Mathematica Slovaca

Jozef Eliáš

A note on the differential equation $y^{(n)}(x) + f(x)y^{\alpha}(x) = 0,\ 0 < \alpha < 1$

Mathematica Slovaca, Vol. 34 (1984), No. 2, 135--140

Persistent URL: http://dml.cz/dmlcz/128882

Math. Slovaca 34, 1984, No. 2, 135—140

A NOTE ON THE DIFFERENTIAL EQUATION $y^{(n)}(x)+f(x)y^{\alpha}(x)=0,\quad 0<\alpha<1$

JOZEF ELIAŠ

Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

In this paper we shall consider the nonlinear differential equation

$$y^{(n)}(x)+f(x)y^{\alpha}(x)=0,\quad n>1,\quad 0<\alpha<1, \tag{1}$$

where $\alpha=p/q$ and p, q are odd natural numbers and the function $f(x)$ is continuous in the considered interval.

In the following part of the paper we shall need the following lemma.

Lemma. *Let $y(x)$ be a solution of equation (1) defined on the interval $\langle x_1, x_2)(x_1\geqq x_0)$ such that it satisfies the initial conditions:*

$$y^{(i)}(x_1)=y_i,\quad i=n-1,,n-2,\ \dots,\ 2,\ 1,\ 0 \tag{2}$$

where y_i are arbitrary real numbers and $y^{(0)}(x)=y(x)$. Then

$$y^{(i)}(x)=\sum_{k=i}^{n-1}y_k\,\frac{(x-x_1)^{k-i}}{(k-i)!}-\int_{x_1}^{x}\frac{(x-t)^{n-i-1}}{(n-i-1)!}f(t)y^{\alpha}(t)\,\mathrm{d}t, \tag{3}$$

holds for $x\geqq x_i\geqq x_0$ and $i=n-1,\ n-2,\ \dots,\ 1,0$.

Proof. Integrating (1) from x_1 to $x(x\geqq x_1)$ we have

$$y^{(n-1)}(x)=y^{(n-1)}(x_1)=\int_{x_1}^{x}f(t)y^{\alpha}(t)\,\mathrm{d}t.$$

According to (2), $y^{(n-1)}(x_1)=y_{n-1}$, then the last equality has the form

$$y^{(n-1)}(x)=y_{n-1}-\int_{x_1}^{x_2}f(t)y^{\alpha}(t)\,\mathrm{d}t.$$

Integrating the last equality from x_1 to x and utilizing the initial condition $y^{(n-2)}(x_1)=y_{n-2}$ we obtain

$$y^{(n-2)}(x)=y_{n-2}+y_{n-1}\frac{(x-x_1)}{1!}-\int_{x_1}^{x}\mathrm{d}\xi\int_{x_1}^{\xi}f(s)y^{\alpha}(s)\,\mathrm{d}s.$$

Changing the order of integration we get

$$y^{(n-2)}(x)=y_{n-2}+y_{n-1}\frac{(x-x_1)}{1!}-\int_{x_1}^{x}\frac{(x-t)}{1!}f(t)y^{\alpha}(t)\,\mathrm{d}t.$$

If we repeat the above argument we obtain that Lemma holds for $i=n-1$, $n-2, \ldots, 1, 0$.

Theorem 1. *Let the function $f(x)$ be continuous on the interval $\langle x_0, \infty)$. Then every solution of the differential equation (1) can be extended to the whole interval $\langle x_0, \infty)$.*

Proof. Let $y(x)$ be the solution of equation (1), defined on the interval $\langle x_1, x_2)(x_1 \geqq x_0)$ such that it satisfies the initial conditions (2). From (3), for $i=0$, we have

$$y(x)=y_0+y_1\frac{(x-x_1)}{1!}+y_2\frac{(x-x_1)^2}{2!}+\ldots+y_{n-1}\frac{(x-x_1)^{n-1}}{(n-1)!}-$$

$$-\frac{1}{(n-1)!}\int_{x_1}^{x}(x-t)^{n-1}f(t)y^{\alpha}(t)\,\mathrm{d}t.$$

From here, for $x-x_1\geqq 1$, we get

$$|y(x)|\leqq(x-x_1)^{n-1}\left(|y_0|+|y_1|+\ldots+|y_{n-1}|+\int_{x_1}^{x}|f(t)||y(t)|^{\alpha}\,\mathrm{d}t\right). \qquad (4)$$

In case $x-x_1<1$, we get the following estimate

$$|y(x)|\leqq|y_0|+|y_1|+\ldots+|y_{n-1}|+\int_{x_1}^{x}|f(t)||y(t)|^{\alpha}\,\mathrm{d}t$$

and then we proceed in the same way as in the case $x-x_1\geqq 1$.

