding considerations. Moreover, any countably generated σ-algebra $\mathcal{A}$ has the property $\operatorname{card}(\mathcal{A}) \leq \operatorname{card}(\mathbb{R})$ (cf. [4; p. 26]), i.e. $\mathcal{P}(\mathbb{R})$ is not countably generated.

One should point out that it cannot be decided in ZFC whether $\mathcal{P}(\Omega)$ with $\operatorname{card}(\Omega) = \aleph_1$ is not countably generated, since there are models of ZFC such that the continuum hypothesis respectively Martin's axiom together with the negation of the continuum hypothesis holds true (see [5; p. 232]). In the first mentioned case, $\mathcal{P}(\Omega)$ is not countably generated according to the same argument yielding that $\mathcal{P}(\mathbb{R})$ is not countably generated (cf. [4; p. 26]). In the model of ZFC where Martin's axiom together with the negation of the continuum hypothesis holds true, $\mathcal{P}(\Omega)$ with $\operatorname{card}(\Omega) = \aleph_1$ does not have a minimal generator (cf. [7; p. 39]), which implies that $\mathcal{P}(\Omega)$ with $\operatorname{card}(\Omega) = \aleph_1$ is countably generated (cf. [7; p. 37, Proposition 54]). The interest in the question whether $\mathcal{P}(\Omega)$ with $\operatorname{card}(\Omega) = \aleph_1$ is not countably generated is justified by the equivalence to the hypothesis that any metric space with countably generated Borel σ-algebra is separable (cf. [6]), which cannot be proved in ZFC by the preceding considerations.

Theorem 2 might also be illustrated by the following example:

EXAMPLE (ULAM MEASURABLE CARDINALS). A set Ω is called *Ulam measurable* if there exists some $\{0,1\}$-valued probability measure P on the set $\mathcal{P}(\Omega)$ of all subsets of Ω such that $P(\{\omega\}) = 0$ for all $\omega \in \Omega$ is fulfilled. According to the preceding theorem the property of Ω to be Ulam measurable is equivalent to the property of Ω that there exists some maximal σ-ideal of $\mathcal{P}(\Omega)$ being not countably generated.

In the sequel it will be shown that the property of a σ-algebra to be of countably generated type is preserved under the operations concerning traces and direct products. However, the property of a σ-algebra to be of countably generated type is not preserved under inclusion since the σ-algebra $\mathcal{A}$ of subsets of a set Ω with $\operatorname{card}(\Omega) = \aleph_1$, generated by the singletons $\{\omega\}$, $\omega \in \Omega$, i.e. $\mathcal{A} = \{A \subset \Omega : A \text{ or } \Omega \setminus A \text{ is countable}\}$ is not of countably generated type, whereas $\mathcal{P}(\Omega)$ is of countably generated type according to a theorem of Ulam (cf. [5; p. 303, Lemma 27.7]). The preceding example about countably generated σ-algebras tells that one might replace Ω by $\mathbb{R}$.

