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REMARKS ON THE ZERO-ONE LAW

HARRY I. MILLER*—BOŠKO ŽIVALJEVIĆ

1. Introduction

The beautiful theorem of Kolmogorov, often called the zero-one law ([1], pg. 247), states the following:

Theorem. *If* $(X_n)_{n=1}^{\infty}$ *is a sequence of independent random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$, *and if*

$$A \in \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \dots),$$

then either

$$P(A)=0 \quad \text{or} \quad P(A)=1.$$

Here $\sigma(X_n, X_{n+1}, \dots)$ is the smallest σ-algebra of subsets of Ω containing all sets of the form $X_i^{-1}((a, \infty))$, where a is any real number and $i \in \{n, n+1, \dots\}$.

The following corollary of Kolmogorov's Theorem can be obtained by considering characteristic functions of independent events.

Corollary. *If* $(A_n)_{n=1}^{\infty}$ *is an independent sequence of events (in a probability space, say* $(\Omega, \mathcal{F}, P)$*), then for each event A in the tail σ-field* $\bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \dots)$, $P(A)$ *is either* 0 *or* 1.

Here $\sigma(A_n, A_{n+1}, \dots)$ is the smallest σ-algebra containing the sets A_i, $i \geq n$.

It is not difficult to show that the last mentioned result implies the following:

Theorem A. *If* $A \subset [0, 1)$ *is a Lesbesgue measurable „tail set", then the Lesbesgue measure of A is either* 0 *or* 1.

Definition. $A \subset [0, 1)$ *is called a „tail set" if and only if* $x \in A$ *and* $x \sim_T y$ *implies* $y \in A$.

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Here $x \sim_T y$ $(x, y \in [0, 1))$ means that there exists a positive integer N such that $x_i(x) = x_i(y)$ for every $i \geq N$, where for each $a \in [0, 1)$

$a = \sum_{i=1}^{\infty} x_i(a) 2^{-i}$ is the unique binary expansion of a (i.e. $x_i(a) \in \{0, 1\}$ for each i) with $\sum_{i=1}^{\infty} x_i(a) < \infty$ in case a is of the form $\frac{m}{2^n}$.

Theorem A can be shown to follow from the zero-one law of Kolmogorov with X_n taken to be the function x_n (i.e. the n^{th} binary digit function) for each n. Also Theorem A can be obtained from the Corollary given above with A_n given by $A_n = \{x \in [0, 1) : x_n(x) = 1\}$ for each n.

The following Baire set analogue of Theorem A holds ([4], pg. 85):

Theorem B. *If $A \subset [0, 1)$ is a „tail set" possessing the property of Baire, then either A or $(0, 1) \setminus A$ is a set of the first Baire category.*

Definition. *A subset A of a topological space X is said to possess the Baire property, or be a Baire set, if A can be written in the form:*

$A = (G \setminus P) \cup Q$, where G is an open set and P and Q are sets of the first Baire category.

The relationship between measurable sets and Baire sets is carefully studied in Oxtoby's book „Measure and Category" [4].

For completeness we shall offer the proofs of Theorems A and B in outlines.

Proof of Theorem A. If A is a „tail set" $(A \subset [0, 1))$, then for each n the sets

$$\left\{A \cap \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right\}_{k=1}^{2^n}$$

are congruent and therefore if A is Lesbesgue measurable, each of these sets has the same Lesbesgue measure, namely $\frac{m(A)}{2^n}$, where m denotes the Lesbesgue measure. Therefore A and each set $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ are independent (two Lesbesgue measurable subsets B and C of $[0, 1)$ are said to be independent if $m(B \cap C) = m(B)m(C)$) since

$$m\left(A \cap \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right) = m(A) \cdot m\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right)$$

for each positive integer n and each k, $1 \leq k \leq 2^n$. From this it follows that A and any set that is the union of sets of the form

$$\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$$

are independent. Since any measurable set can be approximated by sets of this form it can be shown that A and any Lesbesgue measurable subset B of $[0, 1)$ are independent and therefore

$$P(A) = P(A \cap A) = P(A)P(A),$$

completing the proof.

