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Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

ON ALMOST CONTINUOUS ADDITIVE FUNCTIONS[1]

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(*Communicated by Ľubica Holá*)

ABSTRACT. It is proved that every additive function is the sum of two almost continuous (in Stallings' sense) additive functions and the limit of a sequence (of a transfinite sequence) of almost continuous additive functions. Moreover, it is shown that the maximal additive family for the set of all almost continuous additive functions having the graphs of the second category is contained in the class of continuous additive functions.

Let $\mathbb{R}$ be the set of all reals. A function $g\colon (a,b) \to \mathbb{R}$ is said to be *almost continuous* (*in Stallings' sense* [5]) if for every open set $D \subset \mathbb{R}^2$ containing the graph $G(g)$ of the function g there is a continuous function $h\colon (a,b) \to \mathbb{R}$ with $G(h) \subset D$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *additive* (see, e.g., [4]) if it satisfies Cauchy's equation

$$f(x+y) = f(x) + f(y)\,, \qquad x, y \in \mathbb{R}\,.$$

It is well known that there exists additive almost continuous function $f\colon \mathbb{R} \to \mathbb{R}$ which is not continuous ([3], see also [2]). In this article, I prove that every additive function is the sum of two additive almost continuous functions and the limit of a sequence (of a transfinite sequence) of additive almost continuous functions, and I investigate the maximal additive family for the class of all additive almost continuous functions with the graphs of the second category.

Throughout the article, I assume the Continuum Hypothesis CH and all considered functions are real and of real variable.

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Let

$$\mathcal{A}dd = \{f : f \text{ is additive}\},$$
$$\mathcal{AC} = \{f : f \text{ is almost continuous}\},$$
$$\mathcal{C} = \{f : f \text{ is continuous}\},$$
$$\Theta = \{f : \text{ for every } g \in \mathcal{A}dd \cap \mathcal{AC},\ f + g \in \mathcal{A}dd \cap \mathcal{AC}\}.$$

Denote by $\mathbb{Q}$ the set of all rationals. Every linear basis in $\mathbb{R}$ over $\mathbb{Q}$ is called a Hamel basis in $\mathbb{R}$. Let

$$I_1, \dots, I_n, \dots \tag{1}$$

be a sequence of all open intervals with rational endpoints, let ω_1 denote the first uncountable ordinal number, and let

$$x_1, \dots, x_\alpha, \dots, \qquad \alpha < \omega_1, \tag{2}$$

be a transfinite sequence of all reals.

Remark 1. There is a Hamel basis $H \subset \mathbb{R}$ such that for every open interval $I \subset \mathbb{R}$ the intersection $I \cap H$ is of the cardinality continuum.

P r o o f. Let

$$0 \neq x_0^1 \in \mathbb{R}$$

be arbitrary, and for every $\alpha < \omega_1$ and $n \geq 1$ let x_α^n be the first element x_β of the sequence (2) such that

$$x_\beta \in I_n$$

and the set

$$\{x_\beta^n\}_{\beta<\alpha,\ n\geq 1} \cup \{x_\alpha^k\}_{k\leq n}$$

is linearly independent over $\mathbb{Q}$. Then the set

$$H = \{x_\alpha^n : \alpha < \omega_1,\ n \geq 1\}$$

is a Hamel basis such that

$$\operatorname{card}(H \cap I) = c$$

for every open interval I. □

If f is a function, we mean by a *blocking set* of f a closed set $K \subset \mathbb{R}^2$ such that $G(f) \cap K = \emptyset$ and $G(g) \cap K \neq \emptyset$ for every continuous function $g \colon \mathbb{R} \to \mathbb{R}$. An *irreducible blocking set* (IBS) K of f is a blocking set of f such that no proper subset of K is a blocking set ([3]).

It is known that f is almost continuous if and only if it has no blocking set. Moreover, if f is not almost continuous, then there is an (IBS) K of f, and the x-projection $\operatorname{pr}_x(K)$ of K is a non-degenerate connected set ([3]).

Let

$$K_1, \dots, K_\alpha, \dots, \qquad \alpha < \omega_1, \tag{3}$$

be a transfinite sequence of all irreducible blocking sets in $\mathbb{R}^2$.

