Alexander Wilce

Perspectivity and congruence in partial abelian semigroups

Mathematica Slovaca, Vol. 48 (1998), No. 2, 117--135

Persistent URL: http://dml.cz/dmlcz/128774

Math. Slovaca, **48** (1998), No. 2, 117–135

Mathematica Slovaca

PERSPECTIVITY AND CONGRUENCE IN PARTIAL ABELIAN SEMIGROUPS

Alexander Wilce

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. To any subset M of a partial abelian semigroup L, we associate a relation of *perspectivity* $\sim_M$, where $a \sim_M b$ if and only if there exists some element $c \in L$ such that $a \oplus c$ and $c \oplus b$ both exist and belong to M. If $\sim_M$ is a (faithful) congruence, we say that M is *algebraic*. The theory of pairs (L, M), where L is a partial abelian semigroup and M is a fixed algebraic subset provides a natural generalization of the theory of manuals.

Introduction

This paper concerns the common partial-algebraic background of the various "orthostructures" that have been considered in the literature on quantum logic: Orthomodular posets, orthoalgebras, D-posets (or effect algebras), and their respective non-unital variants.

The most familiar of these are orthomodular partially ordered sets (cf. [9]). If L is an orthocomplemented poset, one calls elements $a, b \in L$ *orthogonal* if and only if $a \leq b'$; L is *orthomodular* if and only if for all orthogonal pairs $a, b \in L$,

(i) $a \vee b$ exists, and
(ii) $(a \vee b) \wedge b' = a$.[1)]

It is usual to refer to the join guaranteed by (i) as the *orthogonal sum* of a and b, and accordingly to write $a \oplus b$ for $a \vee b$ when $a \leq b'$. Evidently, this partial binary operation is commutative, and it is not difficult to show that the orthomodular law (ii) is equivalent to the condition that the partial binary operation $\oplus$ be both associative and cancellative. Thus, any OMP $(L, \leq, ', 0, 1)$ gives rise to a partial abelian semigroup $(L, \oplus)$. We can recover the partial

AMS Subject Classification (1991): Primary 03G12, 81P10; Secondary 06F99, 08A55.
Key words: effect algebra, orthoalgebra, orthomodular poset, partial abelian semigroup, perspectivity.

[1)] Concerning (ii), note that $b \leq (a \vee b)''$, so that $(a \vee b) \wedge b' = \big((a \vee b)' \vee b\big)'$ exists by virtue of (i).

ordering from the orthogonal sum, since $a \le b$ if and only if $\exists c \in L$ (namely, $b \wedge a'$) with $a \oplus c = b$.

While it is a matter of taste whether to view an OMP as primarily an order-theoretic or primarily a partial-algebraic object, this is no longer the case when we consider more general orthostructures. An *orthoalgebra* ([2], [6]) is a cancellative partial abelian semigroup $(L, \oplus)$ possessing a *unit element* $\mathbf{1}$ such that,

(i) $\forall a \in L \ \exists! a' \in L \quad a \oplus a' = \mathbf{1}$,
(ii) $\forall a \in L \quad \exists a \oplus a \implies a = \mathbf{0} := \mathbf{1}'$.

If we relax condition (ii) and instead require only

(iii) $\forall a \in L \quad \exists a \oplus \mathbf{1} \implies a = \mathbf{0} := \mathbf{1}'$,

we obtain the objects variously called *weak orthoalgebras* [7], *D-posets* [3], [4], [11], or *effect algebras* [5], [8]. We shall use the last term. Any orthoalgebra or effect algebra L may be partially-ordered by declaring $a \le b$ if and only if $b = a \oplus c$ for some c. $(L, \le, ')$ is then an involutive poset (orthocomplemented if L is an orthoalgebra), the operation $\oplus$ is order-preserving in both arguments, and $a \oplus b$ is defined if and only if $a \le b'$. However, $a \oplus b$ need not be the join of a and b; hence, the partial order is definitely subordinate to the partial binary operation.

Our aim in this paper is to place these structures in a broader partial-algebraic context, by developing some of the relevant general theory of partial abelian semigroups. This offers, besides generality, much in the way of unity and clarity of presentation. In particular, we are able to re-interpret a number of known results concerning orthoalgebras and D-algebras as instances of universal constructions involving cancellative partial abelian semigroups.

An outline of the paper is as follows: In Section 1, we collect basic definitions, examples, and some elementary results concerning the natural pre-order on a partial abelian semigroup. In Section 2, we discuss certain "faithful" congruences on partial abelian semigroups, showing that any PAS supports a canonical such congruence, and that the corresponding quotient has a certain universal property. Using this result, it is shown that any cancellative, unital partial abelian semigroup has a universal effect-algebraic homomorphic image. In Section 3, we consider a special sort of faithful congruence which we call a *perspectivity*. In Section 4, we obtain a representation theorem for cancellative, unital partial abelian semigroups generalizing that of Foulis and Randall for orthoalgebras (cf. [9]) and the analogous representation for D-algebras of Dvurečenskij and Pulmannová [4]. We also show that every unital PAS has a universal cancellative, unital homomorphic image. Similar arguments are deployed in Section 5 to construct a tensor product for cancellative, unital partial abelian semigroups, extending work of Bennett and Foulis [2] and Dvurečenskij [3].

Some of our results have been announced previously ([13], [14]).

1. Preliminary results

By a *partial abelian semigroup* (*PAS*) we mean a structure $(L, \perp, \oplus)$, where $\perp$ is a binary relation on L, and $\oplus$ is a partially defined binary operation with domain $\perp$ satisfying

$$p \oplus q = q \oplus p, \tag{1}$$

$$(p \oplus q) \oplus r = p \oplus (q \oplus r). \tag{2}$$

These identities are to be understood as asserting that if the term on *either* side is defined, so is that on the other, and the two are equal.

The associativity condition is quite strong, and in fact does most of the work in what follows. It may be well to consider, for contrast, two examples of partially-defined commutative binary operations that are associative in a weak sense, but not in the sense of (2). First, let S be an arbitrary semigroup. Define $a \perp b$ if and only if $ab = ba$, and set $a \circ b = ab$ if this is the case. Certainly, if *both* $a \circ (b \circ c)$ and $(a \circ b) \circ c$ exist, they are equal; however, the existence of the one by no means implies that of the other. Next, consider an arbitrary partially ordered set $(P, \leq)$. Let $a \perp b$ if and only if $a \vee b$ exists in P. Again, if both $a \vee (b \vee c)$ and $(a \vee b) \vee c$ exist, they are equal; however, one may construct simple examples in which $b \vee c$ and $a \vee (b \vee c)$ exist while $a \vee b$ does not.

A *zero* in a PAS L is an element $\mathbf{0}$ such that $p \perp \mathbf{0}$ for all $p \in L$ and $p \oplus \mathbf{0} = p$. The usual argument shows that a zero, if any exists, is unique. The formal adjunction of a zero to a PAS presents no difficulty; we shall therefore assume henceforth that *every PAS* (*and, in particular, every abelian semigroup*) *possesses a zero.*

We say that L is *cancellative* if and only if for all $a, b, c \in L$,

$$a \perp c \perp b \ \ \& \ \ a \oplus c = b \oplus c \implies a = b.$$

L is *positive* if and only if for all $a, b \in L$, $a \oplus b = \mathbf{0} \implies a, b = \mathbf{0}$. (Note that if L contains no zero, then the formal adjunction of a zero yields a positive PAS.)

1.1. Examples.

(a) Let L be any complete $\wedge$-semilattice with 0. For $a, b \in L$, set $a \perp b$ if and only if $\exists c \in L$ with $a, b \leq c$. Then $a \perp b \implies a \vee b$ exists in L, and $(L, \perp, \vee)$ is a positive PAS, but need not be cancellative.

(b) Let L be an orthomodular poset. As usual, set $p \perp q$ if and only if $p \leq q'$ and $p \oplus q = p \vee q$ for $p \perp q$. Then $(L, \perp, \oplus)$ is a positive, cancellative PAS.