Proof of Theorem B. Suppose $A \subset [0, 1)$ is a "tail set" possessing the Baire property. If A is not a set of the first category, then A can be written in the form $A = (G \backslash P) \cup Q$, where G is a non-empty open set and P and Q are sets of the first Baire category. Since $G \neq \emptyset$ and an open A contains some set of the form

$$\left[\frac{k_0-1}{2^n}, \frac{k_0}{2^n}\right) \backslash P_{k_0},$$

where P_{k_0} is of the first Baire category and

$$P_{k_0} \subset \left[\frac{k_0-1}{2^n}, \frac{k_0}{2^n}\right).$$

Therefore, as A is a "tail set", each of the sets

$$\left\{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \cap A\right\}_{k=1}^{2^n}$$

is congruent and therefore

$$A \supset \bigcup_{k=1}^{2^n} \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \backslash P_k,$$

where P_k is congruent to P_{k_0} for each k.

Therefore $[0, 1) \backslash A \subset \bigcup_{k=1}^{2^n} P_k$ is a set of the first Baire category.

In this paper we show that the hypotheses that A is measurable in theorem A and that A is a Baire set in Theorem B are not redundant. We give two proofs, one using a standard analysis and the other using a non-standard analysis, of the fact that if $A \subset [0, 1)$ is a "tail set", then A need not be Lesbesgue measurable, nor a Baire set. In addition questions about general equivalence relations on $[0, 1)$, having countable equivalance classes rather than only the equivalence relation $\sim_T$ considered in our introduction in connection with Theorems A and B, are considered.

2. Results

Theorem 1. *There exists a "tail set" A, $A \subset [0, 1)$, that is non-measurable and lacks the property of Baire.*

First (standard) proof. Our proof imitates the proof of Theorem 5.3 (due to F. Bernstein) on page 23 in [4]. Let c denote the cardinal number of the continuum (i.e. the real line). By the well-ordering principle and the fact that the class $\mathfrak{A}$ of uncountable closed subsets of [0, 1) has cardinality c, $\mathfrak{A}$ can be indexed by the ordinal numbers less than ω_c, where ω_c is the first ordinal having c predecessors, that is can be written as

$$\mathfrak{A} = \{U_\alpha: \alpha < \omega_c\}.$$

Assume further that [0, 1), and therefore each member of $\mathfrak{A}$ has been well ordered.

Let $O_1 = \{p \in [0, 1): p \sim p_1\}$, where p_1 is the first element in U_1 (the first set in $\mathfrak{A}$) and $\sim_T$ is the equivalence relation on [0, 1) given in the introduction. Let q_1 denote the first element in $U_1 \backslash P_1$ (which is nonempty since the cardinality of U_1 is c (Lemma 5.1, p.g. 23., [4])) and P_1 is countable. Let $Q = \{q \in [0, 1): q \sim_T q_1\}$. Let p_2 be the first member of $U_2 \backslash (P_1 \cup Q_1)$, again this set is non-empty by the above remarks.

Set $P_2 = \{p \in [0, 1): p \sim_T p_2\}$. Let q_2 denote the first element in $U_2 \backslash (P_1 \cup P_2 \cup Q_1)$ and let $Q_2 = \{q \in [0, 1): q \sim_T q_2\}$. Suppose that $1 < \alpha < \omega_c$, and that the equivalence classes (of $\sim_T$) P_β and Q_β have been defined for all $\beta < \alpha$ in such a way that:

a) $P_\beta \cap U_\beta \neq \emptyset$ and $Q_\beta \cap U_\beta \neq \emptyset$ for all $\beta, \beta < \alpha$.

b) $P_{\beta_1} \cap P_{\beta_2} = \emptyset$, $Q_{\beta_1} \cap Q_{\beta_2} = \emptyset$, and $P_{\beta_1} \cap Q_{\beta_2} = \emptyset$ for all $\beta_1, \beta_2 < \alpha$, $\beta_1 \neq \beta_2$.