APPLICATION (CONSERVATION WITH RESPECT TO TRACES OF σ-ALGEBRAS). Let $\mathcal{A}$ denote a σ-algebra of subsets of a set $\Omega \neq \emptyset$ and let A stand for some non-empty subset of Ω. Then the trace $\mathcal{A} \cap A$ of $\mathcal{A}$ with respect to A is of countably generated type if $\mathcal{A}$ is already of countably generated type. This follows from the observation that one arrives by some $\{0,1\}$-valued probability measure P on $\mathcal{A} \cap A_0$ according to P_0 defined by $P_0(A) = P(A \cap A_0)$, $A \in \mathcal{A}$, at some $\{0,1\}$-valued probability measure P_0 on $\mathcal{A}$. Therefore, $P_0(A) = 1$ is valid for some atom $A \in \mathcal{A}$ of $\mathcal{A}$. Moreover, $A \cap A_0$ is an atom of $\mathcal{A} \cap A_0$, since $B \cap A_0 \subset A \cap A_0$ for some $B \in \mathcal{A}$ satisfying $B \cap A_0 \neq \emptyset$ implies $A \subset B$. The other case $B \subset A^c$ cannot occur on account of $B \cap A_0 \subset A \cap A_0$ and $B \cap A_0 \neq \emptyset$. Finally, $A \subset B$ yields $B \cap A_0 = A \cap A_0$. Hence $A \cap A_0$ is an atom of $\mathcal{A} \cap A_0$. A similar consideration shows that the σ-algebra $\sigma(\mathcal{A} \cup \{A_0\})$ generated by $\mathcal{A} \cup \{A_0\}$ is of countably generated type if $\mathcal{A}$ is of countably generated type. For the proof one might start from some $\{0,1\}$-valued probability measure P' on $\sigma(\mathcal{A} \cup \{A_0\})$, where $\mathcal{A}$ is of countably generated type. Then P introduced as the restriction $P'|_{\mathcal{A}}$ of P' to $\mathcal{A}$ is some $\{0,1\}$-valued probability measure on $\mathcal{A}$ satisfying $P(A_1) = 1$ for some atom $A_1 \in \mathcal{A}$ of $\mathcal{A}$. Therefore, $A_1 = (A_1 \cap A_0) \cup (A_1 \cap A_0^c)$ leads to $P'(A_1 \cap A_0) = 1$ or $P'(A_1 \cap A_0^c) = 1$, where $A_1 \cap A_0$ or $A_2 \cap A_0$ is an atom of $\sigma(\mathcal{A} \cup \{A_0\})$ according to the preceding considerations. In particular, for every σ-algebra $\mathcal{A}$ of subsets of a set $\Omega \neq \emptyset$ being not of countably generated type there exists a system $\mathcal{A}_i$, $i \in I$, of sub-σ-algebras of $\mathcal{A}$ being of countably generated type such that the σ-algebra $\mathcal{S}\Big(\bigcup_{i \in I} \mathcal{A}_i\Big)$ is not of countably generated type and where for any pair $\mathcal{A}_i, \mathcal{A}_j$, $i,j \in I$, $i \neq j$, the inclusion $\mathcal{A}_i \subset \mathcal{A}_j$ or $\mathcal{A}_j \subset \mathcal{A}_i$ is valid. Otherwise, according to the lemma of Zorn, there would exist some maximal sub-σ-algebra of $\mathcal{A}$ of countably generated type, which is impossible according to the preceding result.

