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STRONG SUBDIRECT PRODUCTS OF MV-ALGEBRAS

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ABSTRACT. In this paper we investigate the Dedekind completion of a strong subdirect product of MV-algebras.

1. Introduction

Strong subdirect products of lattices and of pseudo MV-algebras have been investigated in [8].

In the present paper we apply this notion for dealing with Dedekind completions of MV-algebras.

We recall that the Dedekind completion $D(\mathcal{A})$ of an MV-algebra $\mathcal{A}$ is an MV-algebra if and only if $\mathcal{A}$ is archimedean (cf. [5], or [3; p. 436]; instead of "Dedekind completion" the term "MacNeille completion" has also been used).

Let $\mathcal{A}$ be an archimedean MV-algebra. We prove the following result:

(A) Suppose that $\mathcal{A}$ is a strong subdirect product of MV-algebras $\mathcal{A}_i$ $(i \in I)$. Then its Dedekind completion $D(\mathcal{A})$ is isomorphic to the direct product of MV-algebras $D(\mathcal{A}_i)$.

In [8], b-atomic MV-algebras have been dealt with; for the definition, cf. Section 2 below. We apply (A) and [8; Theorem 4.2]; we obtain:

(B) The following conditions for $\mathcal{A}$ are equivalent:
- (i) $\mathcal{A}$ is b-atomic.
- (ii) $D(\mathcal{A})$ is a direct product of linearly ordered MV-algebras.

A particular case of a b-atomic MV-algebra is the atomic MV-algebra. In this connection, cf. [3; Theorem 6.4.20], where the Dedekind completion of an archimedean atomic MV-algebra has been considered.

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2. Preliminaries

For the definition of an MV-algebra, several equivalent systems of axioms have been applied (cf., e.g., [1], [2], [4]).

We will use the definition from [2]; thus an MV-algebra is an algebraic structure $\mathcal{A} = (A; , \oplus, \neg, 0)$ of type $(2,1,0)$ satisfying the axioms MV1–MV6 from [2]. We put $\neg 0 = 1$.

We also apply the well-known results on the relations between MV-algebras and abelian lattice ordered groups (cf. [2]). Hence there is an abelian lattice ordered group G with a strong unit u such that A is the interval $[0,u]$ of G, $1 = u$ and for each $x, y \in A$ we have $x \oplus y = (x+y) \wedge u$, $\neg x = u - x$; we put $\mathcal{A} = \Gamma(G, u)$.

We denote by $\ell(\mathcal{A})$ the lattice $(A; \vee, \wedge)$, where the operations $\vee$ and $\wedge$ are defined as in G. The lattice $\ell(\mathcal{A})$ is distributive.

An element $0 < b \in A$ is called *basic* if the interval $[0,b]$ of the lattice $\ell(\mathcal{A})$ is a chain. The set of all basic elements of A is denoted by $B(\mathcal{A})$. An MV-algebra is said to be b-*atomic* if for each $0 < a \in A$ there exists $b \in B(\mathcal{A})$ such that $b \leqq a$.

If $b \in A$ and the interval $[0,b]$ is a two-element set, then b is an *atom* of $\mathcal{A}$. The MV-algebra $\mathcal{A}$ is *atomic* if for each $0 < a \in A$ there exists an atom b with $b \leqq a$. If $\mathcal{A}$ is atomic, then it is b-atomic, but not conversely, in general.

The direct product of MV-algebras is defined in the usual way; we apply the symbols

$$\mathcal{A} \times \mathcal{B}, \qquad \prod_{i \in I} \mathcal{A}_i .$$

Consider a homomorphism

$$\varphi \colon \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i = \mathcal{A}^0$$

of the MV-algebra $\mathcal{A}$ into the MV-algebra $\mathcal{A}^0$. For $x \in A$ and $i \in I$ we denote

$$x_i = \varphi(x)_i \, ;$$

x_i is said to be the *component* of x in $\mathcal{A}_i$ (under the mapping φ). We write also $x_i = x(\mathcal{A}_i)$.

If φ is a bijection, then it is said to be a *direct product decomposition* of $\mathcal{A}$.

If for each $i \in I$ and each element x^i of A_i (= the underlying set of $\mathcal{A}_i$) there exists $x \in A$ with $x_i = x^i$, then φ is called a *subdirect product decomposition* of $\mathcal{A}$.

An analogous terminology and notation will be applied for lattices.