Let p_α be the first element of $U_\alpha \backslash \bigcup_{\beta<\alpha} (P_\beta \cup Q_\beta)$, which is a non-empty set since the cardinality of U_α is c and $\bigcup_{\beta<\alpha} P_\beta \cup Q_\beta$ is the union of less than c-many countable sets and so has the cardinality less than c.

Let $P_\alpha = \{p \in [0, 1): p \sim_T p_\alpha\}$. Let q_α be the first element in

$$U_\alpha \backslash \left\{ \bigcup_{\beta<\alpha} (P_\beta \cup Q_\beta) \cup P_\alpha \right\} \quad \text{and let}$$

$Q_\alpha = \{q \in [0, 1): q \sim_T q_\alpha\}$.

Then clearly the collections of sets $\{P_\beta\}_{\beta<\alpha}$ and $\{Q_\beta\}_{\beta\leq\alpha}$ satisfy conditions a) and b) with $<$ replaced by $\leq$ everywhere. Therefore by transfinite induction it follows that there exist two collections of equivalence classes (of $\sim_T$), $\{P_\alpha\}_{\alpha<\omega_c}$ $\{Q_\alpha\}_{\alpha<\omega_c}$ satisfying a) and b).

Put

$$A = \bigcup_{\alpha<\omega_c} P_\alpha.$$

Since $p_\alpha \in A \cap U_\alpha$ and $q_\alpha \in ([0, 1) \backslash A) \cap U_\alpha$ for each $\alpha < \omega_c$, the set A, which is clearly a "tail set", has the property that both it and its relative complement ($[0, 1) \backslash A$) neet every uncountable closed subset of [0, 1). From this it follows, exactly as in the proof of Theorem 5.4 on pg. 24 in [4] that A is non-measurable and lacks the property of Baire.

We now will give a non-standard proof of Theorem 1. The notations used here, the usual ones of non-standard analysis, can be found in [2] or [3].

Second (non-standard proof). Let U denote the standard universe with the individuals set R of real numbers. N denotes the set of natural numbers, Z the set of integers and $P[0, 1)$ the collection of all subsets of $[0, 1)$. Then we have

$$U:=(\forall n\in N)(\exists A_n\in P[0,1))(\exists B_n\in P[0,1))(F_1\wedge F_2\wedge F_3\wedge F_4)$$

where:

$$F_1=(\forall x\in A_n)(\forall m\in Z)\left(x+\frac{m}{2^n}\in[0,1)\Rightarrow x+\frac{m}{2^n}\in A_n\right)$$

$$F_2=(\forall x\in[0,1))(x\in A_n\Leftrightarrow 1-x\in B_n)$$

$$F_3=(A_n\cup B_n=[0,1)\setminus I_n)$$

$$F_4=(A_n\cap B_n=\emptyset)$$

and

$$I_n=\left\{\frac{m}{2^{n+1}}:\ 0\leq m<2^{n+1}\right\}.$$

To see that sets $(A_n)_{n\in N}$ and $(B_n)_{n\in N}$ exist one need only consider the following elementary examples:

$$A_n=\bigcup_{k=0}^{2^n-1}\left(\frac{2k}{2^{n+1}},\ \frac{2k+1}{2^{n+1}}\right),\quad B_n=\bigcup_{k=0}^{2^n-1}\left(\frac{2k+1}{2^{n+1}},\ \frac{2k+2}{2^{n+1}}\right).$$

Clearly F_1 and F_2 imply that

$$U:=F_1'$$

$$F_1'=(\forall x\in B_n)(\forall m\in Z)\left(x+\frac{m}{2^n}\in[0,1)\Rightarrow x+\frac{m}{2^n}\in B_n\right).$$

