Math. Slovaca, **44** (1994), No. 2, 237–244

Mathematica Slovaca

Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

THE TENSOR PRODUCT OF GROUPOIDS

LADISLAV SATKO

(Communicated by Tibor Katriňák)

ABSTRACT. The tensor product in the class of all groupoids is studied, and the greatest semigroup image of this tensor product is described.

In this paper, we deal with the tensor product in the class of all groupoids and its connection with the tensor product in the class of all semigroups. We show that the tensor product in the class of all semigroups can be obtained by the tensor product in the class of all groupoids and the greatest semigroup image of a groupoid, respectively. The tensor product in the class of all (commutative) semigroups was defined in [2], [4], [5], [6]. The tensor product on a variety of universal algebras was introduced in [1], and its properties are studied in [3]. Therefore, in this paper, we only recall the definition of the tensor product in the class of all groupoids and the existence theorem without proof. First we deal with the construction of the greatest semigroup image of a groupoid. Here, a groupoid is a nonempty set with one binary operation.

DEFINITION 1. *Let G be an arbitrary groupoid. The greatest semigroup image of the groupoid G is the semigroup $S(G)$ with the following property: There exists a surjective homomorphism $\vartheta\colon G \to S(G)$ such that for any homomorphism $\delta\colon G \to S$ onto an arbitrary semigroup S there exists a unique homomorphism $\varphi\colon S(G) \to S$ such that $\delta = \varphi \circ \vartheta$.* (Fig. 1)

AMS Subject Classification (1991): Primary 20L99.
Key words: Groupoid, Tensor product, Greatest semigroup image of groupoid.

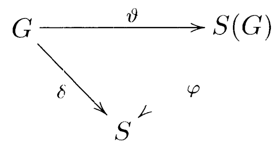

Figure 1.

It is well known that the greatest semigroup image $S(G)$ exists for any groupoid G. The greatest semigroup image of a groupoid is given by the least semigroup congruence on the groupoid. One of the possible constructions of classes of the least semigroup congruence on a groupoid G is described in [7]. For convenience of the reader, we give this construction. We start with some notations.

Let G be an arbitrary groupoid, and $A, B \subseteq G$. The set

$$AB = \{ab \in G \mid a \in A, \ b \in B\}$$

is the *set product of* A *and* B. For any $g_1, g_2, \dots, g_n \in G$, we define the following sets:

$$\begin{aligned}
[g_1] &= \{g_1\}, \\
[g_1, g_2] &= \{g_1 g_2\}, \\
[g_1, g_2, g_3] &= \{g_1(g_2 g_3), (g_1 g_2) g_3\}, \\
[g_1, g_2, g_3, g_4] &= \{g_1\big(g_2(g_3 g_4)\big), g_1\big((g_2 g_3) g_4\big), (g_1 g_2)(g_3 g_4), \\
&\qquad \big((g_1 g_2) g_3\big) g_4, \big(g_1(g_2 g_3)\big) g_4\}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
[g_1, g_2] &= [g_1][g_2], \\
[g_1, g_2, g_3] &= [g_1][g_2, g_3] \cup [g_1, g_2][g_3], \\
[g_1, g_2, g_3, g_4] &= [g_1][g_2, g_3, g_4] \cup [g_1, g_2][g_3, g_4] \cup [g_1, g_2, g_3][g_4].
\end{aligned}$$

If we suppose that the sets $[g_{i_1}, g_{i_2}, \dots, g_{i_k}]$ are defined for any k such that $1 \le k \le n-1$, we can inductively define

$$\begin{aligned}
[g_1, g_2, \dots, g_n] &= [g_1][g_2, \dots, g_n] \cup [g_1, g_2][g_3, \dots, g_n] \cup \dots \cup [g_1, \dots, g_{n-1}][g_n] \\
&= \bigcup_{i=1}^{n-1} [g_1, \dots, g_i][g_{i+1}, \dots, g_n].
\end{aligned}$$

