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Mathematica Slovaca, Vol. 29 (1979), No. 4, 333--345

Persistent URL: http://dml.cz/dmlcz/128905

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INTEGRAL REPRESENTATIONS OF NON-LINEAR FUNCTIONALS

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The purpose of the present paper is to establish the integral representation of some non linear functionals L on $\mathscr{L}$ ($\mathscr{L}$ is a linear lattice of real functions on a space X) in the form

$$Lf = \int f \, d\mu \quad \text{for } f \text{ in } \mathscr{L},$$

where μ is a suitable set function depending on the functional L.

When we consider such a general integral representation, several questions arise. What is the suitable family of sets $\mathscr{D}$ on which μ should be defined? Is this representation unique? What conditions (imposed on μ) guarantee that $\int f \, d\mu$ can be defined?

In papers [3] and [4] we presented the theory of integration with respect to some non-additive set functions; namely to the pre-measure (a non negative, monotone, at an empty set vanishing set function μ). By the help of this theory of integration we are able to prove the representation theorems for a special type of nonlinear order continuous functionals, namely for pre-linear, strong sublinear and strong superlinear functionals. These are natural generalizations of a non-negative linear functional.

In the original Riesz representation theorem [2] the space $\mathscr{L}$ is the space of all continuous real functions on the interval $\langle 0,1 \rangle$ and L is a bouded linear functional. Many authors have extended and generalized the classical studies of Riesz for the case when X is a special (e.g. compact) topological space, $\mathscr{L} = \mathscr{C}(X)$ is the space of all continuous functions on X and L is linear. So the question arises whether in the case $\mathscr{L} = \mathscr{C}(X)$ or $\mathscr{C}_0(X)$ a nonlinear L may be represented as an integral. We show that the answer is positive for a so-called strong sublinear order continuous functionals.

To prove the last mentioned problem we shall need the Daniell extension scheme for strong sublinear functionals. We show that this extension scheme works also in this case, but the method used is only a slight modification of the method given by Riečan (see [1]).

The last part of this work deals with the representation theorem for not necessarily continuous functionals.

§ 0. Preliminary

Let X be a non empty space. By an *affine pre-lattice* of functions we mean a family of functions $\mathscr{L}$ defined on X with following property

(i) If $f \in \mathscr{L}$, then $af \in \mathscr{L}$ for every real a.

(ii) If $f \in \mathscr{L}$, then $f \wedge a$ and $f - f \wedge a$ are in $\mathscr{L}$ for every non-negative real a.

Lemma 1. *If $\mathscr{L}$ is an affine pre-lattice, and $f \in \mathscr{L}$, then f^+ and f^- and also in $\mathscr{L}$.*

Proof. This follows from the definition of an affine pre-lattice, since

$$f^+ = f - f \wedge 0 \quad \text{and} \quad f^- = -(f \wedge 0).$$

A functional L defined on an affine pre-lattice $\mathscr{L}$ will be said to be *pre-linear* iff

(i) L is monotone ($f \leqq g \Rightarrow \mathrm{L}f \leqq \mathrm{L}g$).

(ii) L is homogeneous ($\mathrm{L}(af) = a\, \mathrm{L}f$).

(iii) L is additive in a horizontal sense, i.e.,

$$\mathrm{L}f = \mathrm{L}(f \wedge a) + \mathrm{L}(f - f \wedge a)$$

for $f \in \mathscr{L}$ and for a non-negative real a.

A functional L is said to be *order continuous* or only continuous iff

(i) $f_n \nearrow f \geqq g\, (f_n, g \in \mathscr{L})$ implies $\lim_n \mathrm{L}f_n \geqq \mathrm{L}g$.

(ii) $f_n \searrow f \leqq g\, (f_n, g \in \mathscr{L})$ implies $\lim_n \mathrm{L}f_n \leqq \mathrm{L}g$.

Lemma 2. *If $\mathscr{L}$ is an affine pre-lattice, L is a pre-linear functional on $\mathscr{L}$, then*

$$\mathrm{L}f = \mathrm{L}f^+ - \mathrm{L}f^-$$

for every f in $\mathscr{L}$.

A *pre-space* is a pair $(X, \mathscr{D})$, where $\mathscr{D}$ is a family of subsets of X containing the empty set.

An extended real valued, monotone, at empty set vanishing set function defined on $\mathscr{D}$ is called a *pre-measure.*

A pre-measure μ is called *continuous* iff

(i) $A_n \nearrow A \supset B\, (A_n, B \in \mathscr{D})$, implies $\lim_n \mu(A_n) \geqq \mu(B)$.

(ii) $A_n \searrow A \subset B$, $\mu(A_1) < \infty$ $(A_n, B \in \mathscr{D})$ implies $\lim_n \mu(A_n) \leqq \mu(B)$.

The function $f: X \to \langle -\infty, \infty \rangle$ is $\mathscr{D}$ — *measurable* or only *measurable* if the sets $\{x\,;\, f(x) \geqq a\}$ and $\{x\,;\, f(x) \leqq -a\}$ are in $\mathscr{D}$ for every $a > 0$.

We denote by $\mathscr{L}(\mathscr{D})$ the set of all $\mathscr{D}$-measurable functions. $f \in \mathscr{L}(\mathscr{D})$ is called a *simple function* if the range of f is finite. The set of all simple functions from $\mathscr{L}(\mathscr{D})$ will be denoted by $\mathscr{L}_s(\mathscr{D})$.