From inequality (4), if we raise both its sides to the power α and multiply them by $f(x)$, we obtain

$$\frac{|f(x)||y(x)|^{\alpha}}{\left(|y_0|+|y_1|+\ldots+|y_{n-1}|+\int_{x_1}^{x}|f(t)||y(t)|^{\alpha}\,\mathrm{d}t\right)}\leqq(x-x_1)^{(n-1)\alpha}|f(x)|.$$

Integrating the last inequality from x_1 to x, we get

$$\left(|y_0|+|y_1|+\ldots+|y_{n-1}|+\int_{x_1}^{x}|f(t)||y(t)|^{\alpha}\,\mathrm{d}t\right)^{1-\alpha}\leqq$$

$$\leqq(1-\alpha)\int_{x_1}^{x}(t-x_1)^{(n-1)\alpha}|f(t)|\,\mathrm{d}t+(|y_0|+|y_1|+\ldots+|y_{n-1}|)^{1-\alpha}$$

and finally we obtain the inequality

$$|y(x)| \leqq (x-x_1)^{n-1} \left\{(1-\alpha)\int_{x_1}^{x} (t-x_1)^{n-1}|f(t)|\,dt + \right.$$

$$\left. + (|y_0|+|y_1|+\ldots+|y_{n-1}|)^{(1-\alpha)}\right\}^{1/1-\alpha}.$$

Since the right side of the last inequality is defined and continuous for all $x \geqq x_1$, the solution $y(x)$ is bounded in the interval $\langle x_1, x_2)$.

Now we prove that $y^{(i)}(x)$, $i=n-1, \ldots, 2, 1$ are bounded. From (3) it follows

$$|y^{(i)}(x)| \leqq \sum_{k=i}^{n-1} |y_k| \frac{(x-x_1)^{k-i}}{(k-i)!} + \int_{x_1}^{x} \frac{(x-t)^{n-i-1}}{(n-i-1)!} |f(t)||y(t)|^{\alpha}\,dt,$$

for $i=n-1, n-2, \ldots, 2, 1$.

Hence $y^{(i)}(x)$, $i=n-1, \ldots, 2, 1$, is bounded for all $x \geqq x_1$. Since $y(x)$ and $y^{(i)}(x)$ are bounded for all $x \geqq x_1$, the solution $y(x)$ can be extended to the whole interval $\langle x_1, \infty)$ (see [2], page 24—27). In the interval $(x_1, x_2\rangle$, the consideration is analogous. The proof of Theorem 1 is completed.

Corollary 1. *If $n=2$, we get Theorem 1 from paper* [1].

Remark. The following estimate follows from inequality (5). We suppose that $\int^{\infty} x^{\alpha(n-1)}|f(x)|\,dx < \infty$, then for every solution $y(x)$ of equation (1) there exists a constant K such that $|y(x)| \leqq Kx^{n-1}$ for all $x \geqq x_1$.

Theorem 2. *Let the function $f(x)$ be continuous on the interval $\langle x_0, \infty)$ and $\int^{\infty} x^{(\alpha+1)(n-1)}|f(x)|\,dx < \infty$. Then for every solution $y(x)$ of equation (1) there exist $\lim_{x\to\infty} y^{(i)}(x)$, $i=1, 2, \ldots, n-1$ and*

$$y^{(n-i)}(x) = c_1 \frac{x^{i-1}}{(i-1)!} + c_2 \frac{x^{i-2}}{(i-2)!} + \ldots + c_{i-1}x + c_i + o(1), \tag{6}$$

where $i=1, 2, \ldots, n$ and $c_1, \ldots, c_i$ are suitable constants.