3. Strong subdirect products of lattices

Let L be a lattice. It is well-known that the Dedekind completion $D(L)$ of L is defined uniquely up to isomorphism and that there exists a canonical embedding of L into $D(L)$.

A lattice L is said to be a *regular sublattice* of a lattice L' if L is embedded into L' such that the embedding preserves all existing joins and meets in L.

3.1. LEMMA. ([10]) *Let L and L' be lattices such that L' is complete and*

(i) *L is a regular sublattice of L';*

(ii) *for each $a \in L'$ there exist subsets X and Y of L such that the relation*

$$\sup X = a = \inf Y$$

are valid in L'.

Then L' is a Dedekind completion of L.

3.1.1. COROLLARY. *Let L and L' be as in* 3.1. *Let $p, q \in L$, $p \leqq q$. We denote*

$$P_1 = \{x \in L : \ p \leqq x \leqq q\},$$
$$P_2 = \{x \in L' : \ p \leqq x \leqq q\}.$$

Then P_2 is a Dedekind completion of P_1.

We recall a definition from [8].

Assume that L^0 is a direct product of lattices L_i $(i \in I)$. For each $i \in L_i$ let 0^i be the least element of L_i. The elements of L^0 are denoted as $(x_i)_{i \in I}$.

For any fixed $i \in I$ we put

$$\overline{L}_i = \left\{x \in L^0 : \ x_j = 0^j \text{ for each } j \in I \setminus \{i\}\right\}.$$

Let L^1 be a sublattice of L^0. For $i \in I$ we denote

$$\overline{L}'_i = \left\{x \in L^1 : \ x_i = 0^i\right\}.$$

The lattice L^1 is said to be a *strong subdirect product* of the lattices L_i $(i \in I)$ if the relation

$$L^1 = \overline{L}_i \times \overline{L}'_i \tag{1}$$

is valid for each $i \in I$.

In more details, the relation (1) is understood in the sense that the following conditions are valid:

(a) $\overline{L}_i \subseteq L^1$;

(b) the morphism $\varphi_i(x) = \left(x_{i1}, x_{i2}\right)$ is an isomorphism of L^1 onto $\overline{L}_i \times \overline{L}'_i$, where
$x_{i1} \in \overline{L}_i$, $\left(x_{i1}\right)_i = x_i$,
$x_{i2} \in \overline{L}'_i$, $\left(x_{i2}\right)_j = x_j$ for each $j \in I \setminus \{i\}$.

Assume that L^1 is a strong subdirect product of lattices L_i $(i \in I)$. Further, suppose that for each $i \in I$, L_i has the greatest element 1^i. Then there exist elements e^i, e^{*i} in L^0 such that

$$(e^i)_i = 1^i\,, \qquad (e^i)_j = 0^j \quad \text{for } j \in I \setminus \{i\}\,,$$
$$(e^{*i})_i = 0^i\,, \qquad (e^{*i})_j = 1^j \quad \text{for } j \in I \setminus \{i\}\,.$$

For each $x \in L^0$ and $i \in I$ we denote by $\overline{x}_i$ the element of $\overline{L}_i$ such that

$$(\overline{x}_i)_i = x_i\,.$$

It is obvious that the mapping

$$x_i \to \overline{x}_i$$

is an isomorphism of the lattice L_i onto the lattice $\overline{L}_i$.

3.2. LEMMA. *For each $x \in L^0$, the relation*

$$x = \bigvee_{i \in I} \overline{x}_i \tag{2}$$

is valid in L^0.

Proof. For each $i, j \in I$ we have $(\overline{x}_i)_j \leqq x_j$. Hence x is an upper bound of the set $\{\overline{x}_i\}_{i \in I}$. Let $y \in L^0$, $y \geqq \overline{x}_i$ for each $i \in I$. Thus $y_j \geqq (\overline{x}_i)_j$ for each $i, j \in I$, whence $y_i \geqq x_i$ for each $i \in I$. Therefore (2) holds. □

We have clearly

$$\overline{x}_i = x \wedge e^i \qquad \text{for each} \quad i \in I\,.$$

For $x \in L^0$ and $i \in I$ let $\overline{x}_i^*$ be the element of L^0 such that

$$(\overline{x}_i^*)_i = x_i\,,$$
$$(x_i^*)_j = 1^j \qquad \text{for each} \quad j \in I \setminus \{i\}\,.$$

Further, we denote

$$\overline{L}_i^* = \{\overline{x}_i^* :\ x \in L^1\}\,.$$

Then $\overline{L}_i^*$ is a sublattice of L^1.