For any $k \geq 3$ we define a relation $\mathscr{E}_k$ on the groupoid G in the following way: For any $a, b \in G$, $a\,\mathscr{E}_k\,b$ if and only if either $a = b$ or there exist $g_1, g_2, \ldots, g_k \in G$ such that $\{a, b\} \subseteq [g_1, \ldots, g_k]$. If $\mathscr{F}_k$ is the transitive closure of the relation $\mathscr{E}_k$, then $a\,\mathscr{F}_k\,b$ if and only if there exist $x_0, x_1, \ldots, x_n \in G$ such that $x_0 = a$, $x_n = b$ and $x_{i-1}\,\mathscr{E}_k\,x_i$ for any $i = 1, 2, \ldots, n$.

If G is a groupoid such that $G = G^2$, then $\mathscr{F}_3 \subseteq \mathscr{F}_4 \subseteq \mathscr{F}_5 \subseteq \ldots$. However, these inclusions are not valid in general. Therefore, for an arbitrary groupoid and $k \in \{3, 4, 5, \ldots\}$, we define the following equivalence relations:

$$\begin{aligned}
&\mathscr{H}_3 = \mathscr{F}_3\,,\\
&\mathscr{H}_4 = \mathscr{H}_3 \vee \mathscr{F}_4 = \mathscr{F}_3 \vee \mathscr{F}_4\,,\\
&\mathscr{H}_5 = \mathscr{H}_4 \vee \mathscr{F}_5 = \mathscr{F}_3 \vee \mathscr{F}_4 \vee \mathscr{F}_5\,,\\
&\quad\ldots\\
&\mathscr{H}_k = \mathscr{H}_{k-1} \vee \mathscr{F}_k = \mathscr{F}_1 \vee \mathscr{F}_2 \vee \cdots \vee \mathscr{F}_k\,,\\
&\quad\ldots
\end{aligned}$$

We recall that the join $\mathscr{H}_k$ of equivalences $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$ can be defined in the following way: $a'\,\mathscr{H}_k\,a$ if and only if there exist

$$\begin{aligned}
x_{12} &= \ a', x_{13}, \ldots, x_{1k}\,,\\
x_{1k} &= x_{22}, x_{23}, \ldots, x_{2k}\,,\\
x_{2k} &= x_{32}, x_{33}, \ldots, x_{3k}\,,\\
&\ldots\\
x_{(n-1)k} &= x_{n2}, x_{n3}, \ldots, x_{nk} = a
\end{aligned}$$

such that

$$\begin{gathered}
x_{12}\,\mathscr{F}_3\,x_{13},\ x_{13}\,\mathscr{F}_4\,x_{14},\ x_{14}\,\mathscr{F}_5\,x_{15},\ \ldots,\ x_{1(k-1)}\,\mathscr{F}_k\,x_{1k}\,,\\
x_{22}\,\mathscr{F}_3\,x_{23},\ x_{23}\,\mathscr{F}_4\,x_{24},\ x_{24}\,\mathscr{F}_5\,x_{25},\ \ldots,\ x_{2(k-1)}\,\mathscr{F}_k\,x_{2k}\,,\\
\ldots\\
x_{n2}\,\mathscr{F}_3\,x_{n3},\ x_{n3}\,\mathscr{F}_4\,x_{n4},\ x_{n4}\,\mathscr{F}_5\,x_{n5},\ \ldots,\ x_{n(k-1)}\,\mathscr{F}_k\,x_{nk}\,.
\end{gathered}$$

Obviously, $\mathscr{H}_3 \subseteq \mathscr{H}_4 \subseteq \mathscr{H}_5 \subseteq \ldots$. Moreover, in the case $G = G^2$, $\mathscr{H}_k = \mathscr{F}_k$ for any $k \in \{3, 4, \ldots\}$.

