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ON QUASI-UNIFORM CONVERGENCE OF A SEQUENCE OF S.Q.C. FUNCTIONS

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ABSTRACT. It is proved that every almost everywhere continuous function is the limit of a quasi-uniformly convergent sequence of Darboux strongly quasi-continuous functions.

Let $\mathbb{R}$ be the set of all reals and let μ_e (μ) denote the outer Lebesgue measure (the Lebesgue measure) in $\mathbb{R}$. Denote by

$$d_u(A,x) = \limsup_{h\to 0^+} \mu_e\big(A\cap(x-h,x+h)\big)/2h$$
$$\big(\, d_l(A,x) = \liminf_{h\to 0^+} \mu_e\big(A\cap(x-h,x+h)\big)/2h \,\big)$$

the *upper* (*lower*) *density of the set* $A\subset\mathbb{R}$ *at a point* x. A point $x\in\mathbb{R}$ is called a *density point of the set* $A\subset\mathbb{R}$ if there exists a Lebesgue measurable set $B\subset A$ such that $d_l(B,x)=1$. The family

$$\mathcal{T}_d = \{A\subset\mathbb{R};\ A \text{ is measurable and every point } x\in A \text{ is a density point of } A\}$$

is a topology, called the *density topology* ([1]).

A function f is said to be *strongly quasi-continuous* (in short *s.q.c.*) *at a point* x if for every set $A\in\mathcal{T}_d$ containing x and for every positive real η there is an open interval I such that $I\cap A\neq\emptyset$ and $|f(t)-f(x)|<\eta$ for all $t\in A\cap I$ ([2]).

Let $f\colon\mathbb{R}\to\mathbb{R}$ and $x\in\mathbb{R}$. If there is an open set U such that $d_u(U,x)>0$ and the restricted function $f\big|(U\cup\{x\})$ is continuous at x, then f is s.q.c. at x ([3]).

By an elementary proof, we obtain:

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Remark 1. The limit f of a uniformly convergent sequence $(f_n)_n$ all of whose terms are s.q.c. at a point x is also s.q.c. at x.

A sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ is said to be *quasi-uniformly convergent* to a function f on $\mathbb{R}$ ([7]) if

$$\forall \eta>0\, \forall m\, \exists p\, \forall x \quad \bigl(\min\bigl(|f_{m+1}(x)-f(x)|,\dots,|f_{m+p}(x)-f(x)|\bigr) < \eta\bigr).$$

It is known ([2], [3]) that every s.q.c. function f is almost everywhere continuous (with respect to μ). Then, the limit of a quasi-uniformly convergent sequence of s.q.c. functions is almost everywhere continuous.

We shall prove the following:

THEOREM 1. *If a function $f\colon \mathbb{R} \to \mathbb{R}$ is almost everywhere continuous then there is a quasi-uniformly convergent sequence of Darboux s.q.c. functions $g_n\colon \mathbb{R} \to \mathbb{R}$ such that $f = \lim\limits_{n\to\infty} g_n$.*

Proof. Let cl denote the closure operation, let $\operatorname{int}(X)$ denote the interior of X and let

$$B = \bigl\{y \in \mathbb{R};\ \mu\bigl(\operatorname{cl}(f^{-1}(y)\bigr) > 0\bigr\}.$$

We can suppose that $\mu\bigl(\operatorname{cl}(f^{-1}(0))\bigr) = 0$, since, in the opposite case, we can consider instead the function $g = f - a$, where the constant a is such that $\mu\bigl(\operatorname{cl}(f^{-1}(a))\bigr) = 0$. Since f is almost everywhere continuous, the set B is countable. Let $E(B)$ be the linear space over the field $\mathbb{Q}$ of all rationals generated by the set B. Since the set $E(B)$ is countable, there exists a positive number $c \in \mathbb{R}\setminus E(B)$. Denote by $\mathbb{Z}$ the set of all integers and by $\mathbb{N}$ the set of all positive integers. Let $k \in \mathbb{Z}$ and let $n \in \mathbb{N}$ be integers. If