In [3] we introduced a notion of integration with respect to a pre-measure as follows.

Let $\mathscr{F}$ be a family of all finite subsets of $\langle -\infty, \infty\rangle$ which contains zero. Let $F \in \mathscr{F}$ with $F=\{b_m<b_{m-1}< \dots <b_0=0=a_0<a_1< \dots <a_n\}$, and let f be a $\mathscr{D}$-measurable function. We put

$$S(f, F)=\sum_{i=1}^{n}(a_i-a_{i-1})\mu(\{x\,;f(x)\geqq a_i\})+$$
$$+\sum_{j=1}^{m}(b_j-b_{j-1})\mu(\{x\,;f(x)\leqq b_j\})$$

if the right-hand side expression contains no expression of the type $\infty-\infty$.

Since $\mathscr{F}$ is directed by inclusion, the triple $(S(f, F), \mathscr{F}, \supset)$ is a net. We put

$$\mathscr{I}f=\mathscr{I}_\mu f=\int f\,\mathrm{d}\mu=\lim_{F\in\mathscr{F}} S(f, F)$$

if the limit exists.

f is called *integrable* iff $\mathscr{I}_\mu f$ is finite. We denote by $\mathscr{L}_1=\mathscr{L}_1\,(X, \mathscr{D}, \mu)$ the set of all integrable functions.

The main properties of $\mathscr{I}_\mu$ are:

1° $\mathscr{I}_\mu\chi_A=\mu(A)$ for A in $\mathscr{D}$.

2° $\mathscr{I}_\mu f=\sup\{\mathscr{I}_\mu g\,;\,g\in\mathscr{L}_s(\mathscr{D}),\,g\leqq f\}$ for $f\geqq 0$.

3° $\mathscr{L}_1$ is an affine pre-lattice.

4° $\mathscr{I}_\mu$ is a pre-linear functional on $\mathscr{L}_1$.

5° If μ is order continuous on $\mathscr{D}$, so is $\mathscr{I}_\mu$ on $\mathscr{L}_1$.

§ 1. Representation theorems on measurable functions

In the following two assertions $\mathscr{L}$ will be an affine pre-lattice of elements of $\mathscr{L}(\mathscr{D})$ with $\mathscr{L}_s(\mathscr{D})\subset\mathscr{L}$. We shall say explicitly whether $\mathscr{L}$ has any other properties.

Lemma 3. *Let* L *be a pre-linear functional on* $\mathscr{L}$. *Let* μ *be a set function on* $\mathscr{D}$ *defined by*

$$\mu(A)=\mu_{\mathrm{L}}(A)=\mathrm{L}\chi_A.$$

Then μ *is a pre-measure and*

$$\mathrm{L}f\geqq\mathscr{I}_\mu f$$

for f *in* $\mathscr{L}^+$.

Proof. Since L is monotone and $\mathrm{L}\chi_A\geqq 0$, it is clear that μ is a pre-measure. Let

$g \in \mathscr{L}_s^+(\mathscr{D})$ and let $a_1, a_2, \ldots, a_n$ be the range of g ; then $g = \sum_{i=1}^{n}(a_i - a_{i-1})\chi_{A_i}$, where $A_i = \{x\,;\, g(x) \geqq a_i\}$ and $a_0 = 0$. By the horizontal additivity and homogenity of L

$$\mathrm{L}g = \sum_{i=1}^{n}(a_i - a_{i-1})\, \mathrm{L}\chi_{A_i} = \sum_{i=1}^{n}(a_i - a_{i-1})\mu(A_i) = \mathscr{I}_\mu g\,.$$

Thus by Lemma 2 $\mathrm{L}g = \mathscr{I}_\mu g$ for every $g \in \mathscr{L}_s(\mathscr{D})$. Let f be in $\mathscr{L}^+$ and let g be in $\mathscr{L}_s(\mathscr{D})$ with $g \leqq f$; then

$$\mathrm{L}f \geqq \mathrm{L}g = \mathscr{I}_\mu g$$

and so

$$\mathrm{L}f \leqq \sup\{\mathscr{I}_\mu g\,;\, g \in \mathscr{L}_s(\mathscr{D}),\, g \leqq f\} = \mathscr{I}_\mu f$$

by 2°.

Theorem 4. *Let* L *be an order continuous pre-linear functional on* $\mathscr{L}$*. Then there exists a unique order continuous pre-measure* $\mu = \mu_{\mathrm{L}}$ *on* $\mathscr{D}$ *such that*

$$\mathrm{L}f = \mathscr{I}_\mu f$$

for f *in* L.

Proof. Let μ_{L} be the same as in the last lemma. Let f be in $\mathscr{L}^+$. By Proposition 13 of [3] there exists a sequence of non-negative $\mathscr{D}$-simple functions $f_n \in \mathscr{L}^+$ with $f_n \nearrow f$. By the preceding lemma $\mathrm{L}f_n = \mathscr{I}_\mu f_n$. By the continuity of L and the theorem of Beppo—Levi (see [3]), we get for $\mathscr{I}_\mu$

$$\mathrm{L}f = \lim_n \mathrm{L}f_n = \lim_n \mathscr{I}_\mu f_n = \mathscr{I}_\mu f.$$

Thus we showed that

$$\mathrm{L}f = \mathscr{I}_\mu f$$

for f in $\mathscr{L}^+$.