Transforming the above expression by the $*$ — transformation we have:

$$*U:=(\forall n\in *N)(\exists A_n\in *P[0,1))(\exists B_n\in *P[0,1))[*F_1\wedge *F_2\wedge *F_3\wedge *F_4]$$

where:

$$*F_1=(\forall x\in A_n)(\forall m\in *Z)\left(x+\frac{m}{2^n}\in *[0,1)\Rightarrow x+\frac{m}{2^n}\in A_n\right)$$

$$*F_2=(\forall x\in *[0,1))(x\in A_n\Leftrightarrow 1-x\in B_n)$$

$$*F_3 = (\forall A_n \cup B_n = *[0, 1) \setminus *I_n)$$

$$*F_4 = F_4$$

and

$$*U: = *F_1'$$

$$*F_1' = (\forall x \in B_n)(\forall m \in *Z)\left(x + \frac{m}{2^n} \in *[0, 1) \Rightarrow x + \frac{m}{2^n} \in B_n\right).$$

Let $v \in *N \setminus N$ and set $A_v' = A_v \cap [0, 1)$. We now proceed to show that A_v' is a "tail set" that is nonmeasurable and lacks the Baire property.

First A_v' is a "tail set". This is true because

$$x \in A_v' \quad \text{and} \quad x + \frac{m}{2^n} \in [0, 1)\,(n \in N, 0 \leq m \leq 2^n)$$

implies $x \in A_v$ and $x + \dfrac{2^{v-n}m}{2^v} \in [0, 1)$ and therefore by $*F_1$ $x + \dfrac{2^{v-n}m}{2^v} \in A_v$. However, it is clear that $x + \dfrac{m}{2^n}$ is a standard element and therefore

$$x + \frac{m}{2^n} \in A_v'.$$

In an analogous way, using $*F_1'$, we conclude that

$$B_v' = B_v \cap [0, 1) \quad \text{is a "tail set".}$$

Because of $*F_3$ we have

$$A_v' \cup B_v' = [0, 1) \setminus D, \text{ where } D = \left\{\frac{m}{2^n}: \; m, n \in N\right\},$$

since $*I_v \cap [0, 1) = D$. Furthermore from F_4 we conclude that $A_v' \cap B_v' = \emptyset$. Condition $*F_2$ implies that A_v' and B_v' are congruent and therefore $m(A_v') = m(B_v')$ if A_v' is measurable. In addition, the Baire categories of A_v' and B_v' are the same.

If A_v' is measurable, then because of Theorem A we have: either the measure of A_v' is zero or one. If $m(A_v') = 0$, then $m(B_v') = 0$ and therefore $1 = m([0, 1)) = m(A_v' \cup B_v' \cup D) = 0$, if $m(A_v') = 1$, then $m([0, 1)) = m(A_v' \cup B_v' \cup D) = 2$.

If A_v' is a Baire set, then Theorem B implies that either $[0, 1) \setminus A_v' = B_v' \cup D$ or A_v' is a set of the first Baire category. But since these two sets have the same Baire category this would imply that $[0, 1)$ is of the first Baire category. Therefore A_v' is not a Baire set.

Remark 1. Suppose that a non-standard extension $*U$ of the superstructure U has been given by the non-principle ultrafilter D over the set of natural numbers and that v denotes the equivalence class of sequences determined by the identity sequence i (i.e. $i\colon N\to N$ and $i(k)=k$ for each k).

Then

$$A_v=\left(\prod_{i\in N} A_i\right)\Big/D,\quad \text{that is } A_v$$

consists of all classes (mod D) of sequences $a\colon N\to[0, 1)$ such that $a(n)\in A_n$ for each n.

In this case A_v' consists of classes of sequences a

$a\colon N\to[0, 1)$, which are D equivalent with some sequence $\hat{x}\colon N\to[0, 1)$, $\hat{x}(n)=x$ for every $n\in N$ and $x\in[0, 1)$.

A_v' can be written in standard form as follows

$$A_v'=\bigcup_{I\in D}\bigcap_{k\in I} A_k.$$

When we consider all non-principle ultra-filters on the set of natural numbers and a fixed infinite natural number v as above, then we obtain different sets A_v'. In fact in this case the intersection of all these sets A_v' is $\bigcap_{n\in N} A_n$, that is all points in $[0, 1)$, excluding those of the form $\frac{m}{2^n}$ $(m, n\in N)$, whose bose 4 representation contains only zeroes and twos. This set is a nowhere dense set of measure zero.