The converse implication that the property of $\sigma(\mathcal{A} \cup \{A_0\})$ to be of countably generated type might be carried over to the σ-algebra $\mathcal{A}$ of subsets of a set $\Omega \neq \emptyset$, where A_0 is some subset of Ω, follows from the observation that the trace of $\sigma(\mathcal{A} \cup \{A_0\})$ with respect to A_0 and A_0^c coincides with $\mathcal{A} \cap A_0$ and $\mathcal{A} \cap A_0^c$, respectively. Moreover, it will be shown that $\mathcal{A}$ is already of countably generated type if $\mathcal{A} \cap A_0$ and $\mathcal{A} \cap A_0^c$ have this property, which shows that $\mathcal{A}$ is of countably generated type if $\sigma(\mathcal{A} \cup \{A_0\})$ is of countably generated type. In the case where $\mathcal{A} \cap A_0$ and $\mathcal{A} \cap A_0^c$ are of countably generated type, let P stand for some $\{0,1\}$-valued probability measure on $\mathcal{A}$. Then $P^*(A_0) = 1$ or $P^*(A_0^c) = 1$ is valid, where P^* denotes the outer measure of P. In particular, $P^*(A_0^c) = 0$ implies $P_*(A_0) = 1$ and $P^*(A_0) = 0$ yields $P_*(A_0^c) = 1$ with P_* being the inner measure of P. Moreover, one arrives by the property of $\mathcal{A} \cap A_0$ and $\mathcal{A} \cap A_0^c$ to be of countably generated type at atoms $A_1 \cap A_0 \in \mathcal{A} \cap A_0$ of $\mathcal{A} \cap A_0$ and $A_2 \cap A_0^c \in \mathcal{A} \cap A_0$ of $\mathcal{A} \cap A_0^c$ with $A_j \in \mathcal{A}$, $j = 1,2$, satisfying $P^*(A_1 \cap A_0) = 1$ as well as $P^*(A_2 \cap A_0) = 1$ in the case $P^*(A_0) = 1$ and $P^*(A_0^c) = 1$, since P_j, $j = 1,2$, defined by $P_1(A) = P^*(A \cap A_0)$, $A \in \mathcal{A}$, and $P_2(A) = P^*(A \cap A_0^c)$, $A \in \mathcal{A}$, are in the case under consideration $\{0,1\}$-valued probability measures on $\mathcal{A}$. Moreover, $P^*(A_0^c) = 0$ yields on account of $P_*(A_0) = 1$ the existence of some $A_3 \in \mathcal{A}$ satisfying $A_3 \subset A_0$ and $P(A_3) = 1$, from which, in the case $P^*(A_0) = 1$ and $P^*(A_0^c) = 0$, follows that $A_1 \cap A_3 \in \mathcal{A}$ is an atom of $\mathcal{A}$ satisfying $P(A_1 \cap A_3) = 1$ on account of $A_1 \cap A_3 \neq \emptyset$ and $A_1 \cap A_3 \subset A_1 \cap A_0$ with $A_1 \cap A_0 \in \mathcal{A} \cap A_0$ being some atom of $\mathcal{A} \cap A_0$, i.e. $A_1 \cap A_3 = A_1 \cap A_0$. A similar argument yields the existence of some $A_4 \in \mathcal{A}$ satisfying $A_4 \subset A_0^c$ and $P(A_4) = 1$ in the case $P^*(A_0^c) = 1$ and $P^*(A_0) = 0$, i.e. $P_*(A_0^c) = 1$ such that $A_2 \cap A_4 \in \mathcal{A}$ is an atom of $\mathcal{A}$ with $P(A_2 \cap A_4) = 1$. It remains to show the existence of some atom $A \in \mathcal{A}$ of $\mathcal{A}$ satisfying $P(A) = 1$ in the case $P^*(A_0) = 1$ and $P^*(A_0^c) = 1$. It will be proved that $A_1 \cap A_2$ plays the role of A, where $A_j \in \mathcal{A}$, $j = 1,2$, have been already introduced. First of all $P(A_1 \cap A_2) = 1$ is valid on account of $P(A_j) = 1$, $j = 1,2$, since $P^*(A_1 \cap A_0) = 1$ and $P^*(A_2 \cap A_0^c) = 1$ holds true. Now $A_1 \cap A_2 \neq \emptyset$ together with $A_1 \cap A_2 = (A_1 \cap A_2 \cap A_0) \cup (A_1 \cap A_2 \cap A_0^c)$ yields $A_1 \cap A_2 \cap A_0 \neq \emptyset$ or $A_1 \cap A_2 \cap A_0^c \neq \emptyset$, from which one derives $A_1 \cap A_2 \cap A_0 = A_1 \cap A_0$ or $A_1 \cap A_2 \cap A_0^c = A_2 \cap A_0^c$, since $A_1 \cap A_0 \in \mathcal{A} \cap A_0$ is an atom of $\mathcal{A} \cap A_0$ and $A_2 \cap A_0^c \in \mathcal{A} \cap A_0$ is an atom of $\mathcal{A} \cap A_0^c$. In the case $A_1 \cap A_2 \cap A_0 = A_1 \cap A_0$, one arrives together with $A_1 \cap A_2 \cap A_0^c = \emptyset$ at $A_1 \cap A_2 = A_1 \cap A_2 \cap A_0$, from which follows that $A_1 \cap A_2 \in \mathcal{A}$ is indeed some atom of $\mathcal{A}$ with $P(A_1 \cap A_2) = 1$, since $A_1 \cap A_2 \cap A_0 = A_1 \cap A_0 \in \mathcal{A} \cap A_0$ is an atom of $\mathcal{A} \cap A_0$ satisfying $P^*(A_1 \cap A_0) = 1$. Finally, the case where $A_1 \cap A_2 \cap A_0 = A_1 \cap A_0$ and $A_1 \cap A_2 \cap A_0^c = A_2 \cap A_0^c$ is valid has to be studied, which occur if $A_1 \cap A_2 \cap A_0 \neq \emptyset$ and $A_1 \cap A_2 \cap A_0^c \neq \emptyset$ holds true in the case $P^*(A_0) = 1$ and $P^*(A_0^c) = 1$. For this purpose let $B \in \mathcal{A}$ be non-empty and satisfy $B \subset A_1 \cap A_2$. It will be shown that $B = A_1 \cap A_2$ is valid, i.e. $A_1 \cap A_2 \in \mathcal{A}$ is indeed an atom of $\mathcal{A}$. Now $B \neq \emptyset$ together with $B = (B \cap A_0) \cup (B \cap A_0^c)$ leads to $B \cap A_0 \neq \emptyset$ or $B \cap A_0^c \neq \emptyset$. Therefore, $B \subset A_1 \cap A_2$ results in $B \cap A_0 = A_1 \cap A_2 \cap A_0$ as well as $B \cap A_0^c = A_1 \cap A_2 \cap A_0^c$, since the case $B \cap A_0 = \emptyset$ or $B \cap A_0^c = \emptyset$ cannot happen, because $B \cap A_0^c = \emptyset$ implies $B = B \cap A_0$, i.e. $B \subset A_1 \cap A_2 \cap A_0$ is valid. Hence $A_1 \cap A_2 \cap A_0 = A_1 \cap A_0$ shows that $B = A_1 \cap A_0$ holds true, since $A_1 \cap A_0 \in \mathcal{A} \cap A_0$ is an atom of $\mathcal{A} \cap A_0$ and $B \neq \emptyset$. In particular, $P(B) = 1$ on account of $P^*(A_1 \cap A_0) = 1$ is valid. Now one arrives by $P(B) = 1$ together with $B \subset A_0$ at $P_*(A_0) = 1$, i.e. $P^*(A_0^c) = 0$, whereas in the case under consideration $P^*(A_0^c) = 1$ holds true. A similar contradiction yields the assumption $B \cap A_0 = \emptyset$, namely $P^*(A_0) = 0$, whereas $P^*(A_0) = 1$ and $P^*(A_0^c) = 1$ is valid in the case under consideration. In this case, $B \cap A_0 = A_1 \cap A_2 \cap A_0$ and $B \cap A_0^c = A_1 \cap A_2 \cap A_0^c$ has been proved for any non-empty set $B \in \mathcal{A}$ satisfying $B \subset A_1 \cap A_2$. Hence $B = (B \cap A_0) \cup (B \cap A_0^c) = (A_1 \cap A_2 \cap A_0) \cup (A_1 \cap A_2 \cap A_0^c) = A_1 \cap A_2$ has been shown, i.e. $A_1 \cap A_2 \in \mathcal{A}$ is indeed an atom of $\mathcal{A}$ with $P(A_1 \cap A_2) = 1$.