If now f is from $\mathscr{L}$, then the proof follows by

$$\mathrm{L}f = \mathrm{L}f^+ - \mathrm{L}f^- = \mathscr{I}_\mu f^+ - \mathscr{I}_\mu f^-.$$

It remains to show that μ is monotone and order continuous on $\mathscr{D}$, but this is an easy consequence of the order continuity of L and the definition of μ. The unicity of μ is trivial.

We turn now our attention to the case when $\mathscr{D}$ and $\mathscr{L}$ have some special properties. Until Corollary 6 $\mathscr{D} = \mathscr{S}$ will be a ring, $\mathscr{L} \subset \mathscr{L}(\mathscr{S})$ will be a linear lattice with Stone's condition containing $\mathscr{L}_s(\mathscr{S})$ and L is an order continuous strong sublinear (superlinear) functional on $\mathscr{L}$, i.e., L is pre-linear and

$$\mathrm{L}(f+g) \leqq \mathrm{L}(f \wedge g) + \mathrm{L}(f \vee g) \leqq \mathrm{L}f + \mathrm{L}g$$
$$(\mathrm{L}(f+g) \geqq \mathrm{L}(f \wedge g) + \mathrm{L}(f \vee g) \geqq \mathrm{L}f + \mathrm{L}g)$$

for f and g from $\mathscr{L}^+$.

Theorem 5. *Under the above conditions for $\mathscr{L}$ and* L *there exists a unique order continuous strong submeasure (strong supermeasure) μ on $\mathscr{S}$ such that*

$$\mathrm{L}f = \mathscr{I}_\mu f$$

for every f in $\mathscr{L}$.

Proof. Since $\mathscr{L}$ is an affine pre-lattice and L is pre-linear it remains only to show the strong subadditivity (superadditivity) of μ.

$$\mu(A \cap B) + \mu(A \cup B) = \mathrm{L}\chi_{A \cap B} + \mathrm{L}\chi_{A \cup B} = \mathrm{L}(\chi_A \wedge \chi_B) + \\ + \mathrm{L}(\chi_A \vee \chi_B) \leqq \mathrm{L}\chi_A + \mathrm{L}\chi_B = \mu(A) + \mu(B).$$

The case of a strong superlinear operator is similar.

Corollary 6. *Let l be an order continuous monotone linear functional on $\mathscr{L}$. Then there exists a unique σ-additive measure μ on $\mathscr{S}$ such that*

$$lf = \mathscr{I}_\mu f$$

for every f in $\mathscr{L}$.

§ 2. The Daniell extension scheme

In paper [1] Riečan proved that the Daniell extension scheme works also in the case of subadditive functionals. In this paragraph we shall show that this is true also when the extended functional is a strong sublinear map defined on a lattice of functions satisfying Stone's condition.

Let $\mathscr{L}$ be a set of functions and let L be a functional defined on $\mathscr{L}$. L is said to be *exhausting* if $f_n \leqq f_{n+1} \leqq \ldots \leqq f$, f_n, $f \in \mathscr{L}$ and $\lim_n \mathrm{L}f_n < \infty$ implies

$$\lim_n \mathrm{L}(f_{n+1} - f_n) = 0.$$

In this paragraph L will be a linear lattice of functions with Stone's condition, and L will be a continuous, strong sublinear and exhausting functional on $\mathscr{L}$.

Let $\mathscr{K}$ be a family of functions with $\mathscr{K} \supset \mathscr{L}$. We say that $\mathscr{K}$ is L_σ *full* iff $0 \leqq g \leqq f$, $f_n \nearrow f$, $f_n \in \mathscr{L}$ $(n = 1, 2, \ldots)$ and $\lim_n \mathrm{L}f_n = 0$ implies $g \in \mathscr{K}$.

Theorem 7. *There exists a unique continuous strong sublinear extension $\bar{\mathrm{L}}$ of* L *to the smallest L_σ full, conditionally complete σ-lattice $\bar{\mathscr{L}} \supset \mathscr{L}$.*

We give first the construction of $\bar{\mathrm{L}}$, the proof will follow in the subsequent lemmas.

We put

$$\mathscr{L}_\sigma = \{g\,;\, \exists g_n \in \mathscr{L},\, g_n \nearrow g,\, \lim_n \mathrm{L}g_n < \infty\}$$
$$\mathscr{L}_\delta = \{h\,;\, \exists h_n \in \mathscr{L},\, h_n \searrow h,\, \lim_n \mathrm{L}h_n > -\infty\}$$

and define

$$\mathrm{L}_\sigma(g)=\lim_n \mathrm{L}g_n \quad g_n\nearrow g,\ g_n\in\mathscr{L}.$$
$$\mathrm{L}_\delta(h)=\lim_n \mathrm{L}h_n \quad h_n\searrow h,\ h_n\in\mathscr{L}.$$

