Mathematica Slovaca

Jozef Rovder

A note on comparison theorems for third-order linear differential equations

Mathematica Slovaca, Vol. 26 (1976), No. 4, 323--327

Persistent URL: http://dml.cz/dmlcz/128806

Math. Slov., 26, 1976, No. 4, 323—327

A NOTE ON COMPARISON THEOREMS FOR THIRD — ORDER LINEAR DIFFERENTIAL EQUATIONS

JOZEF ROVDER

In this paper we prove some comparison theorems for the differential equation of the third-order

$$(a) \qquad y''' + b(x)y' + c(x)y = 0\,,$$

where $b(x)$, $c(x)$ and $b'(x)$ are continuous functions in $(0, \infty)$.

As usual, a solution of (a) is called nonoscillatory iff it has no zeros for arbitrarily large x and (a) is said to be nonoscillatory iff all its nontrivial solutions are nonoscillatory.

The following theorem is analogous to Theorem 2 in [4] for differential equations of class V_1.

Theorem 1. *Suppose the coefficients of* (a) *satisfy the assumption* $2c(x) - b'(x) \geq 0$ *in* $(0, \infty)$. *Let* (a) *be nonoscillatory. Then there exists a number* $\gamma > 0$ *such that the equation* (a) *has no solution with more than two zeros in* $[\gamma, \infty)$.

Proof. Since the equation (a) is nonoscillatory, there exists a solution $y(x)$ of (a) and a number $\gamma > 0$ such that $y(\gamma) = 0$, $y(x) \neq 0$ for $x > \gamma$. Let $z(x)$ be a solution of (a) with the properties $z(\gamma) = z'(\gamma) = 0$, $z''(\gamma) \neq 0$. If $y'(\gamma) \neq 0$, then from Theorem 4 in [1] it follows $z(x) \neq 0$ for $x > \gamma$. If $y'(\gamma) = 0$. then $z(x) = cy(x)$ and so $z(x) \neq 0$ for $x > \gamma$. Consequently, the equation (a) always has a solution $z(x)$ such that $z(\gamma) = z'(\gamma) = 0$, $z(x) > 0$ in (γ, ∞), $\gamma > 0$.

Now we show that every solution of (a) has not more than two zeros in $[\gamma, \infty)$. At first, consider the solution of (a) with a zero at γ. Let $u(x)$ be a solution of (a) such that $u(\gamma) = u(x_1) = u(x_2) = 0$, $\gamma \leq x_1 \leq x_2$. If $\gamma = x_1 < x_2$, then $u(x) = cz(x)$ and so $u(x) \neq 0$ for $x > \gamma$. Also the case $\gamma < x_1 = x_2$ leads to a contradiction with the identity

$$[yy'' - \tfrac{1}{2}y'^2 + \tfrac{1}{2}b(x)y^2]' = -\tfrac{1}{2}[2c(x) - b'(x)]y^2\,.$$

Now let $\gamma < x_1 < x_2$. Suppose $u(x) > 0$ in (x_1, x_2). Then there exist a number $c > 0$ and $\tau \in (x_1, x_2)$ such that the solution $z(x) - cu(x)$ has a double zero at τ and a simple zero at γ, which is in contradiction with the above identity. So every solution of (a) with a zero at γ has not more than two zeros in $[\gamma, \infty)$.

Finally we prove that every solution $v(x)$ of (a) such that $v(\gamma)\neq 0$ has not more than two zeros in $[\gamma, \infty)$. Suppose to the contrary that $v(x_1)=v(x_2)=v(x_3)=0$, $\gamma<x_1\leqslant x_2<x_3$. (As we have showed above, the case $x_1<x_2=x_3$ leads to a contradiction.) Let $v(x)>0$ in (x_2, x_3). Let $w(x)$ be a solution of (a) such that $w(\gamma)=w(x_1)=0$, $w(x)<0$ in (γ, x_1). Then $w(x)>0$ in (x_1, ∞). Then by Lemma 2 in [1], there exist numbers $c>0$ and $\tau\in(x_1, x_2)$ such that the solution $w(x)-cv(x)$ of (a) has a double zero at τ and a simple zero at x_1 which contradicts the above id ntity. Theorem is proved completely.