(c) Let X be an set, and $\mathcal{E}$, a collection of subsets of X with the property that if $A \in \mathcal{E}$ and $B \subseteq A$, then $B \in \mathcal{E}$. For $A, B \in \mathcal{E}$, let $A \perp B$ if and only if $A \cap B = \emptyset$ and $A \cup B \in \mathcal{E}$; in this case, set $A \oplus B = A \cup B$. Then $(\mathcal{E}, \perp, \oplus)$ is a cancellative, positive PAS.

(d) Let $\mathfrak{F}$ be a collection of non-negative integer-valued functions on a set X with the property that if $f \in \mathfrak{F}$ and $g \leq f$, then $g \in \mathfrak{F}$. For $f, g \in \mathfrak{F}$, let $f \perp g$ if and only if $f + g \in E$, in which case define $f \oplus g = f + g$. Note that if all the functions in $\mathfrak{F}$ are $\{0,1\}$-valued, we recover example (c).

(e) Let A be an ordered abelian group with positive cone A_+. For any $e \in A_+$, the interval $[0,e] = \{a \in A \mid 0 \leq a \leq e\}$ becomes a PAS if we take $a \perp b$ to mean $a + b \leq e$, and set $a \oplus b = a + b$ in this case.

In any PAS L, we may define a relation $a \leq b$ if and only if $\exists x \in L$ such that $a \oplus x = b$. If L is cancellative, then this element x is unique, and we denote it by $b - a$.

1.2. Lemma. *Let L be a PAS. For $p, q \in L$, let $p \leq q \iff \exists r \in L \ \ p \oplus r = q$. Then*

(a) *The relation $\leq$ is reflexive and transitive, i.e., a pre-ordering.*
(b) *If $\leq$ is a partial ordering, then L is positive.*
(c) *If L is cancellative and positive, then $\leq$ is a partial ordering.*

Proof.

(a) Since $a \oplus \mathbf{0} = a$, $a \leq a$ for any $a \in L$. Transitivity follows from the associativity of $\oplus$.

(b) For any $a \in L$, we have $\mathbf{0} \leq a$. Hence, if $a \oplus b = \mathbf{0}$ and $\leq$ is a partial ordering, $a, b \leq \mathbf{0} \leq a, b$, whence, $a = b = \mathbf{0}$.

(c) Suppose L is cancellative and positive, and suppose that $a, b \in L$ with $a \leq b \leq a$. Then $a = b \oplus y = (a \oplus x) \oplus y$ for some $x, y \in L$. By associativity, $a = a \oplus (x \oplus y)$, whence, by cancellation, $x \oplus y = \mathbf{0}$. Positivity now yields $x = y = \mathbf{0}$, whence, $a = b$. □

Let us say that two elements a, b of a PAS are *strongly orthogonal* if and only if $a \perp b$ and for all $z \leq a, b$, $z \perp z \implies z = \mathbf{0}$. The following generalizes a standard result concerning orthoalgebras (cf. [6]):

1.3. Proposition. *Let L be a positive cancellative PAS. Then a and b are strongly orthogonal if and only if $a \oplus b$ is a minimal upper bound for $a, b \in L$.*

Proof. Suppose a, b are strongly orthogonal. Certainly $a, b \leq a \oplus b$. Let $a, b \leq c \leq a \oplus b$: Then $c = a \oplus x = b \oplus y$ and $a \oplus b = c \oplus z = a \oplus (x \oplus z)$, whence, by cancellation, $x \oplus z = b$. By the same token, $a \oplus b = b \oplus (y \oplus z)$, whence, $y \oplus z = a$. Thus $z \leq a, b$, and (as $a \perp b$) $z \perp z$, whence, $z = \mathbf{0}$. It follows that $c = a \oplus b$. Conversely, suppose $\mathbf{0} \neq z \leq a, b$ and $z \perp z$. Then $a = x \oplus z$, $b = y \oplus z$ for some $x, y \in L$. Let $c = x \oplus y \oplus z$, and observe that $a, b \leq c$ and $a \oplus b = c \oplus z > c$. □

We call an element u of a PAS L a *unit* if and only if for every $a \in L$, there exists at least one $b \in L$ such that $a \oplus b = u$. A *unital* PAS is a PAS with a distinguished unit, which we denote generically by $\mathbf{1}$.

Evidently, u is a unit if and only if $a \leq u$ for every $a \in L$. If $\leq$ is a partial ordering, L has at most one unit. On the other hand, *every* element of an abelian group is a unit.

1.4. LEMMA. *Let L be a unital PAS with unit $\mathbf{1}$.*

(a) *L is cancellative if and only if*

$$\forall a{\in}L\ \exists!\, a'{\in}L \qquad a \oplus a' = \mathbf{1}. \tag{3}$$

(b) *If L is a cancellative, then L is positive if and only if*

$$\forall a{\in}L \qquad a \perp \mathbf{1} \implies a = \mathbf{0}. \tag{4}$$

Proof.

(a) Clearly, (3) holds in any cancellative unital PAS. Conversely, if L satisfies (3) and $a, b, c \in L$ with $a \oplus b = a \oplus c = d \in L$, then $d' \oplus (a \oplus b) = \mathbf{1} = d' \oplus (a \oplus c)$. By associativity, $(d' \oplus a) \oplus b = \mathbf{1} = (d' \oplus a) \oplus c$. But then $b = (d' \oplus a)' = c$. Thus, L is cancellative.

(b) Suppose L is cancellative. If L is positive and $a \oplus \mathbf{1}$ exists, then $a \oplus (a \oplus \mathbf{1})' = \mathbf{1}' = \mathbf{0}$; by positivity, $a = \mathbf{0}$. Cancellativity plays no role in the converse: Suppose (4) holds and $a \oplus b = \mathbf{0}$. Then $a \oplus b \perp \mathbf{1}$, whence, $a \perp \mathbf{1}$ and $b \perp \mathbf{1}$, whence, $a = b = \mathbf{0}$. □

Thus, a positive, cancellative unital PAS is the same thing as an *effect algebra* as defined in [3].

1.5. Remark. Effect algebras go by several aliases in the literature. In [7], they are called *weak orthoalgebras*; in [3] it is shown that effect algebras are essentially the same things as the *D-posets* defined in [11] and elsewhere (the latter being defined in terms of a relative difference operation, rather than a partial sum). In [1], Dinges uses the term *D-semigroup* for a (not necessarily unital) cancellative, positive PAS. In one respect, our usage is more general than that of the cited papers: We count the degenerate PAS $\{\mathbf{0}\}$ as an effect algebra (with $\mathbf{1} = \mathbf{0}$).

An *orthoalgebra* (cf. [5], [6]) is a cancellative, unital PAS satisfying the condition that $\forall a{\in}L \quad a \perp a \implies a = \mathbf{0}$. Clearly, this is stronger than condition (4) of Lemma 1.4, so every orthoalgebra is an effect algebra.

By way of example, note that any orthomodular poset, regarded as a PAS as in Example 1.1(b), is an orthoalgebra. On the other hand, if A is an ordered abelian group and $e \in A_+$, the interval $[0, e]$, regarded as a PAS in the manner of Example 1.1 (e), is an effect algebra but seldom an orthoalgebra. This last is a special case of the following observation: If L is cancellative and positive, then for any $e \in L$ and any $p, q \leq e$, define $p \perp_e q$ if and only if $p \oplus q \leq e$. This is evidently a PAS, and inherits cancellativity and positivity from L; since e serves as a unit, $\big([0,e], \perp_e, \oplus, 0, e\big)$ is an effect algebra. The converse is evidently true as well: If $[0,e]$ is an effect algebra for every $e \in L$, then L is cancellative and positive.

It is natural to wonder whether one can adjoin a unit to a PAS. For a cancellative PAS, this is always possible. The following construction is essentially due to A. Mayet [12]; cf. also [10].