Finally we put

$$\bar{\mathscr{L}}=\{f;\ \forall\varepsilon>0\ \exists g\in\mathscr{L}_\sigma,\ h\in\mathscr{L}_\delta,\ h\leqq f\leqq g,\ \mathrm{L}_\sigma(g-h)<\varepsilon\}$$
$$\bar{\mathrm{L}}f=\inf\{\mathrm{L}_\sigma g\ ;\ g\geqq f,\ g\in\mathscr{L}_\sigma\},\quad \text{for}\quad f\in\bar{\mathscr{L}}.$$

Lemma 8. *The definitions of* L_σ *and* L_δ *are correct.*

Proof. Let $f\in\mathscr{L}_\sigma$, and $f_n\nearrow f$, $g_n\nearrow f$, where f_n, $g_n\in\mathscr{L}$. Then

$$\mathrm{L}f_n=\lim_m \mathrm{L}(f_n\wedge g_m)$$

and so

$$\lim_n \mathrm{L}f_n=\lim_n \lim_m \mathrm{L}(f_n\wedge g_m)\leqq\lim_n Lg_m.$$

The other relation may be obtained in this way too. The proof of the correctness of L_δ is similar.

Lemma 9. L_σ *and* L_δ *are strong subadditive, positively homogeneous and horizontal additive functionals. Moreover*

$$\mathrm{L}_\sigma(-f)=-\mathrm{L}_\delta f \quad and \quad \mathrm{L}_\delta(-f)=-\mathrm{L}_\sigma f.$$

Proof. If $f,g\in\mathscr{L}_\sigma^+$, then there exists f_n, $g_n\in\mathscr{L}$ such that $f_n\nearrow f$ and $g_n\nearrow g$, and we may assume that f_n, $g_n\geqq 0$.

Then

$$\mathrm{L}(f_n\wedge g_n)+\mathrm{L}(f_n\vee g_n)\leqq\mathrm{L}f_n+\mathrm{L}g_n\ ;$$

limiting this we get

$$\mathrm{L}_\sigma(f\wedge g)+\mathrm{L}_\sigma(f\vee g)\leqq\mathrm{L}_\sigma f+\mathrm{L}_\sigma g.$$

Let now $a\geqq 0$. Then clearly $\mathrm{L}_\sigma(af)=a\ \mathrm{L}_\sigma f$, obviously $f_n\wedge a\nearrow f\wedge a$ and $f_n-f_n\wedge a\nearrow f-f\wedge a$ and so from

$$\mathrm{L}f_n=\mathrm{L}(f_n\wedge a)+\mathrm{L}(f_n-f_n\wedge a)$$

we obtain

$$\mathrm{L}_\sigma f=\mathrm{L}_\sigma(f\wedge a)+\mathrm{L}_\sigma(f-f\wedge a).$$

The other assertions of the lemma are trivial.

Lemma 10. $\bar{\mathscr{L}}$ *is a linear lattice with Stone's condition.*

Proof. Let $\varepsilon>0$. Let f, $g\in\bar{\mathscr{L}}$; then there exist elements f_1, $g_1\in\mathscr{L}_\delta$ and f_2, $g_2\in\mathscr{L}_\sigma$ such that $f_1\leqq f\leqq f_2$, $g_1\leqq g\leqq g_2$ and $\mathrm{L}_\sigma(f_2-f_1)$, $\mathrm{L}_\sigma(g_2-g_1)<\varepsilon/2$. Then $f_1+g_1\leqq f+g\leqq f_2+g_2$, $f_1+g_1\in\mathscr{L}_\delta$, $f_2+g_2\in\mathscr{L}_\sigma$ and $\mathrm{L}_\sigma((f_2+g_2)-(f_1+g_1))<\varepsilon$, since L_σ is subadditive on $\mathscr{L}_\sigma$ and so $f+g\in\bar{\mathscr{L}}$.

The proof of the assertions $f \vee g, f \wedge g, f-g, f-f \wedge g, af \in \bar{\mathscr{L}}$ is similar. Let now $a \geqq 0$; then $f_1 \wedge a \leqq f \wedge a \leqq f_2 \wedge a$; $f_1 \wedge a$ is a member of $\mathscr{L}_\delta$ and $f_2 \wedge a$ is from $\mathscr{L}_\sigma$ and $f_2 \wedge a - f_1 \wedge a \leqq f_2 - f_1$, and so by the monotonicity of L_σ we have

$$L_\sigma(f_2 \wedge a - f_1 \wedge a) \leqq L(f_2 - f_1) < \varepsilon/2.$$

Hence $f \wedge a$ is from $\bar{\mathscr{L}}$.

Lemma 11. *$\bar{L}$ is a strong sublinear functional on $\bar{\mathscr{L}}$ and*

$$\bar{L}f = \sup \{L_\delta h \,;\, h \leqq f,\, h \in \mathscr{L}_\delta\}.$$

Proof. I. We prove first the second assertion. Let $h \leqq f \leqq g$, $L_\sigma(g-h) < \varepsilon$, $h \in \mathscr{L}_\delta$, $g \in \mathscr{L}_\sigma$. Then

$$\bar{L}f \leqq L_\sigma g \leqq L_\sigma(g-h) + L_\delta h < \varepsilon + L_\delta h$$

and so

$$\bar{L}f \leqq \sup \{L_\delta h \,;\, h \leqq f,\, h \in \mathscr{L}_\delta\}.$$

The opposite inequality is trivial.