By analogous method as in the proof of 3.2 we obtain:

3.2.1. LEMMA. *For each $x \in L^0$, the relation*

$$x = \bigwedge_{i \in I} \overline{x}_i^*$$

is valid in L^0.

Also, we have

$$\overline{x}_i^* = x \vee e^i = \overline{x}_i \vee e^i\,.$$

By a simple calculation we can verify

3.3. Lemma. *The mapping defined by $\overline{x}_i \to \overline{x}_i^*$ is an isomorphism of the lattice $\overline{L}_i$ onto the lattice $\overline{L}_i^*$.*

Denote

$$D(L_i) = D_i, \qquad D = \prod_{i \in I} D_i.$$

Hence both L^0 and L^1 are sublattices of the lattice D, and D is a complete lattice.

3.4. Lemma. *Let $\emptyset \neq X \subseteq L^0$ and suppose that the relation $y = \sup X$ is valid in L^0. Let $i \in I$. Then*

$$y_i = \sup\{x_i\}_{x \in X}$$

holds in L_i.

Proof. For each $x \in X$ we have $x \leqq y$, whence $x_i \leqq y_i$. Let $z \in L_i$ be such that $z \geqq x_i$ for each $x \in X$. There exists $z_0 \in L^0$ with $(z_0)_i = z$ and $(z_0)_j = y_j$ whenever $j \in I \setminus \{i\}$. Thus $(z_0)_j \geqq x_j$ for each $j \in I$ and each $x \in X$. Hence $z_0 \geqq x$ for each $x \in X$. Therefore $z_0 \geqq y$ and $(z_0)_i \geqq y_i$. This yields that $z \geqq y_i$, whence $y_i = \sup\{x_i\}_{x \in X}$. □

3.5. Lemma. *Let $\emptyset \neq X \subseteq L^0$, $y \in L^0$ and suppose that for each $i \in I$ the relation $\sup\{x_i\}_{x \in X} = y_i$ is valid in L_i. Then $y = \sup X$ holds in L^0.*

Proof. We have $y \geqq x$ for each $x \in X$. Let $v \in L^0$, $v \geqq x$ for each $x \in X$. Hence $v_i \geqq x_i$ for each $i \in I$ and each $x \in X$. Thus $v_i \geqq y_i$ for each $i \in I$ and therefore $v \geqq y$. Thus $y = \sup X$. □

Analogously we can verify the assertions which are dual to 3.4 or to 3.5, respectively.

3.6. Lemma. *Let L^1 and L^0 be as above. Then L^1 is a regular sublattice of L^0.*

Proof. Let $\emptyset \neq X \subseteq L^1$, $y \in L^1$ and suppose that the relation

$$\sup X = y$$

holds in L^1. Let $z \in L^0$, $z \geqq x$ for each $x \in X$.

Take any fixed $i \in I$. Since L^1 is a strong subdirect product of lattices L_i, the relation (1) above is valid.

We apply 3.4 for the direct product decomposition (1) (i.e., we have now L^1 instead of L^0). Thus the relation

$$y_{i1} = \sup\{x_{i1}\}_{x \in X}$$

is valid in $\overline{L}_i$. In view of the above mentioned isomorphism between $\overline{L}_i$ and L_i we obtain, that

$$y_i = \sup\{x_i\}_{x\in X}$$

holds in L_i.

In view of 3.2 we infer that the relations

$$z = \bigvee_{i\in I} \overline{z}_i\,, \qquad y = \bigvee_{i\in I} \overline{y}_i$$

are valid in L^0. Further, $z_i \geqq x_i$ for each $i \in I$ and each $x \in X$. Hence $\overline{z}_i \geqq \overline{y}_i$ for each $i \in I$. Then $z \geqq y$. Thus $\sup X = y$ in L^0. Analogously we can verify the dual result. Therefore L^1 is a regular sublattice of L^0. □

3.7. LEMMA. *L^0 is a regular sublattice of the lattice $\prod\limits_{i\in I} D(L_i)$.*

P r o o f . It is obvious that L^0 is a sublattice of the lattice $\prod\limits_{i\in I} D(L_i)$; we denote this lattice by L^d.

