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PRODUCT DECOMPOSITION OF A σ-RING

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Recently, A. D. Joshi of Poona University, India, raised the question: "When is it possible to decompose a given σ-ring of subsets of a Cartesian product $X\times Y$ of abstract sets X and Y as a measure-theoretic product of σ-rings in X and Y?" The present note gives a necessary and sufficient condition for such decomposability in the sense that a certain decomposition is proved to be the only possible. The characterization may be of some interest because once we can decompose a σ-algebra $\mathcal{V}$ on $X\times Y$ endowed with a measure m, we may try to express the measure m as a product of measures in X and Y, thus reducing integration on the measure space $(X\times Y, \mathcal{V}, m)$ to iterated integrals in the most important cases.

1. Notation and Notions

A σ-ring is a nonempty class $\mathcal{U}$ of subsets of an underlying set U such that, for any E, $F\in\mathcal{U}$, the set-theoretic difference $E\setminus F$ is in $\mathcal{U}$ and, for every sequence $\{E_n\}_{n=1}^{\infty}$ of sets in $\mathcal{U}$ we have $\cup_n E_n\in\mathcal{U}$. If U itself is an element of the σ-ring $\mathcal{U}$, then $\mathcal{U}$ is called a σ-algebra. Given any family $\mathcal{X}$ of subsets of U, we denote by $\sigma(\mathcal{X})$ the σ-ring generated by $\mathcal{X}$, i.e. the smallest σ-ring including $\mathcal{X}$. If $\mathcal{S}$ is a σ-ring of subsets of X and $\mathcal{T}$ a σ-ring of subsets of Y, their product $\mathcal{S}\otimes\mathcal{T}$ is the σ-ring (in $X\times Y$) generated by the family of all sets $S\times T$ with $S\in\mathcal{S}$ and $T\in\mathcal{T}$. Given a set $E\subset X\times Y$ and a point $x\in X$, we call $E_x=\{y\in Y\colon (x, y)\in E\}$ the x-section of E and, for a given $y\in Y$, the y-section of E is $E^y=\{x\in X\colon (x, y)\in E\}$. It is known that for $E\in\mathcal{S}\otimes\mathcal{T}$ we always have $E_x\in\mathcal{S}$ and $E^y\in\mathcal{T}$. (Cf. [1].)

2. Main result

Let X, Y be abstract sets and $\mathcal{V}$ a σ-ring of subsets of the Cartesian product $X\times Y$. We shall deal with the nontrivial case $\mathcal{V}\neq\{\emptyset\}$ only, because evidently $\{\emptyset\}=\{\emptyset\}\otimes\mathcal{T}=\mathcal{S}\otimes\{\emptyset\}$ with any σ-rings $\mathcal{S}$ on X and $\mathcal{T}$ on Y and no other decomposition is possible.

Theorem. *A σ-ring $\mathcal{V} \neq \{\emptyset\}$ of subsets of $X \times Y$ is a product of some σ-rings on X and Y if and only if*

$$\mathcal{V} = \sigma(\{E^y\colon E \in \mathcal{V},\ y \in Y\}) \otimes \sigma(\{E_x\colon E \in \mathcal{V},\ x \in X\}).$$

Proof. The "if" part is obvious. Suppose, therefore, that $\mathcal{V} = \mathcal{S} \otimes \mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ are σ-rings on X and Y, respectively. Denote $\mathcal{X} = \sigma(\{E^y\colon E \in \mathcal{V},\ y \in Y\})$, $\mathcal{Y} = \sigma(\{E_x\colon E \in \mathcal{V},\ x \in X\})$. We prove that $\mathcal{S} = \mathcal{X}$ and $\mathcal{T} = \mathcal{Y}$. Let $S \in \mathcal{S}$, take a nonempty $B \in \mathcal{T}$. (If $\mathcal{T} = \{\emptyset\}$, then $\mathcal{V} = \{\emptyset\}$, a contradiction.) Evidently, $S \times B \in \mathcal{S} \otimes \mathcal{T} = \mathcal{V}$ and hence, for $y \in B$, we have $(S \times B)^y = S \in \mathcal{X}$. This proves $\mathcal{S} \subset \mathcal{X}$ and the proof of $\mathcal{T} \subset \mathcal{Y}$ is analogous. To prove that $\mathcal{X} \subset \mathcal{S}$, observe that, for any $E \in \mathcal{V} = \mathcal{S} \otimes \mathcal{T}$ and each $y \in Y$, we have $E^y \in \mathcal{S}$. Therefore $\mathcal{S}$ includes a generator of $\mathcal{X}$ and, being a σ-ring, it includes the whole σ-ring $\mathcal{X}$. Similarly, $\mathcal{Y} \subset \mathcal{T}$ and the proof is complete.

3. Remarks

1. If $X \times Y \in \mathcal{V}$, then $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, so an analogy of the Theorem is true for decomposing a σ-algebra into a product of σ-algebras.

2. An example of a non-decomposable σ-ring can be obtained by considering $X = Y = \{a, b\}$, $\mathcal{V} = \{\emptyset, \{(a, a), (b, b)\}\}$. If there were $\mathcal{V} = \mathcal{S} \otimes \mathcal{T}$, we would have $\{a\} \in \mathcal{S}$, $\{b\} \in \mathcal{T}$, hence $\{(a, b)\} \in \mathcal{V}$, a contradiction.

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РАЗЛОЖЕНИЕ σ-КОЛЬЦА В ПРОИЗВЕДЕНИЕ

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Резюме

В статье доказано необходимое и достаточное условие для того, чтобы σ-кольцо подмножеств произведения двух абстрактных множеств было произведением σ-колец в координатных пространствах. В самом деле, доказано, что σ-кольца, порожденные сечениями множеств из данного σ-кольца, образуют единственное возможное его разложение.