THEOREM 1. *If $f \in \mathcal{A}dd$, then there are $g, h \in \mathcal{A}dd \cap \mathcal{AC}$ such that $f = g + h$.*

Proof. Let H be a Hamel basis from Remark 1. For every $\alpha < \omega_1$ there are points

$$x_\alpha, y_\alpha \in H \cap \mathrm{pr}_x(K_\alpha)$$

such that

$$x_\alpha \neq y_\beta\,, \quad y_\alpha \neq x_\beta\,, \qquad \text{for} \quad \beta \leq \alpha$$

and

$$x_\alpha \neq x_\beta\,, \quad y_\alpha \neq y_\beta\,, \qquad \text{for} \quad \beta < \alpha\,.$$

For every $\alpha < \omega_1$ let $u_\alpha, v_\alpha \in \mathbb{R}$ be points such that

$$(x_\alpha, u_\alpha) \in K_\alpha\,, \qquad (y_\alpha, v_\alpha) \in K_\alpha\,.$$

Define

$$g_1(x) = \begin{cases} u_\alpha & \text{if } x = x_\alpha\,, \ \alpha < \omega_1\,, \\ f(x) - v_\alpha & \text{if } x = y_\alpha\,, \ \alpha < \omega_1\,, \\ 0 & \text{otherwise in } H\,, \end{cases}$$

$$h_1(x) = \begin{cases} f(x) - u_\alpha & \text{if } x = x_\alpha\,, \ \alpha < \omega_1\,, \\ v_\alpha & \text{if } x = y_\alpha\,, \ \alpha < \omega_1\,, \\ f(x) & \text{otherwise in } H\,, \end{cases}$$

and let $g\colon \mathbb{R} \to \mathbb{R}$ ($h\colon \mathbb{R} \to \mathbb{R}$) be the additive extension of g_1 (of h_1) (see, e.g., [4]).

Observe that for every $\alpha < \omega_1$,

$$\big(x_\alpha, g(x_\alpha)\big) = (x_\alpha, u_\alpha) \in K_\alpha$$

and

$$\big(y_\alpha, h(y_\alpha)\big) = (y_\alpha, v_\alpha) \in K_\alpha\,.$$

So, the functions g, h are almost continuous, and, evidently,

$$f = g + h\,.$$

This completes the proof. □

Remark 2. The inclusion

$$\mathcal{A}dd \cap \mathcal{C} \subset \Theta$$

follows from [1].

Lemma 1. *Let $f\colon Z \to \mathbb{R}$ be a function with the graph $G(f)$ of the second category in $\mathbb{R}^2$, and let g be an upper semi-continuous function with domain a non-degenerate interval $J \supset Z$. Then for every countable sets $A, B \subset \mathbb{R}$ there is a point $x \in Z \setminus A$ such that $f(x) + g(x)$ is not in B.*

Proof. Let $(b_k)_k$ be an enumeration of all points of the set B. For $k = 1, 2, \dots$, let

$$A_k = \{x \in Z:\ f(x) + g(x) = b_k\}.$$

If for every $x \in Z \setminus A$ we have

$$f(x) + g(x) \in B,$$

then

$$G(f) \subset \bigcup_k G(f/A_k) \cup \{(x, y):\ x \in A,\ \ y \in \mathbb{R}\},$$

and for every $k = 1, 2, \dots$,

$$G\big((f - b_k)/A_k\big) = G\big((-g)/A_k\big).$$

Since the set $G(-g)$ is nowhere dense, every set

$$G\big((f - b_k)/A_k\big), \qquad k = 1, 2, \dots,$$

is also nowhere dense, and, consequently, every set $G(f/A_k)$, $k \geq 1$, is the same. So, $G(f)$ is of the first category, a contradiction. □

Remark 3. If the graph $G(f)$ of a function $f \in (\mathcal{A}dd \cap \mathcal{AC}) \setminus \mathcal{C}$ is of the second category, then for every open interval $I = (a, b)$ the sets

$$\big\{\big(x, f(x)\big):\ x \in I,\ \ f(x) > 0\big\}$$

and

$$\big\{\big(x, f(x)\big):\ x \in I,\ \ f(x) < 0\big\}$$

are of the second category.