Let $\emptyset \neq X \subseteq L^0$ and suppose that $\sup X = y$ holds in L^0. We remark that for each $t \in L^0$ and each $i \in I$ we have

$$t(L_i) = t\big(D(L_i)\big)\,.$$

According to 3.4, the relation

$$y_i = \sup\{x_i\}_{i\in I}$$

is valid in L_i. In view of 3.1, this relation holds also in $D(L_i)$. Now we apply 3.5 with the distinction that instead of L^0 we consider the lattice L^d. Hence the relation $y = \sup X$ holds in L^d. The corresponding dual result can be proved analogously. Hence L^0 is a regular sublattice of L^d. □

3.8. LEMMA. *Let L^1 and L^d be as above. Then L^1 is a regular sublattice of L^d.*

P r o o f . This is a consequence of 3.6 and 3.7. □

Let $i \in I$. We denote by $\overline{D(L_i)}$ the set of all $x \in L^d$ such that $x_j = x\big(D(L_j)\big) = 0_j$ for each $j \in I \setminus \{i\}$.

In view of the isomorphism between L_i and $\overline{L}_i$ we immediately obtain:

3.9. LEMMA. *$\overline{D(L_i)}$ is the Dedekind completion of the lattice $\overline{L}_i$.*

3.10. LEMMA. *Let $t \in L^d$. Then there exists a subset $X \subseteq L^1$ such that the relation $\sup X = t$ is valid in L^d.*

Proof. Let $i \in I$. The symbol $\bar{t}_i$ is defined analogously as the symbol $\bar{x}_i$ above (cf. 3.2) with the distinction that we now deal with L^d instead of L^0. In view of 3.2 we have

$$t = \bigvee_{i \in I} \bar{t}_i$$

in L^d, where $\bar{t}_i \in \overline{D(L_i)}$.

According to 3.9 and 3.1, for each $\bar{t}_i$ there exists a subset $\{a_{ij}\}$ ($j \in J_i$) of $\overline{L}_i$ such that the relation

$$\bar{t}_i = \bigvee_{j \in J_i} a_{ij}$$

is valid in the lattice $\overline{D(L_i)}$; hence this relation holds in L^d as well. Put

$$X = \{a_{ij}\}_{i \in I, j \in J_i} .$$

Then $t = \sup X$ holds in L^d. □

Now let us suppose that the lattice L^1 satisfies the condition

$$e^{*i} \in L^1 \qquad \text{for each} \quad i \in I. \tag{$*$}$$

Then for each $x \in L^1$ and each $i \in I$, the element

$$\bar{x}_i^* = x \vee e^{*i}$$

belongs to L^1.

By the method dual to that just applied above and by using 3.2.1 instead of 3.2 we conclude:

3.10.1. LEMMA. *Let $t \in L^d$. Then there is a subset $Y \subseteq L^1$ such that the relation $\inf Y = t$ is valid in L^d.*

From 3.1, 3.6, 3.10 and 3.10.1 we obtain:

3.11. PROPOSITION. *Let the lattice L^0 be a direct product of lattices L_i ($i \in I$). Assume that L^1 is a sublattice of L^0 such that*

(i) *L^1 is a strong subdirect product of lattices L_i ($i \in I$);*
(ii) *L^1 satisfies the condition $(*)$.*

Let L^d be the direct product of lattices $D(L_i)$ ($i \in I$). Then L^d is the Dedekind completion of L^1.

3.11.1. Corollary. *Let L^0 and L^d be as in 3.11. Then L^d is the Dedekind completion of L^0.*

We conclude this section by remarking that the considerations contained here remain valid (with the obvious modifications) if instead of the assumption

$$L^0 = \prod_{i\in I} L_i$$

we assume that we are given a direct product decomposition of the lattice L^0

$$\varphi\colon L^0 \to \prod_{i\in I} L_i .$$

4. Auxiliary results

For the sake of completeness, we recall the definition of a particular type of direct product decompositions of an MV-algebra which will be called *internal* (cf. [7]).