Proof. First we prove that $\lim_{x\to\infty} y^{(i)}(x)$ exists. Let $y(x)$ be a solution of equation (1). From (3) for i, $i=1, 2, \ldots, n-1$, we have

$$y^{(i)}(x) = \sum_{k=i}^{n-1} y_k \frac{(x-x_1)^{k-i}}{(k-i)!} - \int_{x_1}^{x} \frac{(x-t)^{n-i-1}}{(n-i-1)!} f(t) y^{\alpha}(t)\,dt. \tag{7}$$

According to Remark, $|y(x)| \leqq Kx^{n-1}$ for all $x \geqq x_1$, because

$$\left|\int_{x_1}^{x} (x-t)^{n-i-1} f(t) y^{\alpha}(t)\,\mathrm{d}t\right| \leqq K^{\alpha}\int_{x_1}^{x} x^{n-i-1}|f(x)|x^{\alpha(n-1)}\,\mathrm{d}x =$$

$$= K^{\alpha}\int_{x_1}^{\infty} |f(x)|x^{\alpha(n-1)+(n-i-1)}\,\mathrm{d}x < \infty.$$

From here it follows that the integral

$$\int_{x_1}^{\infty} \frac{(x-t)^{n-i-1}}{(n-i-1)!} f(t) y^{\alpha}(t)\,\mathrm{d}t$$

exists and from (7) it follows that $\lim_{x\to\infty} y^{(i)}(x)$ exists, $i = 1, 2, \ldots, n-1$.

Now we prove the second part of Theorem 2. According to the first part $\lim_{x\to\infty} y^{(n-1)}(x)$ exists. We denote it by c_1. Integrating (1) from x to ∞, we obtain

$$y^{(n-1)}(x) = c_1 + \int_{x}^{\infty} f(t) y^{\alpha}(t)\,\mathrm{d}t.$$

Integrating the last equality from x_1 to x, we obtain

$$y^{(n-2)}(x) = y^{(n-2)}(x_1) + c_1 x - c_1 x_1 + \int_{x_1}^{x}\left[\int_{s}^{\infty} f(t) y^{\alpha}(t)\,\mathrm{d}t\right]\mathrm{d}s.$$

If we change the order of integration in the last integral, we get

$$y^{(n-2)}(x) = y^{(n-2)}(x_1) + c_1 x - c_1 x_1 + \int_{x_1}^{\infty} (t-x_1) f(t) y^{\alpha}(t)\,\mathrm{d}t +$$

$$+ \int_{x}^{\infty} (x-t) f(t) y^{\alpha}(t)\,\mathrm{d}t.$$

Since

$$\left|\int_{x_1}^{x} (t-x_1) f(t) y^{\alpha}(t)\,\mathrm{d}t\right| \leqq K^{\alpha}\int_{x_1}^{\infty} x^{\alpha(n-1)+1}|f(x)|\,\mathrm{d}x < \infty,$$

the integral

$$\int_{x_1}^{\infty} (t-x_1) f(t) y^{\alpha}(t)\,\mathrm{d}t$$

exists for $x_1 \geqq x_0$. If we denote

$$y^{(n-2)}(x_1) - c_1 x_1 + \int_{x_1}^{\infty} (t-x_1) f(t) y^{\alpha}(t)\,\mathrm{d}t = c_2,$$

then we can write

$$y^{(n-2)}(x)=c_1x+c_2+\int_{x_1}^{\infty}(x-t)f(t)y^{\alpha}(t)\,\mathrm{d}t.$$

Suppose that it has been proved that (6) holds for $i=j-1$, where j is some fixed integer such that $2\leqq j-1\leqq n$ i.e.

$$y^{(n-j+1)}(x)=c_1\frac{x^{j-2}}{(j-2)!}+c_2\frac{x^{j-3}}{(j-3)!}+\ldots+c_{j-1}+\int_x^{\infty}\frac{(x-t)^{j-2}}{(j-2)!}f(t)y^{\alpha}(t)\,\mathrm{d}t.$$