Proof. If $G(f/I)$ is a subset of the first category, then for every bounded open interval J there is a function

$$h(x) = ax + b, \qquad x \in \mathbb{R},\ \ a, b \in \mathbb{Q},\ \ a \neq 0,$$

such that $J \subset h(I)$, and, consequently,

$$G(f/J) \subset aG(f/I) + \big(b, f(b)\big).$$

So, $G(f/J)$ is of the first category for every open interval J, and we obtain a contradiction, since $G(f)$ is of the second category. Suppose that the set

$$\big\{\big(x, f(x)\big):\ x \in I,\ \ f(x) > 0\big\}$$

is of the first category. Since f is a discontinuous additive function with the Darboux property, the set $\{x \in \mathbb{R} :\ f(x) = 0\}$ is dense ([4]). There is an open interval $J \subset I$ such that, at its center z, we have $f(z) = 0$. Observe that, if $x \in J$ and $f(x) > 0$, then

$$f\bigl(z - (x - z)\bigr) = -f(x) < 0 .$$

Since the function

$$h(x,y) = (2z - x, -y) , \qquad x \in I ,\ y \in \mathbb{R},$$

is a homeomorphism, and the set

$$\bigl\{\bigl(x, f(x)\bigr) :\ x \in I ,\ f(x) > 0\bigr\}$$

is of the first category, the set

$$\bigl\{\bigl(x, f(x)\bigr) :\ x \in I ,\ f(x) < 0\bigr\}$$

is the same. But the set

$$\bigl\{\bigl(x, f(x)\bigr) :\ x \in I ,\ f(x) = 0\bigr\}$$

cannot be residual in $I \times \mathbb{R}$, so we have a contradiction. □

Now, let

$$\Omega = \bigl\{f \in \mathcal{A}dd \cap \mathcal{AC} :\ G(f) \text{ is of the second category}\bigr\} \qquad \text{and}$$
$$\Delta = \{f :\ \text{for every } g \in \Omega ,\ f + g \in \Omega\} .$$

PROBLEM 1. Does exist a function $f \in (\mathcal{A}dd \cap \mathcal{AC}) \setminus \mathcal{C}$ with the graph of the first category?

Remark 4. The inclusion

$$\Delta \supset \mathcal{A}dd \cap \mathcal{C}$$

is true.

P r o o f. If $f \in \mathcal{A}dd \cap \mathcal{C}$, then for every function $g \in \Omega$ we have

$$f + g \in \mathcal{A}dd \cap \mathcal{AC}$$

([1]). There is $a \in \mathbb{R}$ such that

$$f(x) = ax , \qquad x \in \mathbb{R} .$$

Since the function

$$F(x,y) = (x, y + ax) , \qquad (x,y) \in \mathbb{R}^2$$

is a homeomorphism, the graph of the function

$$h(x) = g(x) + f(x) = g(x) + ax , \qquad x \in \mathbb{R} ,$$

is of the second category. So, $f + g \in \Omega$, and the proof is completed. □

THEOREM 2. *If the function $f \in \Omega$, then there is a function $h \in \Omega$ such that $f+h$ is not in $\mathcal{AC}$.*

P r o o f. Let

$$H = \{t_0, \dots, t_\alpha, \dots\}$$

be a Hamel basis, let

$$M_0, \dots, M_\alpha, \dots, \qquad \alpha < \omega_1,$$

be a transfinite sequence of all G_δ-sets of the second category in $\mathbb{R}^2$, and let

$$G(u_0), \dots, G(u_\alpha), \dots, \qquad \alpha < \omega_1,$$

be a transfinite sequence of the graphs of all upper semi-continuous functions with domains being non-degenerate intervals such that the domains I_0 of u_0 and I_1 of u_1 have positive endpoints.

There exists a point $(x_0, u_0(x_0)) \in G(u_0)$ such that

$$f(x_0) + u_0(x_0) > 0.$$

Let $(s_0, w_0) \in M_0$ be a point such that the sets $\{x_0, s_0\}$ and $\{f(x_0)+u_0(x_0), f(s_0)+w_0\}$ are linearly independent over $\mathbb{Q}$. Moreover, if x_0, s_0, t_0 are linearly independent over $\mathbb{Q}$, we find v_0 such that $f(t_0)+v_0$, $f(x_0)+u_0(x_0)$, $f(s_0)+w_0$ are linearly independent over $\mathbb{Q}$.

Denote by E_1 (F_1, resp.) the linear subspace over $\mathbb{Q}$ generated by $\{x_0, t_0, s_0\}$ ($\{f(t_0)+v_0, f(x_0)+u_0(x_0), f(s_0)+w_0\}$).