$$kc/2^n \le f(x) < (k+1)c/2^n$$

then let

$$f_n(x) = kc/2^n.$$

Observe that every function f_n, $n \in \mathbb{N}$, is almost everywhere continuous and if $D(f_n)$ denotes the set of all discontinuity points of f_n then $D(f_n)$ is a closed set of measure zero. Moreover, $D(f_n) \subset D(f_{n+1})$ for $n \in \mathbb{N}$ and if $x \in D(f_{k+1}) \setminus D(f_k)$ for some $k \in \mathbb{N}$ then for every $m > k$ the inequality

$$\operatorname{osc} f_m(x) < c/2^{k-1} \tag{1}$$

holds. Let $C(f_n)$, $n \in \mathbb{N}$, be the set of all continuity points of the function f_n, i.e. $C(f_n) = \mathbb{R} \setminus D(f_n)$. For a closed set $X \subset \mathbb{R}$ and for a positive real r denote by $A_r(X)$ the set $\Bigl\{x;\ \operatorname{dist}(x,X) = \inf\limits_{y\in X} |x-y| < r\Bigr\}$. Since the set

$D(f_n)$ is closed and of measure zero, there are disjoint closed intervals $I_{n,k,l,j,i} = [a_{n,k,l,j,i}, b_{n,k,l,j,i}] \subset C(f_n)$, $k \le n$, $j = 1,2$, $i \in \mathbb{N}$, $l \in \mathbb{Z}$, such that:

(2) for every $k < n$, for every $l \in \mathbb{Z}$, for $j = 1,2$ and for every $x \in D(f_1)$ (for every $x \in D(f_{k+1}) \setminus D(f_k)$) we have $d_u\left(\bigcup_{i\in\mathbb{N}} I_{n,1,l,j,i}, x\right) > 0$ $\left(d_u\left(\bigcup_{i\in\mathbb{N}} I_{n,k+1,l,j,i}, x\right) > 0\right)$;

(3) if the limit $\lim_{s\to\infty} a_{n,k,l_s,j_s,i_s}$ of one-to-one sequence $(a_{n,k,l_s,j_s,i_s})_{s\in\mathbb{N}}$ exists then $\lim_{s\to\infty} a_{n,k,l_s,j_s,i_s} = \lim_{s\to\infty} b_{n,k,l_s,j_s,i_s} \in D(f_k)$, $k \le n$, $j_s = 1,2$, $l_s \in \mathbb{Z}$, $i_s \in \mathbb{N}$;

(4) $I_{n,k,l,j,i} \subset A_{1/n}(D(f_k))$ for $k \le n$, $j = 1,2$, $l \in \mathbb{Z}$, $i \in \mathbb{N}$;

(5) for all $k \le n$, $l \in \mathbb{Z}$, $j = 1,2$ and for every $x \in D(f_k)$ the point x is a bilateral accumulation point of the set $\bigcup_{i\in\mathbb{N}} I_{n,k,l,j,i}$.

Next, for all $k \le n$, $j = 1,2$, $l \in \mathbb{Z}$, $i \in \mathbb{N}$ we find a closed interval $J_{n,k,l,j,i} \subset \mathrm{int}(I_{n,k,l,j,i})$ such that for every $x \in D(f_1)$ (for every $x \in D(f_k) \setminus D(f_{k-1})$, $1 < k \le n$) and for all $l \in \mathbb{Z}$ and $j = 1,2$ the inequalities

$$d_u\left(\bigcup_{i\in\mathbb{N}} J_{n,1,l,j,i}, x\right) > 0 \tag{6}$$

$$\left(d_u\left(\bigcup_{i\in\mathbb{N}} J_{n,k,l,j,i}, x\right) > 0\right) \tag{6'}$$

are true.

Let

- $g_{2n-1}(x) = lc/2^n$ if $x \in J_{n,1,l,1,i}$, $l \in \mathbb{Z}$, $i \in \mathbb{N}$;
- $g_{2n-1}(x) = f_n(x) + lc/2^n$ if $x \in J_{n,k,l,1,i}$, $1 < k \le n$, $-2^{n+1-k} \le l \le 2^{n+1-k}$, $i \in \mathbb{N}$;
- g_{2n-1} be linear on all components of the sets $I_{n,1,l,1,i} \setminus \mathrm{int}(J_{n,1,l,1,i})$, $l \in \mathbb{Z}$, $i \in \mathbb{N}$;
- g_{2n-1} be linear on all components of the sets $I_{n,k,l,1,i} \setminus \mathrm{int}(J_{n,k,l,1,i})$, $1 < k \le n$, $-2^{n+1-k} \le l \le 2^{n+1-k}$, $i \in \mathbb{N}$;
- $g_{2n-1}(x) = f_n(x)$ otherwise on $\mathbb{R}$.