1.6. Lemma. *Let L be a cancellative PAS. For each $a \in L$, introduce a symbol $a' \notin L$, and form $L! = L \cup L'$. Extend the relation $\perp$ and the partial operation $\oplus$ to $L!$ as follows:*

(i) $\forall a, b \in L \quad a \perp b' \iff a \leq b$*; in this case,* $a \oplus b' = (b - a)'$.

(ii) $\forall a, b \in L \quad a' \not\perp b'$.

Then $L!$ is a cancellative, unital PAS with unit $\mathbf{1} = \mathbf{0}'$.

P r o o f . It is sufficient to check the associativity of the extended orthogonal sum. There are only two non-trivial cases. Let $a, b, c \in L$. If $a \perp b$ and $(a \oplus b) \perp c'$, then $a \oplus b \leq c$, whence, $c = (a \oplus b) \oplus x$ for a unique x. It follows that $b \leq c$, whence, $b \oplus c' = (c - b)' = (a \oplus x)'$ exists. Hence, $a \leq (a \oplus x)$, i.e., $a \perp (b \oplus c')$, and

$$a \oplus (b \oplus c') = \big((c-b) - a\big)' = x' = \big(c - (a \oplus b)\big)' = (a \oplus b) \oplus c' .$$

The other case, i.e., $(a \oplus b') \perp c$, is handled similarly, and is left to the reader. □

2. Homomorphisms and congruences

A function $f \colon L \to S$ from a PAS L to a PAS S is a *homomorphism* if and only if

(i) $\forall a, b \in L \quad a \perp b \implies f(a) \perp f(b)$ and $f(a \oplus b) = f(a) \oplus f(b)$,

(ii) $f(\mathbf{0}) = \mathbf{0}$.

(note that (ii) is redundant if S is cancellative). A homomorphism f is *faithful* if and only if $f(a) \perp f(b) \implies a \perp b$. We say that f is *unital* if and only if L and S are unital and $f(\mathbf{1}) = \mathbf{1}$.

Note that the trivial homomorphism $f \colon L \to \{\mathbf{0}\}$ is faithful if and only if L is a semigroup, i.e., if and only if $a \perp b$ for all $a, b \in L$. Also note that if $f \colon L \to S$ is faithful, $f(L) \subseteq S$ is a sub-PAS of S.

By a *faithful congruence* on a PAS L, we mean an equivalence relation $\approx$ on L such that for all $a, b, c \in L$, $a \approx b$ and $a \perp c$ imply $b \perp c$ and $a \oplus c \approx b \oplus c$. Denote by $[a]$ the equivalence class of $a \in L$ under a congruence $\approx$ on L; then the partial operation $[a] \oplus [b] = [a \oplus b]$ provided $a \perp b$ is well-defined and makes $L/\approx$ a PAS. The map $[\ ]\colon L \to L/\approx$ is a faithful homomorphism. Conversely, if $f\colon L \to M$ is a faithful surjective homomorphism, the relation $a \approx b \iff f(a) = f(b)$ is a faithful congruence, and L is canonically isomorphic to $L/\approx$.

Remarks. It should be noted that if L is an effect algebra, then any faithful homomorphism $f\colon L \to S$ is injective. The intersection of any set of faithful congruences is again a faithful congruence; however, not every equivalence relation generates a faithful congruence. For instance, the universal relation on L is a faithful congruence if and only if the constant map $f\colon L \to \{\mathbf{0}\}$ is faithful, in which case, as observed above, L is a semigroup.

An *ideal* is a set $J \subseteq L$ such that $\forall a, b \in L$ with $a \perp b$, $a \oplus b \in J$ if and only if $a, b \in J$ (cf. [8]). We call J a *null ideal* if and only if, in addition, $J \subseteq L^{\perp} := \{a \in L \mid \forall b \in L \quad a \perp b\}$. Note that every null-ideal of L is a commutative monoid under $\oplus$. The set $L^{\perp}$ is itself the largest null-ideal of L, and the intersection of null ideals is again a null-ideal; thus, the set of null-ideals of L is a complete lattice, and, in particular, has a smallest element, which we denote by J_0. Note that $J_0 = \{\mathbf{0}\}$ if and only if L is positive.

2.1. Lemma. *Let $\phi\colon L \to S$ be a faithful homomorphism.*

(a) *If $J \subseteq S$ is a null-ideal, so is $\phi^{-1}(J)$.*

(b) *If ϕ is surjective, $\phi^{-1}(\mathbf{0})$ is a null-ideal if and only if S is positive.*

Proof.

(a) Note first that if $a \in L$ and $\phi(a) = \mathbf{0}$, then for all $b \in L$, $\phi(a) \perp \phi(b)$, whence, because ϕ is faithful, $a \perp b$. Thus, $\phi^{-1}(S^{\perp}) \subseteq L^{\perp}$. Now note that for all $a, b \in L$, $a \oplus b \in \phi^{-1}(J)$ if and only if $\phi(a) \oplus \phi(b) \in J$; if J is a null-ideal, this occurs if and only if $\phi(a), \phi(b) \in J$, i.e., if and only if $a, b \in \phi^{-1}(J)$.

(b) If S is positive, $\{\mathbf{0}\}$ is a null-ideal, whence, $\phi^{-1}(\mathbf{0})$ is null by part (a). Conversely, suppose $\phi^{-1}(\mathbf{0})$ is a null-ideal. Let $x, y \in S$. Since ϕ is surjective, $x = \phi(a)$ and $y = \phi(b)$ for some $a, b \in L$. Then $x \oplus y = \mathbf{0} \implies a \oplus b \in \phi^{-1}(\mathbf{0})$, whence, $a, b \in \phi^{-1}(\mathbf{0})$, whence, $x = y = \mathbf{0}$. □

2.2. Theorem. *Let J be a null-ideal of L. Define a relation $\approx$ on L by $a \approx b \iff \exists\, x, y \in J \quad a \oplus x = b \oplus y$. Then $\approx$ is a faithful congruence and $L/\approx$ is positive. If L is cancellative, so is $L/\approx$.*

Proof. The relation $\approx$ is clearly symmetric; since $0 \in J$, it is reflexive. To see that it is transitive, let $a, b, c \in L$ with $a \approx b \approx c$. Then there exist elements

$x, y, u, v \in J$ with $a \oplus x = b \oplus y$ and $b \oplus u = c \oplus v$. Since $u \in L^{\perp}$, $(a \oplus x) \oplus u$ and $(b \oplus y) \oplus u$ exist, and we have $a \oplus (x \oplus u) = (b \oplus u) \oplus y = c \oplus (v \oplus y)$, whence, as $x \oplus u$ and $v \oplus y$ belong to J, $a \approx c$. To see that $\approx$ is a faithful congruence, let $a, b, c \in L$ with $a \approx b$ and $b \perp c$. Let $x, y \in J$ with $a \oplus x = b \oplus y$. Then $y \perp (b \oplus c)$, and we have $(b \oplus c) \oplus y = (b \oplus y) \oplus c = (a \oplus x) \oplus c = (a \oplus c) \oplus x$, whence, $a \oplus c$ exists and $a \oplus c \approx b \oplus c$.