Assume that

$$\varphi\colon \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i \tag{1}$$

is a direct product decomposition of the MV-algebra $\mathcal{A}$. For each $i(1) \in I$ we denote

$$A^0_{i(1)} = \{a \in A : \ a_i = 0_i \ \text{ for each } \ i \in I \setminus i(1)\} .$$

We have $A^0_{i(1)} \subseteq A$ and $0 \in A^0_{i(1)}$. We define the operation $\oplus_{i(1)}$ on $A_{i(1)}$ as follows. Let $a, b \in A^0_{i(1)}$. There exists $c \in A^0_{i(1)}$ such that $(a+b)_{i(1)} = c_i(i)$. We put $a \oplus_{i(1)} b = c$. Further, we set $\neg_{i(1)} a = (\neg a)_{i(1)}$. Then $\mathcal{A}^0_{i(1)} = (A^0_{i(1)}; \oplus_{i(1)}, \neg_{i(1)}, 0_{i(1)})$ is an MV-algebra.

Let $i \in I$ and let x^i be an arbitrary element of A_i. We denote by $\varphi_i(x^i)$ the element of A^0_i such that

$$\bigl(\varphi_i(x^i)\bigr)_i = x^i .$$

Then φ_i is an isomorphism of $\mathcal{A}_i$ onto $\mathcal{A}^0_i$.

Further, for each $x \in A$ we put

$$\varphi^0(x) = (\dots, \varphi_i(x_i), \dots)_{i\in I} .$$

Then φ^0 is an isomorphism of $\mathcal{A}$ onto $\prod\limits_{i\in I} \mathcal{A}^0_i$. We say that φ^0 is an *internal direct product decomposition of the MV-algebra $\mathcal{A}$.*

Now let L be a lattice with the least element 0; consider a direct product decomposition of L having the form

$$\varphi\colon L \to \prod_{i\in I} L_i\,.$$

Similarly as above, for each $i(1) \in I$ we put

$$L^0_{i(1)} = \{x \in L:\ x_i = 0_i \text{ for each } i \in I \setminus \{i(1)\}\}\,.$$

Hence $L^0_{i(1)} \subseteq L$ and $0 \in L^0_{i(1)}$. The lattice operations in $L^0_{i(1)}$ are induced from those in L. For each $i \in I$, the mapping $\varphi_i\colon L_i \to L^0_i$ is defined analogously as in the case of MV-algebras. The definition of the relation

$$\varphi^0\colon L \to \prod_{i\in I} L^0_i$$

is also analogous to that applied for MV-algebras. Then φ^0 is an *internal direct product decomposition of* L.

More generally, this notion can be used for connected partially ordered sets (cf. [9]) and for algebras having an one-element subalgebra (cf. [6]).

To each direct product decomposition φ of an MV-algebra (or of a lattice with the least element) there corresponds an internal direct product decomposition φ^0. When our considerations are made up to isomorphism, then we need not distinguish between a direct product decomposition and the corresponding internal direct product decomposition.

We apply the results of Section 3 for an internal direct product decomposition of the lattice $\ell(\mathcal{A})$.

Again, suppose that (1) is valid. Let $\mathcal{A}^1$ be a subalgebra of $\mathcal{A}$ such that for each $i(1) \in I$ we have

$$A^0_{i(1)} \subseteq A^1\,,$$

where A^1 is the underlying set of the MV-algebra $\mathcal{A}^1$. Consider the partial mapping

$$\varphi^1 = \varphi|_{A^1}\,.$$

Then we say that

$$\varphi^1\colon \mathcal{A}^1 \to \prod_{i\in I} \mathcal{A}_i \tag{2}$$

is a *strong subdirect product decomposition* of the MV-algebra $\mathcal{A}^1$.

This definition is a slight modification of that used in [8] (the difference disappears when we are working 'up to isomorphism'). The results of [8] remain valid also under the present definition.

Assume that (2) is a strong subdirect product decomposition of the MV-algebra $\mathcal{A}^1$. Let $\ell(\mathcal{A}^1)$ and $\ell(\mathcal{A}_i)$ $(i \in I)$ be the corresponding underlying lattices. Then the mapping φ^1 gives, at the same time, a strong subdirect decomposition

$$\varphi^1 : \ell(\mathcal{A}^1) \to \prod_{i \in I} \ell(\mathcal{A}_i) \tag{2'}$$

of the lattice $\ell(\mathcal{A}^1)$.