Let $\mathscr{H}_\infty = \bigcup\limits_{k=3}^{\infty} \mathscr{H}_k = \mathscr{F}_3 \vee \mathscr{F}_4 \vee \cdots \vee \mathscr{F}_k \vee \cdots = \bigvee\limits_{k=3}^{\infty} \mathscr{F}$. In [8], it is proved that $\mathscr{H}_\infty$ is the least semigroup congruence on the groupoid G and $S(G) = G/H_\infty$. To simplify our notation, we further denote $\mathscr{H}_\infty = \sigma$.

Now we describe the tensor product in the class of all groupoids. Let $A \times B$ be the Cartesian product of groupoids A, B. Let G be an arbitrary groupoid. The mapping $\alpha\colon A \times B \to G$ is called a bilinear mapping if $\alpha(a_1 a_2, b) = \alpha(a_1, b)\alpha(a_2, b)$ and $\alpha(a, b_1 b_2) = \alpha(a, b_1)\alpha(a, b_2)$ for any $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$.

DEFINITION 2. *Let A, B be arbitrary groupoids. The tensor product of groupoids A and B is a couple $(\omega, A \otimes B)$, where $A \otimes B$ is a groupoid, and $\omega\colon A \times B \to A \otimes B$ is a bilinear mapping satisfying the following universal property: For any groupoid G and any bilinear mapping $\alpha\colon A \times B \to G$ there exists a unique homomorphism $\varphi\colon A \otimes B \to G$ such that $\alpha = \varphi \circ \omega$.* (Fig. 2)

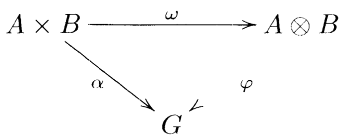

Figure 2.

The groupoid $A \otimes B$ is also called the tensor product of groupoids A and B. The tensor product in the class of all semigroups is defined similarly. Only the word "groupoid" is replaced by the word "semigroup". The notation $A \otimes_s B$ is used for the tensor product in the class of all semigroups.

The existence theorem for the tensor product on an arbitrary variety of universal algebras was proved in [1]. Following the same arguments one can prove a similar theorem for the tensor product in the class of all groupoids. The proof is omitted.

Let A, B be groupoids, and $\mathscr{G}_{A \times B}$ be the free groupoid on the Cartesian product $A \times B$. It is known that there exists an inclusion $\iota\colon A \times B \to \mathscr{G}_{A \times B}$. Elements $\iota(a, b)$ are denoted by $[a, b]$.

THEOREM 1. *Let A, B be groupoids, and $\mathscr{G}_{A \times B}$ be the free groupoid on the Cartesian product $A \times B$. Let $\mathscr{S}$ be a relation on the groupoid $\mathscr{G}_{A \times B}$ such that $[a_1 a_2, b] \mathscr{S} [a_1, b][a_2, b]$ and $[a, b_1 b_2] \mathscr{S} [a, b_1][a, b_2]$ for any $a_1, a_2, a \in A$, $b_1, b_2, b \in B$. Let $\mathscr{T}$ be the least congruence on $\mathscr{G}_{A \times B}$ such that $\mathscr{S} \subseteq \mathscr{T}$. Then $A \otimes B = \mathscr{G}_{A \times B}/\mathscr{T}$, and $\omega\colon A \times B \to A \otimes B$ is a bilinear mapping such that $\omega(a, b) = \mathscr{T}_{[a,b]}$ for any $(a, b) \in A \times B$.*

$\mathscr{T}_{[a,b]}$ is the class of the congruence relation $\mathscr{T}$ containing an element $[a, b]$. In the sequel we denote this class by $a \otimes b$.

Let A, B be semigroups. The tensor product $A \otimes B$ in the class of all groupoids need not be a semigroup. The next theorem shows a connection between a groupoid $A \otimes B$ and the semigroup $A \otimes_s B$.