It follows that there is a faithful surjective homomorphism $\phi\colon L \to L/\approx$ with $\phi^{-1}(\mathbf{0}) = \{a \in L \mid a \approx \mathbf{0}\}$. But if $a \approx \mathbf{0}$, then $a \oplus x = y$ for some $x, y \in J$, whence, since J is a null-ideal, $a \in J$. Thus, $\phi^{-1}(\mathbf{0}) = J$. By part (b) of Lemma 2.1, $L/\approx$ is positive. Finally, note that if L is cancellative and $\phi(a) \oplus \phi(b) = \phi(a) \oplus \phi(c)$, then for some $x, y \in J$ we have $a \oplus b \oplus x = a \oplus c \oplus y$, whence, $b \oplus x = c \oplus y$, i.e., $\phi(b) = \phi(c)$; thus, $L/\approx$ is also cancellative. □

If J is a null-ideal of L, we adopt the notation $\approx_J$ for the faithful congruence of Theorem 2.2, and denote the quotient $L/\approx_J$ by L/J. We write L_+ for L/J_0. L_+ is universal among positive faithful homomorphic images of L in the following sense:

2.3. Corollary. *Any faithful homomorphism $\phi\colon L \to S$ from a PAS L into a positive PAS S factors uniquely through L_+.*

Proof. Let a_+ denote the equivalence class of $a \in L$ under $\approx_{J_0}$. If ϕ: $L \to S$ is a faithful homomorphism and $x \in J_0$, then $x \in \phi^{-1}(\mathbf{0})$ since the latter is a null-ideal. Thus, the map $\phi_+\colon L_+ \to S$ given by $\phi_+\colon a_+ \mapsto \phi(a)$ is well-defined. It is straightforward that ϕ_+ is a faithful homomorphism. □

These matters work out particularly well if L is cancellative and unital:

2.4. Theorem. *Let L be cancellative and unital. Then*

(a) *L_+ is an effect algebra.*

(b) $J_0 = L^{\perp} = \{\mathbf{1}\}^{\perp}$.

(c) *J_0 is an abelian group.*

(d) $\forall a, b \in L$, $a_+ = b_+$ *if and only if* $a \leq b \leq a$.

Proof.

(a) L_+ is cancellative and positive by Theorem 2.2, and the faithful homomorphic image of any unital PAS is unital (since $\phi(\mathbf{1})$ serves as a unit for this image). Thus, L_+ is an effect algebra.

(b) Certainly, $L^{\perp} \subseteq \{\mathbf{1}\}^{\perp}$. Suppose $a \perp \mathbf{1}$. For any $b \in L$, we can find some $b' \in L$ such that $b \oplus b' = \mathbf{1}$, whence, $a \perp b$ by associativity. Thus, $L^{\perp} = \{\mathbf{1}\}^{\perp}$. Now let ϕ be the quotient homomorphism $L \to L_+$. As in the proof of Theorem 2.2, $\phi^{-1}(\mathbf{0}) = J_0 \subseteq L^{\perp}$. By part (a), L_+ is an effect algebra, so $\forall x \in S \quad x \perp \mathbf{1} \implies x = \mathbf{0}$. Thus,

$$\phi^{-1}(\mathbf{0}) = \phi^{-1}(\{\mathbf{1}\}^{\perp}) = \{a \in L \mid \phi(a) \perp \phi(\mathbf{1})\} = \{\mathbf{1}\}^{\perp};$$

hence, $J_0 = \{\mathbf{1}\}^\perp = L^\perp$.

(c) Evidently, J_0 is a cancellative abelian semigroup, so it suffices to produce an inverse for an arbitrary $z \in J_0$. Let $x := (\mathbf{1} \oplus z)'$; by part (b), this belongs to J_0. Now $\mathbf{1} = (\mathbf{1} \oplus z) \oplus x = \mathbf{1} \oplus (z \oplus x)$. Since L is cancellative, $z \oplus x = \mathbf{0}$.

(d) Suppose $a_+ = b_+$. Then for some $x, y \in J_0 = \mathbf{1}^\perp$, $a \oplus x = b \oplus y$. Since $\mathbf{1}^\perp$ is an abelian group, we have $a = (a \oplus x) \oplus (-x) = b \oplus (y - x)$, whence, $b \le a$. Similarly, $a \le b$. Conversely, suppose $a \le b \le a$. Then for some $x, y \in L$, $a \oplus x = b$ and $b \oplus y = a$. Thus,

$$\mathbf{1} = b' \oplus b = b' \oplus \big((b \oplus y) \oplus x\big) = \mathbf{1} \oplus (x \oplus y).$$

Hence, $x \oplus y \in \mathbf{1}^\perp = J_0$. Since J_0 is an ideal, $x, y \in J_0$. But then $a \approx_{J_0} b$, i.e., $a_+ = b_+$. □

Thus, every cancellative unital PAS is an extension of an effect algebra by an abelian group. By way of a simple illustration, if S is an effect algebra and A is an abelian group, we make $L = S \times A$ into a PAS by setting $(p, x) \perp (q, y)$ if and only if $p \perp q$, and, for such a pair, defining $(p, x) \oplus (q, y) = (p \oplus q, x + y)$. Clearly, $L_+ = (S \times A)_+ \simeq S$ and $L^\perp \simeq A$.

2.5. Corollary. *Let L be a cancellative, unital PAS. Any unital homomorphism $\phi\colon L \to S$ from L into an effect algebra S factors uniquely through L_+.*

Proof. If $a \in \mathbf{1}^\perp$, then $\phi(a) \perp \mathbf{1}$ in S, whence, as S is an effect algebra, $\phi(a) = \mathbf{0}$. Hence, $\mathbf{1}^\perp \subseteq \phi^{-1}(\mathbf{0})$. It follows that the map $\phi_+ \colon L_+ \to S$ given by $\phi_+(a_+) = \phi(a)$ is well-defined. Using the fact that the homomorphism $a \mapsto a_+$ is faithful, it is easily verified that ϕ_+ is a unital homomorphism. Clearly, $\phi = \phi_+ \circ [\,]_+$, and, equally clearly, this is the only possible factorization of ϕ through L_+. □

The image of an orthoalgebra or effect algebra under a *faithful* homomorphism is again an orthoalgebra or effect algebra. Let L be a unital PAS, and suppose there exists at least one faithful homomorphism $f\colon L \to S$ from L into an effect algebra S. Let $\approx_e$ be the faithful congruence on S given by $a \approx_e b$ if and only if for every such homomorphism ϕ, $\phi(a) = \phi(b)$. Then $e(L) := L/{\approx_e}$ is easily seen to be an effect algebra, and to be universal among faithful, unital effect-algebraic images of L, in the sense that a faithful unital homomorphism from L into an effect algebra factors uniquely through $e(L)$. If L is cancellative and unital, then, by Corollary 2.5, $e(L)$ exists and coincides with L_+. Note that $e(L) = \{\mathbf{0}\}$ if and only if L is a semigroup.

3. Algebraic sets and perspectivity

There is a well-known representation theory for orthoalgebras in terms of so-called *manuals* or, in more recent usage, *algebraic test spaces* (cf. [4], [6], [9]). In this section, we introduce a generalization of this notion. In Section 4, this will be used to develop a representation for effect algebras in terms of manual-like collections of functions. A related representation is discussed in [3].

Let L be a PAS and $M \subseteq L$. We say that a and b are *perspective* relative to M, writing $a \sim b$, if and only if there exists some $c \in L$ such that $a \perp c$, $b \perp c$, and $a \oplus c$, $c \oplus b \in M$.

Note that every element of M is perspective to every other element of M, and that $a \sim \mathbf{0}$ if and only if $a \oplus b \in M$ for some $b \in M$.

We call a subset $M \subseteq L$ *algebraic* if and only if the relation $\sim$ is a faithful congruence. A *perspectivity* is a faithful congruence arising in this manner from an algebraic set. If $\sim_M$ is the faithful congruence on L induced by an algebraic subset $M \subseteq L$, we write L/M for $L/{\sim_M}$ and $[\ ]_M$ for the canonical surjection $L \to L/M$.

3.1. EXAMPLE. Let $\mathcal{A}$ be any collection of sets and let $L = \{A \mid \exists E \in \mathcal{A}\ A \subseteq E\}$. Then $M = \mathcal{A}$ is algebraic in L if and only if $\mathcal{A}$ is a manual (cf. [9]), and L/M is just the usual Foulis-Randall logic of $\mathcal{A}$.

Call $M \subseteq L$ *dominating* if and only if for every $a \in L$, there exists some $b \in L$ with $a \perp b$ and $a \oplus b \in M$. A unit is by definition an element $\mathbf{1}$ such that $\{\mathbf{1}\}$ is dominating. Notice also that if L is cancellative, an element $u \in L$ is a unit if and only if $\{u\}$ is algebraic.

3.2. LEMMA. *Let M be an algebraic subset of L. Then*

(a) *M is dominating,*
(b) *L/M is cancellative and unital.*

Proof.