Co ol ary 1. *Suppose the inequality $2c(x)-b'(x)\geqslant 0$ $(2c(x)-b'(x)\leqslant 0)$ holds* $(0, \infty)$. *Then (a) is nono cillatory in $(0, \infty)$ if and only if there exists a number $\gamma>0$ uch that the equation (a) is disconjugate in $[\gamma, \infty)$, i.e. the equation (a) has no solution with more than two zeros in $[\gamma, \infty)$.*

Proof. If $2c(x)-b'(x)\geqslant 0$, then the assertion follows from Theorem 1. If $c(x)-b'(x)\leqslant 0$ and (a) is nonoscillatory, then, by Theorem 3 in [1], its adjoint equation is nono cillatory The coefficients of the adjoint equation, denoted by $b(x)$ and $\bar{c}(x)$, satisfy the as umption $2\bar{c}(x)-\bar{b}'(x)\geqslant 0$. Then the adjoint equation of (a) is disconjugate in $[\gamma, \infty)$ for some $\gamma>0$, and by Corollary 3 in [3] the equation (a) is disconjugate in $[\gamma, \infty)$. The sufficient conditions are obvious.

Theorems 6 and 7, Corollaries 1 and 2 in [1] yield the following theorem.

Theorem 2. *Consider the differential equations*

$$(1_i)\qquad y'''b(x)y'+c_i(x)y=0\,,\qquad i=1,2,3,$$

$'(x)$, $c_i(x)$ *are continuous functions in* $(0, \infty)$. *Let the coefficients of* (1_i) *satisfy*

$$(2)\qquad \begin{aligned} b_2(x)&\leqslant b_1(x)\,,\\ 2c_1(x)-b_1'(x)&\leqslant 0\,,\\ 2c_1(x)-b_1'(x)&\leqslant 2c_2(x)-b_2'(x)\leqslant 2c_3(x)-b_3'(x)\,.\\ b_2(x)&\leqslant b_3(x)\,,\\ 2c_3(\ )-b_3'(x)&\geqslant 0 \end{aligned}$$

L t the coefficients of (1_2) satisfy the inequality $2c_2(x)-b_2'(x)\geqslant 0$, or $2c\ (x)-b_2'(x)\leqslant 0$ in $(0, \infty)$, or the equation (1_2) is of class V_1, or class V_2.

Then the equation (1_2) is nonoscillatory if the equation (1_1) and the equation (1_3) are nonoscillatory.

Proof. Let, for instance, $2c_2(x)-b_2'(x)\geqslant 0$. Suppose to the contrary that (1_2) is o cillatory, i e. there exists a solution of (1_2) which has zeros for arbitrarily large x. From the conditions (2) it follows

$$b_3(x)\geqslant b_2(x)\,,\qquad 2c_3(x)-b_3'(x)\geqslant 2c_2(x)-b_2'(x)\geqslant 0\,.$$

Then, by Theorem 6 in [1], the equation (1_3) is oscillatory, which is a contradiction.

In the same way we can prove all cases included in this theorem. (The definitions of the class V_1 and V_2 see in [1] or [4].)

The main aim of this note is to show that Theorem 2 will be valid also, if we omit the assumptions $2c_2(x) - b_2'(x) \geq 0$ $(2c_2(x) - b_2'(x) \leq 0)$, (1_2) is of class V_1 or class V_2 in it. To prove it, we shall use the following theorem (see [2]).

Theorem 3. *Suppose the functions $f(x)$, $g_i(x)$, $i = 1, 2, 3$ are continuous in an interval I. Let for any $x \in I$ be*

$$g_1(x) \leq g_2(x) \leq g_3(x)\,. \tag{3}$$

If the differential equation

$$y''' + f(x)y' + g_i(x)y = 0 \tag{4}$$

is disconugate for $i = 1, 3$, then it is disconjugate for $i = 2$ in I.

Theorem 4. *Suppose the coefficients of* (1_i) *satisfy* (2). *If the equations* (1_1) *and* (1_3) *are nonoscillatory, then the equation* (1_2) *is nonoscillatory in* $(0, \infty)$.

Proof. Consider the differential equations

$$y''' + b_2(x)y' + \bar{c}(x)y = 0\,, \tag{5}$$

$$y''' + b_2(x)y' + \tilde{c}(x)y = 0\,, \tag{6}$$

where the functions $\bar{c}(x)$, $\tilde{c}(x)$ are defined as follows

$$\bar{c}(x) = \begin{cases} c_2(x) \text{ for all } x \in (0, \infty) \text{ such that } 2c_2(x) - b_2'(x) \geq 0, \\ \frac{1}{2} b_2'(x) \text{ for all } x \in (0, \infty) \text{ such that } 2c_2(x) - b_2'(x) < 0, \end{cases}$$

$$\tilde{c}(x) = \begin{cases} c_2(x) \text{ for all } x \in (0, \infty) \text{ such that } 2c_2(x) - b_2'(x) \leq 0, \\ \frac{1}{2} b_2'(x) \text{ for all } x \in (0, \infty) \text{ such that } 2c_2(x) - b_2'(x) > 0. \end{cases}$$

The functions $\bar{c}(x)$ and $\tilde{c}(x)$ defined in this way are continuous in $(0, \infty)$. The coefficients of (5) satisfy the conditions

$$0 \leq 2\bar{c}(x) - b_2'(x) = \max\,[0, 2c_2(x) - b_2'(x)] \leq 2c_3(x) - b_3'(x)\,,$$
$$b_2(x) \leq b_3(x)\,.$$

Since the equation (1_3) is nonoscillatory, then the equation (5) is nonoscillatory by Theorem 2.