Let $i \in I$. In accordance with the notation from Section 3 we denote by e^i the greatest element of the lattice $\ell(\mathcal{A}_i^0)$. Further, let e^{i*} be the element of A such that

$$(e^{i*})_i = 0_i \qquad \text{and} \qquad (e^{i*})_j = 1_j \quad \text{for each } j \in I \setminus \{i\}.$$

Then we have

$$e^i \wedge e^{i*} = 0, \qquad e^i \vee e^{i*} = 1.$$

From these relations we easily obtain

$$e^{i*} = \neg e^i. \tag{3}$$

From (2') we get $e^i \in A^1$ and then (3) yields that e^{i*} also belongs to A^1. Hence we obtain:

4.1. Lemma. *Let $\mathcal{A}^1$ be a strong subdirect product of MV-algebras $\mathcal{A}_i$ $(i \in I)$. Then the lattice $\ell(\mathcal{A}^1)$ satisfies the condition $(*)$ from Section 3, where $L_i = \ell(\mathcal{A}_i)$.*

Consider the relation $\mathcal{A} = \Gamma(G, u)$ mentioned in Section 2. This relation implies:

4.2. Lemma. *Let $a \in A$. Then $\neg a$ is the least element of the set*

$$\{x \in A : \ a \oplus x = 1\}.$$

4.2.1. Lemma. *The operation $\neg$ on A is uniquely determined by the operation $\oplus$ and the partial order $\leqq$ on A.*

4.3. Lemma. *Let $i(1) \in I$ and $a, b \in A^0_{i(1)}$. Then*

$$a \oplus_{i(1)} b = a \oplus b.$$

P r o o f. Consider the direct product decomposition φ^0 of $\mathcal{A}$. For each $x \in A^0_{i(1)}$ we have

$$x(\mathcal{A}^0_{i(1)}) = x.$$

Further, in view of the definition of the operation $\oplus_{i(1)}$ on the set $A^0_{i(1)}$ we get

$$a \oplus_{i(1)} b = (a \oplus b)(\mathcal{A}^0_{i(1)}) .$$

Hence

$$a \oplus_{i(1)} b = a(\mathcal{A}^0_{i(1)}) \oplus b(\mathcal{A}^0_{i(1)}) = a \oplus b .$$

□

We slightly modify the formulation of [7; Theorem 3.5] (cf. also [7; Lemma 3.4]); we obtain:

4.4. PROPOSITION. *Let $\mathcal{A}$ be an MV-algebra, $L = \ell(\mathcal{A})$ and let*

$$\varphi^0 \colon L \to \prod_{i \in I} L^0_i$$

be an internal direct product decomposition of the lattice L. Then the mapping φ^0 yields also an internal direct product decomposition of $\mathcal{A}$

$$\varphi^0 \colon \mathcal{A} \to \prod_{i \in I} \mathcal{A}^0_i$$

such that for each $i \in I$ we have $\ell(\mathcal{A}^0_i) = L^0_i$.

5. Proofs of (A) and (B)

Assume that $\mathcal{A}$ is an archimedean MV-algebra. We apply the notation as above. Let (A) and (B) be as in Section 1.

Proof of (A).

Suppose that $\mathcal{A}$ is a strong subdirect product of MV-algebras $\mathcal{A}_i$ $(i \in I)$. Then the lattice $L = \ell(\mathcal{A}_i)$ is a strong subdirect product of lattices $L_i = \ell(\mathcal{A}_i)$.

Thus in view of 3.11 and 4.1, there is a direct product decomposition

$$\varphi \colon D(L) \to \prod_{i \in I} D_i ,$$

where $D_i = D(L_i)$.

Let us consider the internal direct product decomposition φ^0 corresponding to the direct product decomposition φ

$$\varphi^0 \colon D(L) \to \prod_{i \in I} D^0_i .$$

Consider the Dedekind completion $D(\mathcal{A})$ of the MV-algebra $\mathcal{A}$. Then we have $D(L) = \ell(D(\mathcal{A}))$.

We apply Proposition 4.4 for the MV-algebra $D(\mathcal{A})$ and for the lattice $D(L)$. Hence the mapping φ^0 yields, at the same time, an internal direct product decomposition of $D(\mathcal{A})$

$$\varphi^0 \colon D(\mathcal{A}) \to \prod_{i\in I} \mathcal{X}_i$$

such that for each $i \in I$ we have

$$\ell(\mathcal{X}_i) = D_i^0 .$$

Let $i(1) \in I$. Similarly as in Section 4 we denote by $A_{i(1)}^0$ the set of all $a \in A$ such that

$$a_i = 0_i \qquad \text{for each} \quad i \in I \setminus \{i(1)\} .$$

Further, we define the operations $\oplus_{i(1)}$ and $\neg_{i(1)}$ on the set $A_{i(1)}^0$ in the same way as in Section 4. Let $\mathcal{A}_i^0$ be the corresponding MV-algebra.