(a) Since $\sim_M$ is a faithful congruence, it is reflexive — thus, $a \sim_M a$, whence, for some $b \in L$, $a \oplus b \in M$. Thus, M is dominating.

(b) If $[a]_M \oplus [b]_M = [a]_M \oplus [c]_M$, then for some $x \in L$, $a \oplus b \oplus x$, $a \oplus c \oplus x \in M$, whence, $b \oplus (a \oplus x)$, $c \oplus (a \oplus x) \in M$, whence, $[b]_M = [c]_M$. Thus, L/M is cancellative. Note that for every $e, f \in M$, $e \sim f$. Set $\mathbf{1} = [e]$, where $e \in M$ is arbitrary. Since M is dominating, for every $[a] \in L/M$ there is some $b \in L$ with $a \oplus b \in M$, i.e., $[a] \oplus [b] = \mathbf{1}$. Thus, L/M is unital. □

3.3. LEMMA. *Let M be algebraic in L.*

(a) *If $a \in L$ and $b \in M$, then $a \sim b$ if and only if $b = a \oplus z$, where $z \sim \mathbf{0}$.*
(b) *Set $\overline{M} = \{a \in L \mid \exists b \in M \quad a \sim b\} = \{b \oplus z \mid b \in M,\ z \sim \mathbf{0}\}$. Then $\sim_M = \sim_{\overline{M}}$ (whence, $\overline{M}$ is also algebraic).*

Proof.

(a) If $z \sim \mathbf{0}$, then as $a = a \oplus \mathbf{0}$, $a \sim a \oplus z$. On the other hand, if $a \sim b \in M$, then for some $z \in L$, $a \oplus z$, $z \oplus b \in M$. Since $z \oplus b$, $b \oplus \mathbf{0} \in M$, $z \sim \mathbf{0}$.

(b) It suffices to show that $a \sim_{\overline{M}} b \implies a \sim_M b$. If $a \sim_{\overline{M}} b$, then for some $x \in L$, $a \oplus x$, $x \oplus b$ exist and belong to $\overline{M}$. Hence, there exist elements $z, w \in L$ with $z, w \sim \mathbf{0}$ and $a \oplus x \oplus z$, $b \oplus x \oplus w$ in M. But then we have $a \oplus z \sim b \oplus w$; and since $\sim$ is a faithful congruence, $a \sim a \oplus z$ and $b \oplus w \sim b$. Hence, $a \sim b$. □

If $M = \overline{M}$, we say that M is *closed.*

3.4. Lemma. *Let $M \subseteq L$. The following are equivalent:*

(a) *M is closed and algebraic.*

(b) *M is dominating, and for all $a, b, c \in L$ with $a \sim b$ and $b \perp c$, $b \oplus c \in M \implies a \perp c$ & $a \oplus c \in M$.*

Proof.

(a) $\implies$ (b). If M is algebraic and $a \sim b$ and $b \oplus c \in M$, then $a \oplus c$ exists. Since M is algebraic, it is dominating, whence, for some $z \in L$, $a \oplus c \oplus z \in M$. Suppose $a \oplus x$, $x \oplus b \in M$. Then $c \sim x$, whence, $a \oplus c \oplus z \sim a \oplus x \oplus z$. Since $a \oplus x \in M$ and $(a \oplus x) \oplus z \in M$, $z \sim \mathbf{0}$ by part (b) of Lemma 3.2. Hence $a \oplus c \sim (a \oplus c) \oplus z \in M$, whence, since M is closed, $a \oplus c \in M$.

(b) $\implies$ (a). Conversely, suppose M is dominating and satisfies (b). The relation $\sim_M$ is clearly symmetric, is reflexive since M is dominating, and transitive since $a \sim b \sim c$ implies for some x, y $a \oplus x$, $x \oplus b$, $b \oplus y$ and $y \oplus c$ all lie in M, whence, by hypothesis, $a \oplus y \in M$, whence, $a \sim c$. Now suppose $a \sim b$ and $b \perp c$. Since M is dominating, we can find some $x \in L$ with $b \oplus c \oplus x \in M$. Thus, $a \oplus (c \oplus x) \in M$. But then $a \perp c$ and $(a \oplus c) \oplus x$, $x \oplus (b \oplus c) \in M$, i.e., $a \oplus c \sim b \oplus c$. Thus, $\sim_M$ is a faithful congruence, and M is algebraic. It remains to show that M is closed. To this end, let $a \sim b \in M$. Then as $b \oplus \mathbf{0} = b \in M$, we have $a \oplus \mathbf{0} = a \in M$ as well. □

3.5. Lemma. *Let $\phi\colon L \to S$ be a faithful surjective homomorphism, and let $M \subseteq S$ be algebraic. Then*

(a) *$\phi^{-1}(M)$ is algebraic in L.*

(b) *If M is closed, so is $\phi^{-1}(M)$.*

Proof.

(a) For $a, b \in L$, $a \sim_{\phi^{-1}(M)} b$ if and only if for some $c \in L$, $\phi(a) \oplus \phi(c)$, $\phi(c) \oplus \phi(a) \in M$; since ϕ is faithful and surjective, this occurs if and only if there is some $x \in S$ with $\phi(a) \oplus x$, $x \oplus \phi(b) \in M$, i.e., if and only if $\phi(a) \sim_M \phi(b)$. Thus, $\sim_{\phi^{-1}(M)}$ is the faithful congruence induced by the homomorphism $[\,]_M \circ f$, where $[\,]_M \colon L \to L/M$. In particular, since $\phi(L) \simeq L$, $L/\phi^{-1}(M) \simeq S/M$.

(b) If $a \in L$, there is some $x = \phi(b) \in S$ with $\phi(a) \oplus x = \phi(a \oplus b) \in M$, i.e, there is some $b \in L$ with $a \oplus b \in \phi^{-1}(M)$. Thus, $\phi^{-1}(M)$ is dominating. Now suppose $a \oplus x$, $x \oplus b$, $b \oplus c \in \phi^{-1}(M)$: Then $\phi(a) \oplus \phi(x)$, $\phi(x) \oplus \phi(b)$, $\phi(b) \oplus \phi(c) \in M$. Since M satisfies (b) of Lemma 3.4, $\phi(a) \oplus \phi(c) = \phi(a \oplus c) \in M$, whence, $a \oplus c \in M$. □

3.6. THEOREM. *Let $\phi\colon L \to S$ be a faithful surjective homomorphism from L onto a cancellative, unital PAS S. Then $M := \phi^{-1}(a)$ is a closed algebraic set, and $S \simeq L/M$.*

Proof. Suppose that S is cancellative and unital. Then $\{\mathbf{1}\}$ is a closed algebraic set, whence, by Lemma 3.5, M is closed and algebraic. Suppose now that $a, b \in L$ with $a \sim_M b$. Then for some $c \in L$, $\phi(a) \oplus \phi(c) = \mathbf{1} = \phi(b) \oplus \phi(c)$. Thus, since S is cancellative, $\phi(a) = \phi(b)$. Thus, the map $\overline{\phi}\colon L/M \to S$ given by $\overline{\phi}([a]_M) = \phi(a)$ is well-defined. Clearly, $\overline{\phi}$ is a surjective, faithful homomorphism. Now let $\phi(a) = \phi(b)$. Then for some $y = \phi(c)$, $\phi(a) \oplus y = \phi(b) \oplus y = \mathbf{1}$, i.e., $\phi(a \oplus c) = \phi(b \oplus c) = \mathbf{1}$, whence, $a \oplus c, c \oplus b \in M$. Thus, $a \sim_M b$, and ϕ is an isomorphism. □

As an example, let M be the set of unit elements of a cancellative, unital PAS L. Then $M = \phi^{-1}(\mathbf{1})$, where ϕ is the faithful surjective homomorphism $L \to L^+$ of Theorem 2.2. Hence, M is closed-algebraic and $L/M \simeq L_+$. This observation can be sharpened somewhat. Let $A = L^\perp$ be the null-ideal of L as in Theorem 2.2; A is an abelian group and acts freely and transitively on M. Let $N \subseteq M$: Then N is closed-algebraic if and only if N is the orbit $\{u+x \mid x \in B\}$ of a unit $u \in M$ under a subgroup B of A; moreover, if this is the case, then $(L/N)^\perp \simeq A/B$.