APPLICATION (CONSERVATION WITH RESPECT TO DIRECT PRODUCTS OF σ-ALGEBRAS). Let $\mathcal{A}_j$ stand for σ-algebras of subsets of sets $\Omega_j \neq \emptyset$, $j = 1,2$. Then it will be shown that the direct product $\mathcal{A}_1 \otimes \mathcal{A}_2$ of $\mathcal{A}_j$, $j = 1,2$, is of countably generated type if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ shares this property of $\mathcal{A}_1 \otimes \mathcal{A}_2$. In the case where $\mathcal{A}_j$, $j = 1, 2$, are of countably generated type, and where P is some $\{0,1\}$-valued probability measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$, the equation $P(A) = 1$ has to be shown for some atom $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ of $\mathcal{A}_1 \otimes \mathcal{A}_2$. This follows from the observation that the marginal probability measures P_j on $\mathcal{A}_j$, $j = 1, 2$, of P defined by $P_1(A_1) = P(A_1 \times \Omega_2)$, $A_1 \in \mathcal{A}_1$, $P_2(A_2) = P(\Omega_1 \times A_2)$, $A_2 \in \mathcal{A}_2$, are $\{0,1\}$-valued. Hence $P_j(A_j) = 1$ is valid for some atom $A_j \in \mathcal{A}_j$ of $\mathcal{A}_j$, $j = 1, 2$. Moreover, $A_1 \times A_2 \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is an atom of $\mathcal{A}_1 \otimes \mathcal{A}_2$ satisfying $P(A_1 \times A_2) = 1$.

For the proof of the converse implication, one starts from some $\{0,1\}$-valued probability measures P_j on $\mathcal{A}_j$, $j = 1, 2$, where $\mathcal{A}_1 \otimes \mathcal{A}_2$ is of countably generated type. Now the product measure $P_1 \otimes P_2$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ is $\{0,1\}$-valued, and therefore there exists some atom $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ of $\mathcal{A}_1 \otimes \mathcal{A}_2$ fulfilling $(P_1 \otimes P_2)(A) = 0$. Now every atom $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is of the type $A = A_1 \times A_2$, where $A_j \in \mathcal{A}_j$ are atoms of $\mathcal{A}_j$, $j = 1, 2$. This might be shown by the structure of atoms of countably generated σ-algebras, which has been already described in the example about countably generated σ-algebras, or by means of the fact that sections of sets belonging to $\mathcal{A}_1 \otimes \mathcal{A}_2$ are elements of $\mathcal{A}_j$, $j = 1, 2$. Finally, $(P_1 \otimes P_2)(A_1 \times A_2) = P_1(A_1)P_2(A_2) = 1$ yields $P_j(A_j) = 1$, $j = 1, 2$.

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