By Lemma 1 and Remark 3, there is a point $x_1 \in I \setminus E_1$ such that $0 > f(x_1) + u_1(x_1)$ is not in F_1.

Let $(s_1, w_1) \in M_1$ be a point such that the set $\{s_1, x_1\} \cup E_1$ is linearly independent over $\mathbb{Q}$, and the set $\{f(s_1)+w_1, f(s_1)+u_1(x_1)\} \cup F_1$ is also linearly independent over $\mathbb{Q}$. If the points

$$x_0,\ x_1,\ t_0,\ t_1,\ s_0,\ s_1$$

are linearly independent over $\mathbb{Q}$, we find a point v_1 such that the set $\{f(t_1)+v_1, f(x_1)+u_1(x_1), f(s_1)+w_1\} \cup F_1$ is linearly independent over $\mathbb{Q}$.

Fix an countable ordinal $\alpha > 1$ and suppose that we have defined points x_β, $(s_\beta, w_\beta) \in M_\beta$ and, if necessary, v_β, $1 < \beta < \alpha$, such that

$$f(x_\beta) + u_\beta(x_\beta), \quad f(t_\beta) + v_\beta, \quad f(s_\beta) + w_\beta, \qquad \beta < \alpha,$$

are linearly independent over $\mathbb{Q}$, and x_β, s_β and such t_β for which v_β exist are also linearly independent over $\mathbb{Q}$. Denote by E_α (F_α) the linear subspace over $\mathbb{Q}$ generated by $\{x_\beta : \beta < \alpha\} \cup \{s_\beta : \beta < \alpha\}$ and such t_β for which v_β

exist $(\{f(x_\beta)+u_\beta(x_\beta):\ \beta<\alpha\}\cup\{f(t_\beta)+v_\beta:\ \beta<\alpha$ and v_β is chosen$\}\cup\{f(s_\beta)+w_\beta:\ \beta<\alpha\})$. By Lemma 1, there is a point

$$x_\alpha\in I_\alpha\setminus E_\alpha\,,$$

where I_α is the domain of the function u_α, such that $f(x_\alpha)+u_\alpha(x_\alpha)$ is not in F_α.

Let $(s_\alpha,w_\alpha)\in M_\alpha$ be a point such that the set $\{s_\alpha,x_\alpha\}\cup E_\alpha$ is linearly independent over $\mathbb{Q}$, and the set $\{f(s_\alpha)+w_\alpha,f(x_\alpha)+u_\alpha(x_\alpha)\}\cup F_\alpha$ is also linearly independent over $\mathbb{Q}$. If the set $\{t_\alpha,x_\alpha,s_\alpha\}\cup E_\alpha$ is linearly independent over $\mathbb{Q}$, then we find a real v_α such that the set $\{f(t_\alpha)+v_\alpha,f(s_\alpha)+w_\alpha,f(x_\alpha)+u_\alpha(x_\alpha)\}\cup F_\alpha$ is linearly independent over $\mathbb{Q}$. All points x_α,s_α, $\alpha<\omega_1$, and such points t_α for which v_α exist form a Hamel basis H_1. Let h be the additive extension on $\mathbb{R}$ of the function

$$h_1(x)=\begin{cases}u_\alpha(x_\alpha) & \text{if } x=x_\alpha\,,\ \alpha<\omega_1\,,\\ v_\alpha & \text{if } x=t_\alpha\in H_1\setminus\{x_\beta,s_\beta:\ \beta<\omega_1\}\,,\ \alpha<\omega_1\,,\\ w_\alpha & \text{if } x=s_\alpha\,,\ \alpha<\omega_1\,.\end{cases}$$

Since

$$G(h)\cap G(u_\alpha)\neq\emptyset$$

and

$$G(h)\cap M_\alpha\neq\emptyset$$

for every $\alpha<\omega_1$, the function h is almost continuous ([3]), and its graph $G(h)$ is of the second category. Suppose that

$$f(x)+h(x)=0$$

for some $x>0$. Then

$$x=r_1z_1+\dots r_kz_k\,,$$

where $r_i\in\mathbb{Q}\setminus\{0\}$ and $z_i\in H_1$ for $i\le k$. Thus

$$r_1(f+h)(z_1)+\dots+r_k(f+h)(z_k)=0\,,$$

which is a contradiction with the linear independence of

$$(f+h)(z_1),\dots,(f+h)(z_k)\,.$$

But

$$(f+h)(x_0)>0$$

and

$$(f+h)(x_1)<0\,,$$

so $f+h$ has not the Darboux property. Thus $f+h$ is not almost continuous ([3]), and the function f is not in the collection Ω. This completes the proof. □