II. Let $\varepsilon > 0$. Let f, $g \in \bar{\mathscr{L}}^+$, take f_1, $g_1 \in \mathscr{L}_\sigma^+$ with $\bar{L}f + \varepsilon \geqq L_\sigma f_1$, $f \leqq f_1$ and $\bar{L}g + \varepsilon \geqq L_\sigma g_1$, $g \leqq g_1$. Since $\mathscr{L}_\sigma$ is a lattice and $f \vee g \leqq f_1 \vee g_1$, and $f \wedge g \leqq f_1 \wedge g_1$, we get

$$\begin{aligned} \bar{L}f + \bar{L}g + 2\varepsilon &\geqq L_\sigma f_1 + L_\sigma g_1 \\ &\geqq L_\sigma(f_1 \vee g_1) + L_\sigma(f_1 \wedge g_1) \\ &\geqq \bar{L}(f \vee g) + \bar{L}(f \wedge g). \end{aligned}$$

III. By Lemma 9 and the definition of $\bar{L}$ it is clear that $\bar{L}$ is homogeneous.

IV. Let now $f \in \bar{\mathscr{L}}$ and $a \geqq 0$; then

$$\begin{aligned} \bar{L}f &= \inf \{L_\sigma g \,;\, g \in \mathscr{L}_\sigma,\, g \geqq f\} \\ &= \inf \{L_\sigma(g \wedge a) + L_\sigma(g - g \wedge a) \,;\, g \in \mathscr{L}_\sigma,\, g \geqq f\} \\ &\geqq \inf \{L_\sigma g_1 \,;\, g_1 \in \mathscr{L}_\sigma,\, g_1 \geqq f \wedge a\} + \\ &\quad + \inf \{L_\sigma g_2 \,;\, g_2 \in \mathscr{L}_\sigma,\, g_2 \geqq f - f \wedge a\} \\ &= \bar{L}(f \wedge a) + \bar{L}(f - f \wedge a), \end{aligned}$$

where we used the horizontal additivity of L_σ.

Now we prove the opposite inequality.

$$\begin{aligned} \bar{L}f &= \sup \{L_\delta h \,;\, h \in \mathscr{L}_\delta,\, h \leqq f\} \\ &= \sup \{L_\delta(h \wedge a) + L_\delta(h - h \wedge a) \,;\, h \in \mathscr{L}_\delta,\, h \leqq f\} \\ &\leqq \sup \{L_\delta h_1 \,;\, h_1 \in \mathscr{L}_\delta,\, h_1 \leqq f \wedge a\} + \\ &\quad + \sup \{L_\delta h_2 \,;\, h_2 \in \mathscr{L}_\delta,\, h_2 \leqq f - f \wedge a\} \\ &= \bar{L}(f \wedge a) + \bar{L}(f - f \wedge a). \end{aligned}$$

V. The monotonicity of $\bar{L}$ follows from its definition.

Lemma 12. ([1] *Proposition 3*). *Let* $f_n \in \bar{\mathscr{L}}$ $(n=1, 2, \ldots)$ $f_n \nearrow f$ *and* $\{\bar{L} f_n\}$ *be bounded. Then* $f \in \bar{\mathscr{L}}$ *and*

$$\bar{L} f = \lim_n \bar{L} f_n.$$

Note that also the dual assertion to the last lemma holds.

Proof of Theorem 7. It is easy to see that $\bar{\mathscr{L}}$ is a conditionally complete σ-lattice and that $\bar{\mathscr{L}}$ is L_σ full. Hence it suffices to prove the uniquess of $\bar{L}$ (see [1]). Let H be a strong sublinear, continuous functional on $\bar{\mathscr{L}}$ which is an extension of L. Put

$$\mathscr{N} = \{f \in \bar{\mathscr{L}}; \bar{L} f = Hf\}.$$

By the assumption $\mathscr{N} \supset \mathscr{L}$. Moreover

$$\mathscr{N} \supset \mathscr{L}_\sigma \cup \mathscr{L}_\delta.$$

Indeed, take $f_n \in \mathscr{L}$ with $f_n \nearrow f$. Then

$$\bar{L} f = L_\sigma f = \lim_n L f_n = \lim_n H f_n = Hf.$$

Similarly $\bar{L} f = Hf$ for f in $\mathscr{L}_\delta$.

Let $f \in \bar{\mathscr{L}}$. Then to any $\varepsilon > 0$ there exists g and h such that $g \in \mathscr{L}_\sigma$, $h \in \mathscr{L}_\delta$, $h \leqq f \leqq g$ and $L_\sigma(g-h) < \varepsilon$. Therefore

$$\bar{L} h \leqq \bar{L} f \leqq \bar{L} g = \bar{L}(h + g - h) < \bar{L} h + \varepsilon$$

and

$$\bar{L} h = Hh \leqq Hf \leqq Hg = \bar{L} g < \bar{L} h + \varepsilon.$$

From this it follows $|\bar{L} f - Hf| < \varepsilon$. Therefore $\bar{L} f = Hf$ and so

$$\mathscr{N} \supset \bar{\mathscr{L}}.$$

We shall show now that the continuous extension of L need not exist when L is not exhausting.