Integrating the last equality from x_1 to x we get

$$y^{(n-j)}(x)=y^{(n-j)}(x_1)+c_1\frac{x^{j-1}}{(j-1)!}-c_1\frac{x_1^{j-1}}{(j-1)!}+c_2\frac{x^{j-2}}{(j-2)!}-$$

$$-c_2\frac{x_1^{j-2}}{(j-2)!}+\ldots+c_{j-1}x-c_{j-1}x_1+\int_{x_1}^{x}\left[\int_s^{\infty}\frac{(x-t)^{j-2}}{(j-2)!}f(t)y^{\alpha}(t)\,\mathrm{d}t\right]\mathrm{d}s.$$

If we change the order of integration we get

$$y^{(n-j)}(x)=y^{(n-j)}(x_1)+c_1\frac{x^{j-1}}{(j-1)!}-c_1\frac{x_1^{j-1}}{(j-1)!}+c_2\frac{x^{j-2}}{(j-2)!}-c_2\frac{x_1^{j-2}}{(j-2)!}+$$

$$+\ldots+c_{j-1}x-c_{j-1}x_1+\int_{x_1}^{\infty}\frac{(t-x_1)^{j-1}}{(j-1)!}f(t)y^{\alpha}(t)\,\mathrm{d}t+\int_x^{\infty}\frac{(x-t)^{j-1}}{(x-1)!}f(t)y^{\alpha}(t)\,\mathrm{d}t.$$

Since

$$\left|\int_{x_1}^{x}(t-x_1)^{j-1}f(t)y^{\alpha}(t)\,\mathrm{d}t\right|\leqq K^{\alpha}\int_{x_1}^{\infty}x^{j-1}|f(x)|x^{\alpha(n-1)}\,\mathrm{d}x=$$

$$=K^{\alpha}\int_{x_1}^{\infty}|f(x)|x^{\alpha(n-1)+(j-1)}\,\mathrm{d}x$$

then $\int_x^{\infty}(x-t)^{j-1}f(t)y^{\alpha}(t)\,\mathrm{d}t$ converges and $\int_x^{\infty}(x-t)^{j-1}f(t)y^{\alpha}(t)\,\mathrm{d}t=o(1)$.

If we denote

$$y^{(n-j)}(x_1)-c_1\frac{x_1^{j-1}}{(j-1)!}-\ldots-c_{j-1}x_1+\int_{x_1}^{\infty}\frac{(t-x_1)^{j-1}}{(j-1)!}f(t)y^{\alpha}(t)\,\mathrm{d}t=c_j$$

then we obtain

$$y^{(n-j)}(x)=c_1\frac{x^{j-1}}{(j-1)!}+c_2\frac{x^{j-2}}{(j-2)!}+\ldots+c_{j-1}x+c_j+o(1)$$

for $j=1, 2, \ldots, n-1, n$. This completes the proof of the second part of Theorem 2.

Corollary 2. *If $n=2$, we get Theorem 2 from paper [1].*

REFERENCES

[1] BELOHOREC, Š.: On some properties of the equation $y''(x)+f(x)y^{\alpha}(x)=0$, $0<\alpha<1$, Mat. časopis 17 (1967), 10–19.

[2] ХАРТМАН, Ф.: Обыкновенные дифференциальные уравнения (Russian translation), Мир, Москва 1970.

Received October 15, 1981

Katedra matematiky a deskriptívnej geometrie
Strojnícka fakulta SVŠT
Gottwaldovo nám. 17
880 19 Bratislava

О ДИФФЕРЕНЦИАЛЬНОМ УРАВНЕНИИ $y^{(n)}(x)+f(x)y^{\alpha}(x)=0$, $0<\alpha<1$

Jozef Eliaš

Резюме

В работе рассматривается дифференциальное уравнение

$$y^{(n)}(x)+f(x)y^{\alpha}(x)=0, \quad 0<\alpha<1 \tag{1}$$

где $f(x)\in C[\langle x_0, \infty)]$, $\alpha=p/q$, p, q – нечетные натуральные числа. Доказывается, что каждое решение уравнения (1) может быть продолжено на интервал $\langle x_0, \infty)$. Приведены достаточные условия, чтобы для каждого решения уравнения (1) существовал

$$\lim_{x\to\infty} y^{(n-1)}(x).$$

Для каждого решения уравнения (1) была найдена его асимптотическая форма