PROBLEM 2. Are the following equalities true:

$$\mathcal{A}dd \cap \mathcal{C} = \Theta = \Delta\,?$$

THEOREM 3. *If $f \in \mathcal{A}dd$, then there is a sequence of functions $f_n \in \mathcal{A}dd \cap \mathcal{AC}$, $n \geq 1$, such that $f = \lim_{n\to\infty} f_n$.*

P r o o f. Let $H \subset \mathbb{R}$ be a Hamel basis satisfying the condition from Remark 1. For every $\alpha < \omega_1$ there is a sequence of points

$$x_{\alpha,n} \in H \cap \mathrm{pr}_x(K_\alpha), \qquad n = 1, 2, \dots,$$

such that

$$x_{\alpha,n} \neq x_{\beta,k}$$

if

$$(\alpha, n) \neq (\beta, k), \quad \beta < \alpha, \quad k, n = 1, 2, \dots.$$

For each point $x_{\alpha,n}$ there is a point $y_{\alpha,n}$ such that

$$(x_{\alpha,n}, y_{\alpha,n}) \in K_\alpha, \qquad \alpha < \omega_1, \quad n \geq 1.$$

Define, for $n = 1, 2, \dots$,

$$g_n(x) = \begin{cases} y_{\alpha,k} & \text{if } x = x_{\alpha,k}, \ \alpha < \omega_1, \ k \geq n, \\ f(x) & \text{otherwise in } H, \end{cases}$$

and let f_n be the additive extension of g_n on $\mathbb{R}$. Since

$$(x_{\alpha,n}, y_{\alpha,n}) \in K_\alpha \cap G(f_n)$$

for $\alpha < \omega_1$ and $n \geq 1$, all functions f_n are almost continuous. Moreover, if $x = x_{\alpha,k}$, $\alpha < \omega_1$, $k \geq 1$, then $f_n(x) = f(x)$ for $n > k$, and if $x \in H$, and $x \neq x_{\alpha,k}$ for all $\alpha < \omega_1$ and $k \geq 1$, then $f_n(x) = f(x)$ for all $n \geq 1$. So, $f = \lim_{n\to\infty} f_n$ on H and, consequently, on $\mathbb{R}$. Thus the proof is completed. □

THEOREM 4. *If $f \in \mathcal{A}dd$, then there is a transfinite sequence of functions $f_\alpha \in \mathcal{A}dd \cap \mathcal{AC}$, $\alpha < \omega_1$, such that $\lim_\alpha f_\alpha = f$, i.e.,*

$$\forall x \ \exists \beta{<}\omega_1 \ \forall \omega_1{>}\alpha{>}\beta \qquad f_\alpha(x) = f(x).$$

P r o o f. Let a Hamel basis H be the same as in the proof of Theorem 3. There are pairwise disjoint sets T_α, $\alpha < \omega_1$, such that every set

$$H \cap \mathrm{pr}_x(K_\alpha) \cap T_\alpha, \qquad \alpha < \omega_1,$$

is uncountable. For each $\alpha < \omega_1$ let

$$(x_{\alpha,\beta})_{\beta<\omega_1}$$

be a transfinite sequence of all points of the set

$$H \cap \operatorname{pr}_x(K_\alpha) \cap T_\alpha ,$$

and let

$$g_\alpha(x) = \begin{cases} y_{\alpha,\beta} & \text{if } x = x_{\alpha,\beta}\,, \ \omega_1 > \beta \geq \alpha\,, \\ f(x) & \text{otherwise in } H\,, \end{cases}$$

where $y_{\alpha,\beta}$ are points such that

$$(x_{\alpha,\beta}, y_{\alpha,\beta}) \in K_\beta\,, \qquad \alpha, \beta < \omega_1\,,$$

and let f_α be the additive extension g_α on $\mathbb{R}$. Analogously as in the proof of Theorem 3, we can observe that all functions f_α are almost continuous and

$$\lim_\alpha f_\alpha = f\,.$$

This completes the proof. □

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