Example 13. Let $X = \{0, 1, \ldots, n, \ldots\}$, $\mathscr{L} = \{f; f$ is a real valued function on X and $f(x) = 0$ except on a finite subset of $X\}$. Let $Lf = \max f - \min f$.

Then $\mathscr{L}$ is a linear lattice with Stone's condition and L is a strong sublinear map on $\mathscr{L}$ which is order continuous. The conditionally complete σ-lattice generated by $\mathscr{L}$ is the set of all bounded real functions on X, denote it by $\bar{\mathscr{L}}$. L has only one extension and this is

$$\bar{L} f = \sup f - \inf f.$$

It is clear that this extension is not continuous since if

$$A_n = X - \{0, 1, \ldots, n\},$$

then

$\chi_{A_n} \searrow 0$ and $\lim_n L\chi_{A_n} = 1$.

§ 3. Representation theorems on a lattice of functions

Theorem 14. *Let $\mathscr{L}$ be a linear lattice of functions with Stone's condition. Let* L *be an exhausting, order continuous strong sublinear functional on $\mathscr{L}$. Then there exists a σ-ring $\mathscr{S}$ and a unique continuous strong submeasure μ on $\mathscr{S}$ such that*

$$\mathrm{L}f=\mathscr{I}_{\mu}f$$

for f in $\mathscr{L}$.

Proof. By the preceding theorem there exists a unique extension $\bar{\mathrm{L}}$ of L to the conditionally complete σ-lattice $\bar{\mathscr{L}}$ with Stone's condition containing $\mathscr{L}$. Put

$$\mathscr{S}_0=\{\{x\,;\,f(x)\geqq 1\}\,;\,f\in\bar{\mathscr{L}}\}$$

Let $f\in\bar{\mathscr{L}}$. Put $f_n=n(f\wedge 1-f\wedge(1-1/n))$; then $f_n\in\bar{\mathscr{L}}$ and $f_n\searrow\chi_{\{x\,;\,f(x)\geqq 1\}}$. Since $\bar{\mathscr{L}}$ is a conditionally complete σ-lattice, we get $\chi_{\{x\,;\,f(x)\geqq 1\}}\in\bar{\mathscr{L}}$. In other words we proved that if $A\in\mathscr{S}_0$, then $\chi_A\in\bar{\mathscr{L}}$. And so we may put

$$\mu_0(A)=\bar{\mathrm{L}}\chi_A \quad \text{for } A \text{ in } \mathscr{S}_0.$$

Let $\mathscr{S}$ be a σ-ring generated by $\mathscr{S}_0$. We put

$$\mu(A)=\sup\,\{\mu_0(B)\,;\,B\subset A,\,B\in\mathscr{S}_0\}.$$

I. We prove first that $\mathscr{S}_0$ is a ring with the following property:

If $A_n\in\mathscr{S}_0$ with $A_n\subset A_{n+1}$ and with $\lim_n\mu_0(A_n)<\infty$, then $\cup_n A_n\in\mathscr{S}_0$.

Let $A_n=\{x\,;\,f_n(x)\geqq 1\}$; then $\chi_{A_n}\leqq\chi_{A_{n+1}}$ and $\lim_n \mathrm{L}\chi_{A_n}<\infty$. By the theorem of Beppo—Levy which holds for $\bar{\mathrm{L}}$ on $\bar{\mathscr{L}}$ we get $\lim_n\chi_{A_n}\in\bar{\mathscr{L}}$, from which we obtain

$$\cup_n A_n=\{x\,;\,\lim_n\chi_{A_n}\geqq 1\}\in\mathscr{S}_0.$$

Let now $A,B\in\mathscr{S}_0$; then $\chi_A,\chi_B\in\bar{\mathscr{L}}$. Since $\bar{\mathscr{L}}$ is a linear lattice, we get that the sets $A\cup B=\{x\,;\,\chi_A\vee\chi_B\geqq 1\}$ and $A-B=\{x\,;\,\chi_A-\chi_B\geqq 1\}$ are in $\mathscr{S}_0$.

II. We shall show now that every function $f\in\bar{\mathscr{L}}$ is $\mathscr{S}_0$ measurable. Let $a>0$ and let $f\in\bar{\mathscr{L}}$. Then since $\bar{\mathscr{L}}$ is a linear space, the sets

$$\{x\,;\,f(x)\geqq a\}=\{x\,;\,(1/a)f(x)\geqq 1\},$$
$$\{x\,;\,f(x)\leqq -a\}=\{x\,;\,-(1/a)f(x)\geqq 1\}$$

are from $\mathscr{S}_0$.

III. From Theorem 5 we get that $\bar{\mathrm{L}}f=\mathscr{I}_{\mu_0}f$ for f in $\bar{\mathscr{L}}$, especially

$$\mathrm{L}f=\bar{\mathscr{I}}_{\mu_0}f$$

for f in $\mathscr{L}$, since $\bar{\mathrm{L}}$ is an extension of L.

It is clear that μ is a continuous strong submeasure on $\mathscr{S}$ and that $\mu_0=\mu/\mathscr{S}_0$; hence

$$\mathrm{L}f=\mathscr{I}_{\mu}f$$

for all f in $\mathscr{L}$.