THEOREM 2. *Let A, B be semigroups. The greatest semigroup image $S(A \otimes B)$ is isomorphic to the tensor product $A \otimes_s B$ of semigroups A, B in the class of all semigroups. Thus $S(A \otimes B) \cong A \otimes_s B$.*

Proof. Let $(\omega, A \otimes B)$ be the tensor product of semigroups A, B in the class of all groupoids. Let $\vartheta\colon A \otimes B \to S(A \otimes B)$ be a natural homomorphism. Let C be an arbitrary semigroup. (In this case we consider the semigroup C as a groupoid.) Then for an arbitrary bilinear mapping $\alpha\colon A \times B \to C$ there exists a unique homomorphism $\varphi\colon A \otimes B \to C$ such that $\alpha = \varphi \circ \omega$. Then $\operatorname{Im}\varphi$ is a subsemigroup of the semigroup C, and φ is a homomorphism of the groupoid $A \otimes B$ onto the semigroup $\operatorname{Im}\varphi$. Therefore there exists a unique homomorphism $\varphi^*\colon S(A \otimes B) \to \operatorname{Im}\varphi \subseteq C$ such that $\varphi = \varphi^* \circ \vartheta$. (Fig. 3)

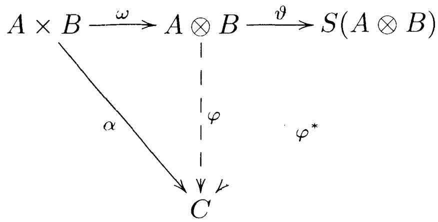

Figure 3.

Hence, to any bilinear mapping $\alpha\colon A \times B \to C$ there exists a unique homomorphism $\varphi^*\colon S(A \otimes B) \to C$ such that $\alpha = \varphi^* \circ (\vartheta \circ \omega)$. Thus the couple $\bigl(\vartheta \circ \omega, S(A \otimes B)\bigr)$ is the tensor product of semigroups A, B in the class of all semigroups. This tensor product is determined up to an isomorphism.

Remark 1. In the rest of the paper, we denote by $\sigma = \mathscr{H}_\infty$ the smallest semigroup congruence on an arbitrary groupoid. It is easy to prove that the isomorphism $S(A \otimes B) \cong A \otimes_s B$ in the preceding theorem is the mapping $\lambda\colon S(A \otimes B) \to A \otimes_s B$ such that $\lambda(\sigma_{a \otimes b}) = a \otimes_s b$ for any $a \in A$ and $b \in B$.

Let $(\omega, A \otimes B)$ be the tensor product of groupoids A and B, and $\xi\colon A \to A'$ and $\zeta\colon B \to B'$ be groupoid homomorphisms. Let $\delta\colon A \times B \to A' \otimes B'$ be a mapping given by $\delta(a,b) = \xi(a) \otimes \zeta(b)$. Evidently, δ is a bilinear mapping. Therefore there exists a unique homomorphism $\varphi\colon A \otimes B \to A' \otimes B'$ such that $\delta = \varphi \circ \omega$.

DEFINITION 3. The above mentioned homomorphism φ is called the *tensor product of homomorphisms ξ and ζ* and is denoted by $\xi \otimes \zeta$.

LEMMA 1. *Let A, B be groupoids, $(\omega, A \otimes B)$ their tensor product, and S an arbitrary semigroup. Let $\gamma\colon A \otimes B \to S$ be a homomorphism. Then the mapping $\alpha\colon S(A) \times S(B) \to S$ such that $\alpha(\sigma_a, \sigma_b) = \gamma(a \otimes b)$ for any $(\sigma_a, \sigma_b) \in S(A) \times S(B)$, is a bilinear mapping.*

P r o o f . First we prove that α is well-defined, e.g. $\alpha(\sigma_a, \sigma_b)$ is independent of a and b in classes σ_a and σ_b.