The following appears in [13]; we include a proof for completeness.

3.7. LEMMA. *Let $M \subseteq L$ be closed and algebraic. The following are equivalent:*

(a) *L/M is positive, hence, an effect algebra.*

(b) $\forall a \in L\ \forall b \in M \quad a \perp b \implies a \oplus b \in M$.

Proof.

(a) $\implies$ (b) : Let $\phi\colon L \to L/M$ be the surjection $a \mapsto [a]_M$. If L/M is positive, then $\phi^{-1}(\mathbf{0})$ is a null-ideal by Theorem 2.2. Thus, if $(a \oplus x) \sim \mathbf{0}$, $x \sim \mathbf{0}$. Now suppose $a \perp b$, where $b \in M$. If $a \perp b \in M$, then since M is dominating, there is some $x \in L$ such that $(a \oplus x) \oplus b \in M$. But then $(a \oplus x) \sim \mathbf{0}$. It follows that $a \sim \mathbf{0}$. Since M is closed, $a \oplus b \in M$.

(a) $\implies$ (b) : Suppose (b) holds, and that $[a]_M \oplus [b]_M = \mathbf{0}$, i.e., that $a \oplus b \sim_M \mathbf{0}$. Then for some $c \in M$, $a \oplus b \oplus c \in M$. It follows that $a \oplus b \perp c$, whence, $b \perp c$ and $b \oplus c \in M$. But then $a \sim \mathbf{0}$. Similarly, $b \sim \mathbf{0}$. □

We shall call a set $M \subseteq L$ *strongly algebraic* if and only if it is algebraic and satisfies the equivalent conditions of Lemma 3.7. We now have the following:

3.8. THEOREM. *Let L be any PAS, and let M be any algebraic subset of L. Then there exists a strongly algebraic subset $N \subseteq L$ with $M \subseteq N$.*

Proof. By Lemma 3.2, L/M is cancellative and unital. By Theorem 2.4, $(L/M)_+$ is an effect algebra. Let $\phi\colon L \to (L/M)_+$ be the map $\phi(a) = ([a]_M)_+$. Then ϕ is a faithful surjective homomorphism, and $M \subseteq \phi^{-1}(\mathbf{1})$. By Theorem 3.6 and Lemma 3.7, $\phi^{-1}(\mathbf{1}) = N$ is a strongly algebraic set with $L/N \simeq (L/M)_+$. □

3.9. COROLLARY. *A PAS L admits a faithful surjective homomorphism onto an effect algebra if and only if L contains an algebraic set.*

4. Summable functions

In this section, we produce the advertised representation theorem. Let L be a PAS. For each $a \in L$, we define a partial function $n \mapsto na$ from $\mathbb{Z}_+$ to L by induction: Define $1a = a$; if na has been defined and $na \perp a$, then set $(n+1)a = na \oplus a$. Let $\rho(a)$ be the greatest $n \in \mathbb{Z}_+$ for which na is defined, if any, and set $\rho(a) = 0$ otherwise. We call $\rho(a)$ the *rank* of $a \in S$. Note that if $n + k \le \rho(a)$, then $(n+k)a = na \oplus ka$.

We now define the *summability* of a finitely-nonzero function $f\colon L \to \mathbb{Z}_+$ and its *sum* $\bigoplus f = \bigoplus\limits_{a\in L} f(a)a \in L$ simultaneously, by recursion on the cardinality of $S_f := L \setminus f^{-1}(0)$, as follows:

(i) If $S_f = \emptyset$ — i.e., if f is identically 0 — then f is summable and $\bigoplus f = \mathbf{0}$.

(ii) If $|S_f| = 1$ with $S_f = \{a\}$, then f is summable if and only if $f(a) \le \rho(a)$; in this case, $\bigoplus f = f(a)a$, where $S_f = \{a\}$.

(iii) If $\#(S_f) > 1$, then f is summable if and only if for any $a \in S_f$, the function f_a given by $f_a(x) = \begin{cases} f(x), & x \neq a \\ 0, & x = a \end{cases}$ is summable and the quantity

$$\bigoplus f := f(a)a \oplus \bigoplus f_a$$

exists and is independent of a.

We call a finite subset $A \subseteq L$ *summable* if and only if the $\mathbb{Z}_+$-valued characteristic function χ_A is summable, setting $\bigoplus A = \bigoplus \chi_A$.

Denote by $\mathfrak{S}(L)$, the collection of summable functions $L \to \mathbb{Z}_+$. Notice that for all $a \in L$, $\chi_{\{a\}}$ is summable and $\bigoplus \chi_{\{a\}} = a$; hence, $\mathfrak{S}(L)$ is non-empty, and the map $\bigoplus: \mathfrak{S}(L) \to L$ is a surjection.

If $f \in \mathfrak{S}(L)$ and $g: L \to \mathbb{Z}_+$ with $g(a) \leq f(a)$ for all $a \in S$, then $g \in \mathfrak{S}(L)$. Hence, $\mathfrak{S}(L)$ is a PAS under the restricted addition $f \oplus g = f + g$ provided this is again in $\mathfrak{S}(L)$. We have the following generalized associative law:

4.1. Lemma. *Let $f, g: L \to \mathbb{Z}_+$ be summable. Then $f + g$ is summable if and only if $\bigoplus f \perp \bigoplus g$, and in this case, $\bigoplus(f + g) = \bigoplus f \oplus \bigoplus g$.*

Proof. A straightforward induction on $\#(S_f \cup S_g)$. □

The substance of the foregoing is that the map $f \mapsto \bigoplus f$ is a faithful homomorphism from $\mathfrak{S}(L)$ into L. Since, as already observed, this map is surjective, we have:

4.2. Theorem. *Every PAS is the faithful homomorphic image of its PAS of summable functions. In particular, every PAS is the faithful homomorphic image of a cancellative, positive PAS.*

If L is a cancellative, unital PAS, we have a somewhat sharper result:

4.3. Corollary. *Let L be a cancellative, unital PAS, and let $\mathfrak{M}$ be the collection of summable functions $f \in \mathfrak{S}(L)$ such that $\bigoplus f = \mathbf{1}$. Then $\mathfrak{M}$ is algebraic in $\mathfrak{S}(L)$ and $L \simeq \mathfrak{S}(L)/\mathfrak{M}$.*

Proof. It is sufficient to notice that $f \sim_{\mathfrak{M}} g$ if and only if $\bigoplus f = \bigoplus g$. □

Notice that a cancellative unital PAS is an orthoalgebra if and only if the rank function ρ is identically 1 on $L \setminus \{0\}$; in this case, $\mathfrak{S}(L)$ consists of $\{0,1\}$-valued functions, i.e., of the characteristic functions of subsets of L. Hence $E(L) = \mathfrak{S}(L)$, and $\mathfrak{M}$ is simply the manual of finite orthopartitions of $\mathbf{1}$ in L, and we recover the standard representation theory of orthoalgebras as logics of manuals as a special case of Corollary 4.3.

If $\phi: L \to S$ is any homomorphism of PASes, we may lift ϕ to a homomorphism $\overline{\phi}: \mathfrak{S}(L) \to \mathfrak{S}(M)$ by setting $\overline{\phi}(f) = \phi\big(\bigoplus f\big)$ for all $f \in \mathfrak{S}(L)$. This will be a faithful homomorphism if and only if ϕ is faithful. More generally, for a given homomorphism $\phi: L \to S$, let us say that a function $f \in \mathbb{Z}_+^L$ is *ϕ-summable* if and only if

$$\bigoplus \phi(f) := \bigoplus_{a \in L} f(a)\phi(a)$$

exists in S (i.e., if and only if $f(a) \leq \rho\big(\phi(a)\big)$, and the function $a \mapsto f(a)\phi(a) \in S$ is summable).