Theorem 15. *Let $\mathscr{L}=\mathscr{C}(X)$ be a family of all continuous functions on a topological space X. Let* L *be an exhausting order continuous strong sublinear functional on $\mathscr{L}$. Then there exists a σ-ring $\mathscr{S}$ and an order continuous strong submeasure μ on $\mathscr{S}$ such that*

$$\mathrm{L}f=\mathscr{I}_\mu f$$

for every f in $\mathscr{S}$.

Proof. The proof is 1) a simple conclusion of the fact that a family of all continuous functions on a topological space is a linear lattice with Stone's condition and 2) the last theorem.

Since every linear order continuous functional is exhausting we have:

Theorem 16. *Let $\mathscr{L}=\mathscr{C}(X)$ be a family of all continuous functions on a topological space X. Let* L *be an order continuous linear monotone functional on $\mathscr{L}$. Then* L *can be represented as an integral with respect to some measure on some σ-ring.*

Theorem 17. *Let X be a locally compact Hausdorff space. Let $\mathscr{C}_0(X)=\mathscr{L}$ be the set of all continuous functions with compact support. Let* L *be an exhausting order continuous strong sublinear functional on $\mathscr{L}$. Then there exists a σ-ring $\mathscr{S}$ of subsets of X containing all compact G_δ sets and an order continuous strong submeasure μ on $\mathscr{S}$ with the properties:*

$$\mu(E)=\inf\{\mu(U);\ E\subset U\in\mathscr{S},\ U \text{ is an open set}\}$$
$$\mu(E)=\sup\{\mu(C);\ C\subset E,\ C\in\mathscr{S},\ C \text{ is compact}\}$$

And moreover for f in $\mathscr{L}$

$$\mathrm{L}f=\mathscr{I}_\mu f.$$

Proof. Let $\mathscr{S}$, $\mathscr{S}_0$ and μ be the same as in the proof of Theorem 14. We shall prove first that the compact G_δ sets are in $\mathscr{S}$.

Let C be a compact G_δ set. Let $C=\cap_n G_n$, G_n being an open and $\bar{G}_n$ a compact set. Let f_n be a continuous function which is 1 on C and vanishes on $X-G_n$. Clearly $f_n\searrow\chi_C$ and $f_n\in\bar{\mathscr{L}}$ for $n=1, 2, 3, \ldots$,. By the theorem of Beppo—Levi $\chi_C\in\bar{\mathscr{L}}$ and so $C\in\mathscr{S}_0$.

We shall show now that μ has the desired property. Let $E\in\mathscr{S}$ with $\mu(E)<\infty$; then $\chi_E\in\bar{\mathscr{L}}^+$ (see the proof of Theorem 14). Let $g_n\in\mathscr{L}_\sigma$, $h_n\in\mathscr{L}_\delta$, $h_n\leqq\chi_E\leqq g_n$ and $L_\sigma(g_n-h_n)\leqq 1/n$.

Let

$$C_n=\{x;\ h_n(x)\geqq 1/n\}$$

and

$$U_n=\{x;\ g_n(x)>1-1/n\}.$$

The sets C_n are closed since h_n is the infimum of continuous functions and the sets U_n are open since g_n is the supremum of continuous functions. Clearly

$$\chi_E \leqq \chi_{U_n} \leqq g_n + (1/n)\chi_{U_n}$$

and so

$$\begin{aligned} \mu(U_n) - \mu(E) &= \bar{\mathrm{L}}\chi_{U_n} - \bar{\mathrm{L}}\chi_E \\ &\leqq \bar{\mathrm{L}}g_n + \mu(U_n)/n - \bar{\mathrm{L}}h_n \\ &\leqq \mathrm{L}(g_n - h_n) + \mu(U_n)/n \\ &\leqq (1 + \mu(U_n))/n, \end{aligned}$$

hence

$$\mu(E) = \inf \{\mu(U);\ U \supset E, U \textit{ is open}, U \in \mathscr{S}\}.$$

Now it is obvious that

$$h_n - h_n \wedge 1/n \leqq \chi_{C_n}.$$

By the mononicity and horizontal additivity of $\bar{\mathrm{L}}$

$$\bar{\mathrm{L}}H_n - \bar{\mathrm{L}}(h_n \wedge 1/n) \leqq \mu(C_n) \leqq \mu(E) \leqq \bar{\mathrm{L}}(g_n),$$

hence

$$\begin{aligned} \mu(E) - \mu(C_n) &\leqq \bar{\mathrm{L}}g_n - \bar{\mathrm{L}}h_n + \bar{\mathrm{L}}(h_n \wedge 1/n) \\ &\leqq \mathrm{L}(g_n - h_n) + \bar{\mathrm{L}}(h_n \wedge 1/n) \\ &\leqq (1 + \mu(E))/n. \end{aligned}$$

And so by the compactness of C_n we get

$$\mu(E) = \sup \{\mu(C);\ C \subset E, C \textit{ — compact}, C \in \mathscr{S}\}.$$

If $\mu(E) = \infty$, then the proof is clear.

We generalized some representation theorems of linear functionals. We do not believe the ordinary theorem about the representation of the linear functional on the space $\mathscr{L}^p$ to be valid, for sublinear functionals, since the most usefull and deepest theorem, the Radon—Nikodym theorem is not valid for strong submeasures.

§ 4. Representation without continuity

We turn our attention to the case of not necessary continuous functionals. We give first the representation theorem for linear functionals.