Let $a' \in \sigma_a$. Therefore there exists $k \in \{1, 2, 3, \dots\}$ such that $a' \mathscr{H}_k a$. As $\mathscr{H}_k = \mathscr{F}_3 \vee \mathscr{F}_4 \vee \dots \vee \mathscr{F}_k$, there exist

$$\begin{aligned} a' &= x_{12}, x_{13}, x_{14}, \dots, x_{1k}, \\ x_{1k} &= x_{22}, x_{23}, \dots, x_{2k}, \\ x_{2k} &= x_{32}, x_{33}, \dots, x_{3k}, \\ &\dots \\ x_{(n-1)k} &= x_{n2}, x_{n3}, \dots, x_{nk} = a \end{aligned}$$

such that

$$\begin{gathered} x_{12} \mathscr{F}_3 x_{13},\ x_{13} \mathscr{F}_4 x_{14},\ x_{14} \mathscr{F}_5 x_{15},\ \dots,\ x_{1(k-1)} \mathscr{F}_k x_{1k}, \\ x_{22} \mathscr{F}_3 x_{23},\ x_{23} \mathscr{F}_4 x_{24},\ x_{24} \mathscr{F}_5 x_{25},\ \dots,\ x_{2(k-1)} \mathscr{F}_k x_{2k}, \\ \dots \\ x_{n2} \mathscr{F}_3 x_{n3},\ x_{n3} \mathscr{F}_4 x_{n4},\ x_{n4} \mathscr{F}_5 x_{n5},\ \dots,\ x_{n(k-1)} \mathscr{F}_k x_{nk}. \end{gathered}$$

Hence $x_{m(i-1)} \mathscr{F}_i x_{mi}$ for $m = 1, 2, \dots, n$ and $i = 3, 4, \dots, k$. Let m and i be fixed. Then $x_{m(i-1)} \mathscr{F}_i x_{mi}$ if and only if there exist $y_0, y_1, \dots, y_r$ such that $y_0 = x_{m(i-1)}$, $y_r = x_{mi}$ and $y_{j-1} \mathscr{E}_i y_j$ for $j = 1, 2, \dots, r$. Therefore $\{y_{j-1}, y_j\} \subseteq [g_1, g_2, \dots, g_i]$ for some $g_1, g_2, \dots, g_i \in A$ and $\gamma(y_{j-1} \otimes b) = \gamma(g_1 \otimes b)\gamma(g_2 \otimes b) \dots \gamma(g_i \otimes b) = \gamma(y_j \otimes b)$ for any $j = 1, 2, \dots r$, as $\gamma(g_1 \otimes b)\gamma(g_2 \otimes b) \dots \gamma(g_i \otimes b)$ is an element of the semigroup S. Therefore $\gamma(x_{m(i-1)} \otimes b) = \gamma(y_0 \otimes b) = \gamma(y_r \otimes b) = \gamma(x_{mi} \otimes b)$. We proved this equality for fixed but arbitrary m and i, and thus $\gamma(x_{m(i-1)} \otimes b) = \gamma(x_{mi} \otimes b)$ for $m = 1, 2, \dots, n$ and $i = 3, 4, \dots, k$. Hence $\gamma(a' \otimes b) = \gamma(x_{12} \otimes b) = \gamma(x_{nk} \otimes b) = \gamma(a \otimes b)$.

Similarly, for $b' \in \sigma_b$, $\gamma(a' \otimes b) = \gamma(a' \otimes b')$, and thus $\gamma(a \otimes b) = \gamma(a' \otimes b) = \gamma(a' \otimes b')$. Hence α is a well-defined mapping.

For arbitrary $\sigma_{a_1}, \sigma_{a_2}, \sigma_a \in S(A)$, $\sigma_{b_1}, \sigma_{b_2}, \sigma_b \in S(B)$, $\alpha(\sigma_{a_1}\sigma_{a_2}, \sigma_b) = \alpha(\sigma_{a_1 a_2}, \sigma_b) = \gamma(a_1 a_2 \otimes b) = \gamma\big((a_1 \otimes b)(a_2 \otimes b)\big) = \gamma(a_1 \otimes b)\gamma(a_2 \otimes b) = \alpha(\sigma_{a_1}, \sigma_b)\alpha(\sigma_{a_2}, \sigma_b)$, and similarly $\alpha(\sigma_a, \sigma_{b_1}\sigma_{b_2}) = \alpha(\sigma_a, \sigma_{b_1})\alpha(\sigma_a, \sigma_{b_2})$. Hence $\alpha\colon S(A) \times S(B) \to S$ is a bilinear mapping.