The following result shows that every unital PAS has a universal cancellative, unital image. The proof is adapted from similar arguments in [2] and [3].

4.4. Theorem. *For any unital PAS L, there exists a cancellative unital PAS $c(L)$ and a unital homomorphism $\gamma\colon L \to c(L)$ such that for any unital homomorphism $\phi\colon L \to S$, S a cancellative, unital PAS, there is a unique homomorphism $\overline{\phi}\colon c(L) \to S$ with $\phi = \overline{\phi} \circ \gamma$.*

Proof. Let $\mathfrak{M} \subseteq \mathbb{Z}_+^L$ consist of those functions $f\colon L \to \mathbb{Z}_+$ such that, for every unital homomorphism $\phi\colon L \to S$ with S cancellative and unital, f is ϕ-summable and $\bigoplus \phi(f) = 1_S$. Let $\mathfrak{F}$ denote the set of functions $g\colon L \to \mathbb{Z}_+$ such that $g \le f$ for some $f \in \mathfrak{M}$. Then $\mathfrak{F}$ is a PAS (cf. Example 1.1(d)). Note that $\chi_{\{1\}} \in \mathfrak{M}$, so this set is non-empty, and indeed, for any $f \in \mathfrak{S}(L)$, $f \in \mathfrak{F}$. Let $f, g \in \mathfrak{F}$ with $f \oplus g \in \mathfrak{M}$. Then $\bigoplus \phi(f \oplus g) = \bigoplus \phi(f) \oplus \bigoplus \phi(g) = 1_S$ for all $\phi\colon L \to S$. Consequently, $f \sim_{\mathfrak{M}} g$ if and only if $\bigoplus \phi(f) = \bigoplus \phi(g)$ for all such homomorphisms ϕ (since S is cancellative). It follows easily that $\mathfrak{M}$ is algebraic in $\mathfrak{F}$. Let $c(L) = \mathfrak{F}/\mathfrak{M}$, and define for $a \in L$ $\gamma(a) = [\chi_a]$. By Lemma 3.2, $c(L)$ is cancellative and unital. Since $\bigoplus \phi(\chi_a \oplus \chi_b) = \phi(a) \oplus \phi(b) = \phi(a \oplus b) = \bigoplus \phi(\chi_{a\oplus b})$ for all ϕ, γ is a homomorphism; clearly, $\gamma(\mathbf{1}) = \mathbf{1}$ since $\chi_1 \in \mathfrak{M}$. Finally, if $\phi\colon L \to S$ with S cancellative and unital, then $\overline{\phi}([f]) = \bigoplus \phi(f)$ is well-defined and gives us the desired homomorphism $c(L) \to S$. □

4.5. Corollary. *Every unital PAS has a universal effect-algebraic image.*

Proof. Let L be a unital PAS and set $e(L) = c(L)_+$. It follows from Theorem 4.2 and Corollary 2.5 that any unital homomorphism from L into an effect algebra S factors uniquely through $e(L)$. □

It should be noted that the trivial PAS $\{0\}$ is cancellative and unital (with $0 = 1$). Under the trivial homomorphism $\phi\colon L \to \{0\}$, every function $f \in \mathbb{Z}_+^L$ is summable. Hence, if L admits no non-trivial unital homomorphisms into a cancellative unital PAS, then $\mathfrak{M} = \mathbb{Z}_+^L$ and $c(L) = \{\mathbf{0}\}$.

5. Tensor products

In this section, we give an account of tensor products of cancellative, unital PASes. Both our results and our methods parallel those of [2].

Let L_1, L_2 and S be PASes. A function $\Phi\colon L_1 \times L_2 \to S$ is a *bimorphism* if and only if for all $a \in L_1$ and for all $b \in L_2$, the functions $\Phi(a, \cdot)\colon L_2 \to S$ and $\Phi(\cdot, b)\colon L_1 \to S$ are homomorphisms. We shall call a finitely non-zero function $f\colon L_1 \times L_2 \to \mathbb{Z}_+$ is *bi-summable* if and only if for any PAS S and any bimorphism $\Phi\colon L_1 \times L_2 \to S$, the sum $\langle \Phi, f\rangle := \bigoplus_{a,b} f(a,b)\Phi(a,b)$ exists (where (a, b) runs over $L_1 \times L_2$).

Let $\mathfrak{S}(L_1, L_2)$ denote the collection of all bi-summable functions on $L_1 \times L_2$. Note that for all $(a, b) \in L_1 \times L_2$, the characteristic function $\chi_{\{(a,b)\}}$ is bi-summable; hence, $\mathfrak{S}(L_1, L_2)$ is non-empty. $\mathfrak{S}(L_1, L_2)$ becomes a PAS if we set $f \oplus g = f + g$ provided that this is again bi-summable.

For notational convenience, we shall henceforth ignore the distinction between an element $a \in L$ and the corresponding summable function $\chi_{\{a\}} \in \mathfrak{S}(L)$, and likewise that between the ordered pair (a, b) and the function $\chi_{\{(a,b)\}}$. Thus, we treat L as a subset of $\mathfrak{S}(L)$ and $L_1 \times L_2$ as a subset of $\mathfrak{S}(L_1, L_2)$.

5.1. LEMMA. *Let L_1 and L_2 be PASes. For $f \in \mathfrak{S}(L)$ and $g \in \mathfrak{S}(L_2)$, define $f \cdot g \in \mathbb{Z}_+^{L_1 \times L_2}$ by $(f \cdot g)(a, b) = f(a)g(b)$. Then*

(a) *$f \cdot g$ is bi-summable;*
(b) *The map $(f, g) \mapsto f \cdot g$ is a bimorphism from $\mathfrak{S}(L) \times \mathfrak{S}(L_2)$ into $\mathfrak{S}(L_1, L_2)$;*
(c) *For every bi-morphism $\Phi\colon L_1 \times L_2 \to S$ there exists a unique homomorphism $\phi\colon \mathfrak{S}(L_1, L_2) \to S$ such that $\phi(f \cdot g) = \Phi(\bigoplus f, \bigoplus g)$.*

Proof. (a) and (b) are straightforward. For (c), let $\Psi\colon \mathfrak{S}(L) \times \mathfrak{S}(L_2) \to S$ be any bimorphism. Define $\phi\colon \mathfrak{S}(L_1, L_2) \to S$ by $\phi(f) = \langle \Phi, f \rangle = \bigoplus_{(a,b)} f(a, b)\Phi(a, b)$, and note that this is a homomorphism. Observe that

$$\phi(f \cdot g) = \bigoplus_{(a,b)} f(a)g(b)\Phi(a, b) = \Phi(f, g).$$

□

If L, L_2 and S are unital, a *unital bimorphism* $\Phi\colon L_1 \times L_2 \to S$ is a bimorphism such that $\Phi(\mathbf{1}, \mathbf{1}) = \mathbf{1}$. A *tensor product* of two cancellative, unital PASes L and L_2 is a pair $(T, \odot)$, where T is a cancellative, unital PAS and $\odot\colon L_1 \times L_2 \to T$ is a unital bimorphism, such that any unital bimorphism $\Phi\colon L_1 \times L_2 \to S$ into a cancellative, unital PAS S lifts uniquely to a unital morphism $\phi\colon L_1 \odot L_2 \to S$ with $\phi(a \odot b) = \Phi(a, b)$ for all $(a, b) \in L_1 \times L_2$.

The usual argument shows that T, if it exists, is unique to within isomorphism; consequently, we speak of *the* tensor product of L and L_2 and denote it by $L_1 \odot L_2$. Note that we allow $L_1 \odot L_2 = \mathbf{0}$ — indeed, this will be the case whenever $L_1 \times L_2$ admits no nontrivial unital bimorphism into a cancellative unital PAS S.

The following result (a close cousin of Theorem 4.4) is essentially due to Bennett and Foulis [2] for orthoalgebras and to Dvurečenskij [3] for D-posets.