Theorem 18. *Let $\mathscr{L}$ be a linear lattice of bounded functions with $\mathscr{L}_s(\mathscr{A}) \subset \mathscr{L} \subset \mathscr{L}(\mathscr{A})$. ($\mathscr{A}$ is an algebra of subsets of a space X). Let l be a linear monotone functional on $\mathscr{L}$.* Then $\nu(A) = l\chi_A$ is a finitely additive measure on $\mathscr{A}$ and

$$lf = \int f \, \mathrm{d}\nu$$

for f in $\mathscr{L}$.

Proof. ν is clearly a finitely additive measure. If $\tilde{f}$ is a simple function, then obviously $l\tilde{f} = \int \tilde{f}\, d\nu$. If f is from $\mathscr{L}^+$ and $\tilde{f} \leqq f$, then $lf \geqq l\tilde{f} = \int \tilde{f}\, d\nu$ and so

$$lf \geqq \sup_{\substack{\tilde{f} \leqq f \\ \tilde{f}-\text{simple}}} \int \tilde{f}\, d\nu = \int f\, d\nu = \mathscr{I}_\nu f$$

We proved that

$$lf \geqq \int f\, d\nu = \mathscr{I}_\nu f$$

for f in $\mathscr{L}^+$. Let f be such a function from $\mathscr{L}^+$ that $lf > \int f\, d\nu$. We may assume that $f \leqq 1$. Then

$$l1 = l(1-f) + lf > \mathscr{I}_\nu(1-f) + \mathscr{I}_\nu f \geqq \mathscr{I}_\nu\, 1,$$

a contradiction. And so $lf = \mathscr{I} f$ for f in $\mathscr{L}^+$. Hence

$$lf = lf^+ - lf^- = \mathscr{I}_\nu f^+ - \mathscr{I}_\nu f^- = \mathscr{I}_\nu f,$$

and so we get

$$lf = \mathscr{I}_\nu f$$

for f in $\mathscr{L}$.

For the rest of the paper $\mathscr{A}$ will be an algebra of subsets of X. $\mathscr{L}$ will be a linear lattice of some bounded functions from $\mathscr{L}(\mathscr{A})$ which contains all simple functions. L will be a strong sublinear functional on $\mathscr{L}$ and μ will be a strong submeasure on $\mathscr{A}$ defined by $\mu(A) = \mathrm{L}\chi_A$ for A in $\mathscr{A}$.

Let us denote by $\mathscr{N}_\mu$ the set of all non negative, finitely additive measures ν on $\mathscr{A}$ with $\nu(A) \leqq \mu(A)$ for A in $\mathscr{A}$, and by $\mathscr{M}_{\mathrm{L}}$ the set of all monotone, linear functionals on $\mathscr{L}$ with $lf \leqq \mathrm{L}f$ for f in $\mathscr{L}^+$.

We shall need the following two lemmas for proving the main theorem of this paragraph.

Lemma 19. *Let f be in $\mathscr{L}^+$; then*

$$\mathscr{I}_\mu f = \max_{\nu \in \mathscr{N}_\mu} \mathscr{I} f.$$

Proof. See Theorem 22 of [4].

Lemma 20. *Let f be in $\mathscr{L}^+$; then*

$$\mathrm{L}f = \max_{l \in \mathscr{M}_{\mathrm{L}}} lf$$

Proof. Let f be in $\mathscr{L}^+$. We define a linear functional l_0 on the linear space $E_0 = \{af\,;\, a \text{ is real}\}$ by $l_0(af) = a\mathrm{L}f$. By Theorem 6 of [5] there exists an extension l of l_0 such that $l \leqq \mathrm{L}$ on $\mathscr{L}^+$ and l is linear and monotone. Clearly $\mathrm{L}f = lf$ and so

$$\mathrm{L}f = \max_{l \in \mathscr{M}_{\mathrm{L}}} lf.$$

Theorem 21. $\mathrm{L}f = \mathscr{I}_\mu f$ *for* f *in* $\mathscr{L}$.

Proof. By Lemma 3

$$\mathrm{L}f \geqq \mathscr{I}_\mu f$$

for f in $\mathscr{L}^+$.

We prove now the opposite inequality. Since f is in $\mathscr{L}^+$, we have

$$\mathrm{L}f = \max_{l \in \mathscr{M}_\mathrm{L}} lf = \max_{l \in \mathscr{M}_\mathrm{L}} \int f \, \mathrm{d}\nu_l \leqq \sup_{\nu \in \mathscr{N}_\mu} \int f \, \mathrm{d}\nu = \mathscr{I}_\mu f.$$

The first equality follows by Lemma 20, the second by Theorem 18 and the last by Lemma 19. Hence $\mathrm{L}f = \mathscr{I}_\mu f$ for f in $\mathscr{L}^+$. The proof for a not necessarily non-negative function is now a simple conclusion of Lemma 2.

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Received January 12, 1978

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ПРЕДСТАВЛЕНИЕ НЕЛИНЕЙНЫХ ФУНКЦИОНАЛОВ В ВИДЕ ИНТЕГРАЛА

Ян Шипош

Резюме

Главная цель, которую мы преследовали – определить по нелинейному функционалу L пред-меру $\mu = \mu_\mathrm{L}$, и получить представление L в виде интеграла.