Definition. *If $A\subset[0, 1)$, $T(A)$ will denote the smallest "tail set" containing A, i.e.*

$$T(A)=\bigcap[B\colon\ B\subset[0, 1),\ A\subset B,\ B \text{ a "tail set"}].$$

Then it is very easy to see that the following two propositions hold.

Proposition 1. *If $A\subset[0, 1)$ is a measurable, then $T(A)$ is measurable and therefore by Theorem A, $m(T(A))=0$ or 1. If $A\subset[0, 1)$ is a Baire set, then $T(A)$ is also a Baire set and therefore by Theorem B, either $T(A)$ or $[0, 1)\setminus T(A)$ is a set of the first Baire category.*

Proof. Set $Q=\left\{\frac{m}{2^n}\colon\ m\in Z,\ n\in N\right\}$, then

$$T(A)=\bigcup[q\oplus A\colon q\in Q],\quad \text{where}\quad q\oplus A=(q+A)\cap[0, 1),$$

and

$$q+A=\{q+a\colon\ a\in A\}.$$

Clearly $q\oplus A$ is measurable if A is measurable as Q is countable. Therefore it follows that $T(A)$ is measurable. The same proof shows that $T(A)$ is a Baire set whenever A is a Baire set.

Proposition 2. *If $A \subset [0, 1)$ is a measurable, then*

a) $m(A)=0$ *implies that* $m(T(A))=0$ *and*
b) $m(A)>0$ *implies that* $m(T(A))=1$.

If $A \subset [0, 1)$ is a Baire set, then
a') *A being a set of the first Baire category implies that $T(A)$ is a set of the first Baire category and*
b') *A being a set of the second Baire category implies that $[0, 1)\backslash T(A)$ is a set of the first Baire category.*

Proof. These results are immediate by Theorems A and B and the fact that A can be written in the form $T(A)=\bigcup[q\oplus A:\ q\in Q]$.

In this paper we have considered the equivalence relation $\sim_T$ on $[0, 1)$, where $x\sim_T y$ if and only if $x_i(x)=x_i(y)$ for all but finitely many i's. Notice that each equivalence class of $\sim_T$ has countably many elements and is dense in $[0, 1)$. The zero-one law (Theorem A) says that any measurable set obtained as the union of equivalence classes of $\sim_T$ must have measure either zero or one. It is natural to ask the following question.

Question: Does there exist an equivalence relation $\sim$ on $[0, 1)$ such that the equivalence classes of $\sim$ are each countable and dense in $[0, 1)$ and such that for each x $(0\leq x\leq 1)$, there exists a subcollection of the equivalence classes of $\sim$ whose union, denoted A_x, is measurable and $m(A_x)=x$?
We now show that it is possible to construct an equivalence relation with the above mentioned properties.

Theorem 2. *There exists an equivalence relation $\sim$ with the properties mentioned in the question above.*

Proof. Let $H\subset[0, 1)$ be a Hamel basis for the real numbers containing a rational number and having measure zero.
$k(H)$, the cardinality of H, is c. Therefore H can be written in the form

$$H=\bigcup_{n=1}^{\infty} H_n, \quad \text{where} \quad k(H_n)=c \quad \text{for}$$

each n and the sets

$$\{H_n\}_{n=1}^{\infty}, \quad \text{are pairwise disjoint.}$$

For each $h\in H$ let $C_h=\{h+r:\ r\in Q\}\cap[0, 1)$, where Q is the set of all rational numbers. Notice that the sets $\{C_h\}_{h\in H}$ are pairwise disjoint since H is a Hamel basis containing a rational number. The interval $[0, 1)$ can be written in the form $[0, 1)=\{x_\alpha:\ \alpha<\omega_c\}$.
Furthermore,