We will investigate the relations between the MV-algebras $\mathcal{X}_{i(1)}$ and $D(\mathcal{A}_{i(1)}^0)$.

a) $D_{i(1)}^0$ is the interval with the endpoints 0 and $e^{i(1)}$ od $D(L)$. Also, the underlying set of $\mathcal{A}_{i(1)}^0$ is the interval with the endpoints 0 and $e^{i(1)}$ of the lattice $\ell(\mathcal{A}) = L$. Thus in view of 3.1.1 we obtain that the underlying lattices of $\mathcal{X}_1$ and of $D(\mathcal{A}_{i(1)}^0)$ are equal.

b) The algebra $\mathcal{A}_{i(1)}^0$ is a subalgebra of $D(\mathcal{A}_{i(1)}^0)$. Further, since $\mathcal{X}_{i(1)}$ is an internal direct factor of $D(\mathcal{A})$, by applying 4.3 we conclude that if $a, b \in A_{i(1)}$, then the operation $\oplus$ used for a and b yields the same result in $\mathcal{X}_{i(1)}$ and in $D(\mathcal{A}_{i(1)}^0)$ (and, in fact, also in $\mathcal{A}$).

c) Next, if a' and b' are any elements of $D(A_{i(1)}^0)$, then there exist subsets X and Y of $A_{i(1)}^0$ such that the relations

$$\sup X = a' , \qquad \sup Y = b'$$

hold in $\ell(D(A_{i(10)}^0))$. Hence in $D(A_{i(1)}^0)$ we have

$$a' \oplus b' = \sup\{x \oplus y : \ x \in X , \ y \in Y\} .$$

In view of a) and b), this equation holds also in $\mathcal{X}_{i(1)}$. Thus the operation $\oplus$ in $D(A_{i(1)}^0)$ coincides with the operation $\oplus$ in $\mathcal{X}_{i(1)}$.

d) In view of a), c) and 4.1 we get

$$\mathcal{X}_{i(1)} = D(\mathcal{A}_{i(1)}^0) .$$

Hence we have a direct product decomposition

$$\varphi^0 \colon D(\mathcal{A}) \to \prod_{i\in I} D(\mathcal{A}_i^0) .$$

Since $\mathcal{A}_i^0$ is isomorphic to $\mathcal{A}_i$ we conclude that (A) is valid. □

Proof of (B).

a) Assume that $\mathcal{A}$ is b-atomic. Then in view of [8; Proposition 4.3], $\mathcal{A}$ is a strong subdirect product of linearly ordered MV-algebras. It is obvious that the Dedekind completion of a linearly ordered set is again linearly ordered. Hence in view of 3.11 and 4.1 we infer that the lattice $D(\ell(\mathcal{A}))$ is a direct product of linearly ordered sets. Since to each direct product decomposition of a lattice there corresponds an internal direct product decomposition, according to 4.4 the MV-algebra $D(\mathcal{A})$ be expressed as a direct product of linearly ordered MV-algebras.

b) Conversely, suppose that $D(\mathcal{A})$ is a direct product of linearly ordered MV-algebras. Without loss of generality we can assume that the direct product under consideration is internal, i.e., we have (under the notation as above)

$$\varphi^0 \colon D(\mathcal{A}) \to \prod_{i\in I} \mathcal{B}_i^0 ,$$

where all $\mathcal{B}_i^0$ are linearly ordered. Let $0 < b \in B$ (we denote by B the underlying set of $D(\mathcal{A})$). Then there exists $i(1) \in I$ such that $0 < b(\mathcal{B}_{i(1)}^0) \in B_i^0$. Thus the interval $[0, b(\mathcal{B}_{i(1)}^0)]$ of the lattice $\ell(D(\mathcal{A}))$ is a chain.

There exists a subset X of A such that the relation

$$\sup X = b(\mathcal{B}_{i(1)}^0)$$

is valid in the lattice $\ell(D(\mathcal{A}))$. Then there is $x \in X$ with $0 < x$. Moreover, the set $X_1 = \{y \in A : \ y \leqq x\}$ is a subset of the above mentioned interval $[0, b(\mathcal{B}_{i(1)}^0)]$; thus X_1 is a chain. Therefore x is a basic element of $\mathcal{A}$. We conclude that the MV-algebra $\mathcal{A}$ is b-atomic. □

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