Likewise, the coefficients of (6) satisfy the conditions

$$0 \geq 2\tilde{c}(x) - b_2'(x) = \min\,[0, 2c_2(x) - b_2'(x)] \geq 2c_1(x) - b_1'(x)\,,$$
$$b_2(x) \leq b_1(x)\,.$$

Then, by Theorem 2, the equation (6) is nonoscillatory since the equation (1_1) is nonoscillatory.

From the definition of $\bar{c}(x)$ and $\bar{c}(x)$ it fillows

$$2\bar{c}(x)-b_2'(x)\leq 2c_2(x)-b_2'(x)\leq 2\bar{c}(x)-b_2'(x)\,,$$

i.e.

$$\bar{c}(x)\leq c_2(x)\leq \bar{c}(x)\,.$$

From the Corollary 1 it follows that the equations (5) and (6) are disconjugate in $[\gamma, \infty)$ for a number $\gamma>0$. Then the equation (1_2) is disconjugate in $[\gamma, \infty)$ by Theorem 3, and so (1_2) is nonoscillatory in $(0, \infty)$.

Remark. From the conditions (2) it follows that if $b_1(x)=b_2(x)=b_3(x)$, i.e. if the equation (1_i) has the same form as (4), then the conditions (2) imply the conditions (3) and hence Theorem 4 generalizes Theorem 3.

Corollary 2. *Let the coefficients of* (a) *satisfy assumptions*

$$b(x)\leq p \quad \textit{and} \quad |2c(x)-b'(x)|\leq q\,,$$

where $p\leq 0$ *and* $q\leq 4/3\sqrt{3}(-p)^{3/2}$, p, q *are constants, or the assumptions*

$$b(x)\leq \frac{p}{x^2} \quad \textit{and} \quad |2c(x)-b'(x)|\leq \frac{\varepsilon}{x^3}\,,$$

where $p\leq 1$ *and* $\varepsilon\leq 4/3\sqrt{3}(1-p)^{3/2}$, p, ε *are constants. Then the equation* (a) *is* nonoscillatory.

Corollary 3. *Let in the equation* (a) *be* $b(x)\equiv 0$. *Then the equation* (a) *is nonoscillatory if*

$$|c(x)|\leq \frac{2}{3\sqrt{3}}\cdot\frac{1}{x^3}\,.$$

Proof. These corollaries are consequences of Theorem 11 in [1].

REFERENCES

[1] ROVDER, J.: Oscillation criteria for third-order linear differential equations. Mat. Čas. 25, 1975, 231—244.

[2] ЛЕВИН, А. Ю.: Неосцилляция решений уравнения $x^{(n)}+p_1(t)x^{(n-1)}+\ldots+p_n(t)x=0$. Успехи матем. наук, XXIV, вып. 2 (146), 43—96, 1969.

[3] GERA, M.: Nichtoszillatorische und oszillatorische Differentialgleichungen dritter Ordnung. Čas. pro pěst. mat. 96, 1971, 278—293.

[4] ŠVEC, M.: Несколько замечаний о линейном дифференциальном уравнении третего порядка. Чех. мат. ж. 15 (90) 1965, 42—49.

Received February 18, 1975

Katedra matematiky a deskriptívnej geometrie
Strojníckej fakulty SVŠT
Gottwaldovo nám. 50
880 31 Bratislava

ЗАМЕЧАНИЕ О ТЕОРЕМАХ СРАВНЕНИЯ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА

Йосеф Ровдер

Резюме

Решение уравнения (a) мы будем называть неколебательным, если существует число a такое, что ето решение неимеет нулей в интервале (a, ∞). Уравнение (a) мы будем называть неколебательным, если все его решения неколебательны, и мы будем называть его без сопряженых точек на I, если каждое его решение имеет на I не более двух нулей.

В работе доказано что если $2c(x) - b'(x) \geqq 0$ ($\leqq 0$) в интервале $(0, \infty)$, потом уравнение (a) ябляется неколебательным на $(0, \infty)$ тогда и только тогда, когда существует число $\gamma > 0$ такое, что уравнение (a) является без сопряженых точек на интервале $[\gamma, \infty)$.

Главным результатом этой работы является

Теорема 4. Пусть коеффициенты уравнения (1_i) удовлетворяют свойствами (2) и пусть уравнения (1_1) и (1_3) ябляются неколебательными на интервале $(0, \infty)$. Тогда уравнение (1_2) является неколебательным на интервале $(0, \infty)$.