5.2. THEOREM. *Any cancellative, unital PASes L_1 and L_2 have a tensor product $L_1 \odot L_2$.*

Proof. Let $\mathfrak{M}$ denote the collection of functions $f \in \mathfrak{S}(L_1, L_2)$ such that for all unital bimorphisms $\Phi\colon L_1 \times L_2 \to S$ from $L_1 \times L_2$ into S is cancellative, unital PAS S, $\bigoplus \Phi(f) = \mathbf{1}_S$. Let $\mathfrak{F}$ be the PAS consisting of all functions $g\colon L_1 \times L_2 \to \mathbb{Z}_+$ such that $g \leq f$ for some $g \in \mathfrak{M}$. Note that $\mathfrak{F} \subseteq \mathfrak{S}(L_1, L_2)$. Observe also that, for all $(a,b) \in L_1 \times L_2$, $\chi_{(a,b)} + \chi_{(a',b')}$ belongs to $\mathfrak{F}$; hence, $L_1 \times L_2 \subseteq \mathfrak{F}$. It is straightforward to verify that $\mathfrak{M}$ is algebraic in $\mathfrak{F} \subseteq \mathfrak{S}(L_1, L_2)$, whence, $L_1 \odot L_2 := \mathfrak{F}/\mathfrak{M}$ is cancellative and unital. Evidently, the map $(a,b) \mapsto a \odot b := [\chi_{(a,b)}]$ is a bimorphism. Suppose now that $\Phi\colon L_1 \times L_2 \to S$ is a unital bimorphism. We may lift Φ to a bimorphism $\overline{\Phi}\colon \mathfrak{S}(L_1) \times \mathfrak{S}(L_2) \to S$ as in the proof of Theorem 4.4. By Lemma 5.1, this factors uniquely through $\mathfrak{S}(L_1, L_2)$. Since Φ is unital, $\overline{\Phi}$ maps $\mathfrak{M}$ to $\{\mathbf{1}_S\}$, whence, descends to a unital homomorphism on $\mathfrak{F}/\mathfrak{M}$. □

Note that if $L_1 \times L_2$ admits no non-trivial bimorphism into a cancellative, unital PAS, then *every* $f \in \mathfrak{S}(L_1, L_2)$ belongs to $\mathfrak{M}$, whence, $L_1 \odot L_2 = \{\mathbf{0}\}$.

5.3. LEMMA. *Let A and B be abelian groups, each understood as a cancellative unital PAS with unit 0. Then $A \odot B$ is the usual tensor product of A and B (in the category of abelian groups).*

P r o o f . It suffices to observe that if Φ is a unital bimorphism from $A \times B$ into a cancellative, unital PAS S, then $\mathbf{0} = \Phi(0,0) = \mathbf{1}$ — whence, S is itself an abelian group with unit $\mathbf{0}$. □

Let us say that $\Phi\colon L_1 \times L_2 \to S$ is a *faithful bimorphism* if and only if both $\Phi(a, \cdot)$ and $\Phi(\cdot, b)$ are faithful homomorphisms for all non-zero $a \in L_1$ and $b \in L_2$. If $L_1 \odot L_2$ exists and $\odot\colon L_1 \times L_2 \to L_1 \odot L_2$ is faithful, we say that L_1 and L_2 *admit a faithful tensor product.* Notice that if $\odot$ is faithful and $L_1 \odot L_2 = \{\mathbf{0}\}$, then both L_1 and L_2 are semigroups — hence, abelian groups, and hence, in fact, both trivial.

The existence of a faithful tensor product imposes rather sharp restrictions on the factors:

5.4. LEMMA. *Suppose L_1 and L_2 admit a faithful tensor product. If either factor contains an element of non-zero rank, then every non-zero element of each factor has rank 1.*

P r o o f . Suppose $a \in L_1$ is non-zero and has non-zero rank. Without loss of generality, we may suppose the rank of a to be 1. Suppose that $b \in L_2 \setminus \{0\}$ has rank other than 1. Then, as $b \perp b$, $(a \odot b) \perp (a \odot b)$. If $x \mapsto x \odot b$ is faithful, we have $a \perp a$, a contradiction. □

As a special case of the foregoing, suppose L_1 is an effect algebra and that there exists a PAS L_2 such that $L_1 \odot L_2$ is a faithful tensor product of L_1 and L_2. Then L_1 is an orthoalgebra.

5.5. THEOREM. *The following statements are equivalent:*

(a) L_1 *and* L_2 *admit a faithful tensor product.*

(b) *There exists a faithful bimorphism* $L_1 \times L_2 \to S$ *for some PAS* S.

Proof. Clearly, (a) implies (b). Conversely, if $\Phi\colon L_1 \times L_2 \to S$ is a bimorphism, then the existence of Φ guarantees that of $L_1 \odot L_2$. Let $\phi\colon L_1 \odot L_2 \to S$ be the unique extension of Φ guaranteed by the definition. Fix $a \in L$ and suppose that $a \odot b \perp a \odot c$ for some $b, c \in L_2$. Then $\Phi(a,b) = \phi(a \odot b) \perp \phi(a \odot c) = \Phi(a,c)$. Since Φ is faithful, $b \perp c$. The same argument shows that $\odot$ is faithful in its first argument as well. Thus, (a) and (b) are equivalent. $\square$

There is no difficulty now in forming tensor products of effect algebras. If L_1 and L_2 are two such, form $L_1 \odot L_2$ as in Theorem 5.2; this is cancellative and unital, hence, by Corollary 2.5, has a universal effect-algebraic image $L_1 \otimes L_2 := (L_1 \odot L_2)_+$ (non-trivial if and only if $L_1 \odot L_2$ is non-trivial) which evidently solves the universal mapping problem for bimorphisms of effect algebras. Thus, we recover Theorem 5.9 of [2] and Theorem 7.2 of [3].

Acknowledgement

The author is indebted to Professor S. Pulmannová for several helpful comments, and especially for pointing out an error in an earlier draft of this paper.

REFERENCES

[1] DINGES, H.: *Zur Algebra der Masstheorie*, Bull. Greek Math. Soc. **19** (1978), 25 97.

[2] BENNETT, M. K.—FOULIS, D. J.: *Tensor products of orthoalgebras*, Order **10** (1993), 271–282.

[3] DVUREČENSKIJ, A.: *Tensor products of difference posets*, Trans. Amer. Math. Soc. **347** (1995), 1043–1057.

[4] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *D-test spaces and difference posets*, Rep. Math. Phys. **34** (1994), 151–170.

[5] FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[6] FOULIS, D. J.—GREECHIE, R. J.—RÜTTIMANN, G. T.: *Filters and supports in orthoalgebras*, Internat. J. Theoret. Phys. **31** (1992), 789–807.

[7] GIUNTINI, R.—GREULING, H.: *Toward a formal language for unsharp properties*, Found. Phys. **19** (1989), 931–945.

[8] GREECHIE, R. J.—FOULIS, D. J.—PULMANNOVÁ, S.: *The center of an effect algebra*, Order **12** (1995), 91–106.

[9] GUDDER, S.: *Quantum Probability*, Academic Press, San Diego, 1988.

[10] HEDLÍKOVÁ, J.—PULMANNOVÁ, S.: *Generalized difference posets and orthoalgebras*, Acta Math. Univ. Comenian. **LXV** (1996), 247–279.

[11] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[12] MAYET-IPPOLITO, A.: *Generalized orthomodular posets*, Demonstratio Math. **24** (1991), 263–274.

[13] PULMANNOVÁ, S.—WILCE, A.: *Representations of D-posets*, Internat. J. Theoret. Phys. **34** (1995), 1689–1696.

[14] WILCE, A.: *Partial abelian semigroups*, Internat. J. Theoret. Phys. **34** (1995), 1807–1812.

Received July 2, 1995
Revised February 26, 1996

Department of Mathematics
and Computer Science
Juniata College
Huntingdon, PA 16652
U. S. A.
E-mail: wilce@juniata.edu