$$\bigcup_{h\in H} C_h=\bigcup_{r\in Q}(r+H)\cap[0, 1), \quad \text{where}$$

$r+H=\{r+h\colon\ h\in H\}$, and therefore $C=\bigcup_{h\in H} C_h$ has measure zero (since $m(H)=0$). For each $n\in N$ let

$$I_n=\left[\frac{2^{n-1}-1}{2^{n-1}},\ \frac{2^n-1}{2^n}\right).$$

Since $k(H_n)=c$, each H_n can be written in the form

$$H_n=\{h^n_\alpha\colon\ \alpha<\omega_c\}.$$

We now proceed to decompose $[0, 1)$ into countable, dense and disjoint subsets that will be the equivalence classes of our equivalence relation.
Let $C^1_1=C_{h^1_1}\cup\{x^1_1\}$ where x^1_1 is the first element (relative to the well-ordering of $[0, 1)$ given above) in I_1 that is not in C.
Let $C^1_2=C_{h^1_2}\cup\{x^1_2\}$ where x^1_2 is the first element in $I_1\backslash(C\cup\{x^1_1\})$.
We can continue this process, so that C^1_α is defined by transfinite induction for each $\alpha<\omega_c$, since C has measure zero and $\bigcup_{\beta<\alpha}\{x^1_\beta\}$ has cardinality less than c as ω_c is the first ordinal having cardinality c. Clearly C^1_α is dense in $[0, 1)$ for each $\alpha<\omega_c$. In addition

$$\bigcup_{\alpha<\omega_c} C^1_\alpha\supset I_1\backslash C \quad\text{and}\quad I_1\cup C\supset \bigcup_{\alpha<\omega_c} C^1_\alpha$$

and therefore

$$m\Big(\bigcup_{\alpha<\omega_c} C^1_\alpha\Big)=\frac{1}{2}$$

since C has measure zero. Furthermore the sets $\{C^1_\alpha\}_{\alpha<\omega_c}$ are pairwise disjoint.

Proceeding to I_2, let $C^2_1=C_{h^2_1}\cup\{x^2_1\}$, where x^2_1 is the first element (relative to the well ordering of $[0, 1)$ given above) in $I_2\backslash C$.
Let $C^2_2=C_{h^2_2}\cup\{x^2_2\}$ where x^2_2 is the first element in $I_2=C\cup\{x^2_1\}$. We continue by transfinite induction as in the $n=1$ case.

By ordinary induction this process can be continued for each $n\in N$ and so we obtain a collection of sets

$$\{C^n_\alpha\colon\ n\in N,\ \alpha<\omega_c\}\quad\text{such that:}$$

a) Each set is countable and dense in $[0, 1)$,
b) The sets in our collection are pairwise disjoint.
c) $m\Big(\bigcup_{\alpha<\omega_c} C^n_\alpha\Big)=\frac{1}{2^n}$ for each $n\in N$.
d) The union of all the sets in our collection is exactly equal to $[0, 1)$.

If $0 \leqslant x \leqslant 1$, then x can be written in the form

$$x = \frac{e_1}{2} + \frac{e_2}{2^2} + \ldots, \quad \text{where} \quad e_n \in \{0, 1\} \text{ for each } n.$$

Take

$$A_x = \bigcup \Big[\bigcup_{\alpha < \omega_c} C_\alpha^n : \; e_n = 1 \Big].$$

Then

$$m(A_x) = x.$$

Remark 2. It would be interesting to characterize those equivalence relations $\sim$ on $[0, 1)$ for which the zero-one law holds; that is, to find necessary and sufficient conditions that $m(A)$ is always either 0 or 1 whenever A is a measurable subset of $[0, 1)$ formed by unions of equivalence classes of $\sim$.

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ЗАМЕЧАНИЯ О НУЛЬ — ЕДИНИЦЕ ЗАКОНЕ

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Резюме

В этой работе даны два доказательства, стандартное и неархимедого, существования остаточного множества (т.е. содержащего все суммы его элементов с бинарными рациональными числами), которые ни не измеримо по Лебегу, ни не является множеством Бэра. Кроме того рассматриваются вопросы о некотором обобщении отношений эквивалентности.