and let

- $g_{2n}(x) = lc/2^n$ if $x \in J_{n,1,l,2,i}$, $l \in \mathbb{Z}$, $i \in \mathbb{N}$;
- $g_{2n}(x) = f_n(x) + lc/2^n$ if $x \in J_{n,k,l,2,i}$, $1 < k \le n$, $-2^{n+1-k} \le l \le 2^{n+1-k}$, $i \in \mathbb{N}$;
- g_{2n} be linear on all components of the sets $I_{n,1,l,2,i} \setminus \mathrm{int}(J_{n,1,l,2,i})$, $l \in \mathbb{Z}$, $i \in \mathbb{N}$;

- g_{2n} be linear on all components of the sets $I_{n,k,l,2,i} \setminus \operatorname{int}(J_{n,k,l,2,i})$, $1 < k \le n$, $-2^{n+1-k} \le l \le 2^{n+1-k}$, $i \in \mathbb{N}$;
- $g_{2n}(x) = f_n(x)$ otherwise on $\mathbb{R}$.

Evidently,

$$\min(|g_{2n-1} - f_n|, |g_{2n} - f_n|) = 0. \tag{7}$$

By (6) and (6') the functions g_{2n-1} and g_{2n} are s.q.c..

Observe that the reduced functions $h_{2n-1} = g_{2n-1} | (\mathbb{R} \setminus D(f_n))$ and $h_{2n} = g_{2n} | (\mathbb{R} \setminus D(f_n))$ are continuous. By (5) we obtain that for every $x \in D(f_n)$ and for every $r > 0$ the images $g_{2n}([x-r, r])$ and $g_{2n}([x, x+r])$ are intervals. So g_{2n} has the Darboux property. Similarly, the function g_{2n-1} has the Darboux property.

Fix a positive real η and $x \in \mathbb{R}$. If $x \in D(f_n)$ for some $n \in \mathbb{N}$ then for $m > n$ we have

$$f_m(x) = g_{2m-1}(x) = g_{2m}(x)$$

and consequently

$$\lim_{n\to\infty} g_{2n-1}(x) = \lim_{n\to\infty} g_{2n}(x) = f(x).$$

So, let $x \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} D(f_n)$. There is $k \in \mathbb{N}$ such that $c/2^{k-1} < \eta$. Let $m \in \mathbb{N}$ be such that $\operatorname{dist}(x, D(f_k)) > 1/m$ and $m > k$. By (4) for $n > m$ we obtain that

$$\max(|g_{2n-1}(x) - f_n(x)|, |g_{2n}(x) - f_n(x)|) \le c/2^n < \eta.$$

Since $\lim_{n\to\infty} f_n(x) = f(x)$, we obtain $\lim_{n\to\infty} g_n(x) = f(x)$. So, by (7), the sequence $(g_n)_n$ quasi-uniformly converges to f. □

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *quasi-continuous at the point* x (*cliquish at the point* x) if for every positive real η and for every open set U containing x there exists a nonempty open set $V \subset U$ such that $|f(t) - f(x)| < \eta$ for $t \in V$ ($\operatorname{osc}_V f < \eta$). In [4] it is proved that every cliquish function f is the limit of a quasi-uniformly convergent sequence of Darboux quasi-continuous functions. This theorem is an immediate consequence of Theorem 1. Indeed, if f is a cliquish function then the set $D(f)$ of all discontinuity points of f is of the first category ([5]) and there is a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that the set $h(D(f))$ is of measure zero ([6]). Thus the function $f \circ h^{-1}$ is almost everywhere continuous and, by Theorem 1, there is a sequence of Darboux s.q.c. functions f_n, $n \in \mathbb{N}$, which converges to $f \circ h^{-1}$ quasi-uniformly. Now, it suffices to observe that all the functions $f_n \circ h$, $n \in \mathbb{N}$, are quasi-continuous with the Darboux property and that the sequence $(f_n \circ h)_n$ converges quasi-uniformly to f.

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