Now, the main theorem of this paper can be formulated.

THEOREM 3. *Let A, B be groupoids and $S(A)$, $S(B)$ their greatest semigroup images. Then $S(A \otimes B) \cong S(A) \otimes_s S(B)$.*

P r o o f. We prove the existence of the homomorphism $\eta\colon A \otimes B \to S(A) \otimes_s S(B)$ such that for any homomorphism $\gamma\colon A \otimes B \to S$ onto an arbitrary semigroup S there exists a unique homomorphism $\varphi\colon S(A) \otimes_s S(B) \to S$ such that $\gamma = \varphi \circ \eta$. (Fig. 4)

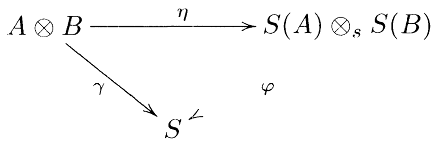

Figure 4.

Let $\psi = \lambda \circ \nu$ be the composition of the natural homomorphism $\nu\colon S(A) \otimes S(B) \to S\big(S(A) \otimes S(B)\big)$ and the isomorphism $\lambda\colon S\big(S(A) \otimes S(B)\big) \to S(A) \otimes_s S(B)$ from Remark 1. Let $\xi\colon A \to S(A)$, $\zeta\colon B \to S(B)$ be natural homomorphisms. If we consider the mapping $\eta = \psi \circ (\xi \otimes \zeta) = \lambda \circ \nu \circ (\xi \otimes \zeta)$, then for any $a \in A$, $b \in B$, $\eta(a \otimes b) = \big(\lambda \circ \nu \circ (\xi \otimes \zeta)\big)(a \otimes b) = (\lambda \circ \nu)\big(\xi(a) \otimes \zeta(b)\big) = (\lambda \circ \nu)(\sigma_a \otimes \sigma_b) = \lambda(\sigma_{\sigma_a \otimes \sigma_b}) = \sigma_a \otimes_s \sigma_b$. The last relation is the consequence of Remark 1.

Let S be an arbitrary semigroup, and $\gamma\colon A \otimes B \to S$ be a homomorphism. Let $\alpha\colon S(A) \times S(B) \to S$ be the mapping from Lemma 1 satisfying the relation $\alpha(\sigma_a, \sigma_b) = \gamma(a \otimes b)$ for any $(\sigma_a, \sigma_b) \in S(A) \times S(B)$. As α is a bilinear mapping, there exists a unique homomorphism $\varphi\colon S(A) \otimes_s S(B) \to S$ such that $\alpha = \varphi \circ \omega$. (Fig. 5)

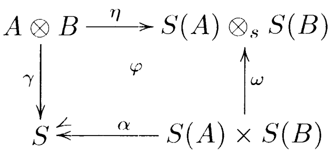

Figure 5.

Then for an arbitrary $a \in A$, $b \in B$, $\gamma(a \otimes b) = \alpha(\sigma_a, \sigma_b) = \varphi \circ \omega(\sigma_a, \sigma_b) = \varphi(\sigma_a \otimes_s \sigma_b) = \varphi\big(\eta(a \otimes b)\big) = \varphi \circ \eta(a \otimes b)$. Since these relations hold for all generating elements of the groupoid $A \otimes B$, $\gamma = \varphi \circ \eta$. Hence, to any homomorphism $\gamma\colon A \otimes B \to S$ onto an arbitrary semigroup S there exists a unique homomorphism $\varphi\colon S(A) \otimes_s S(B) \to S$ satisfying $\gamma = \varphi \circ \eta$. Thus $S(A) \otimes_s S(B)$ is the greatest semigroup image of the groupoid $A \otimes B$ which is determined up to an isomorphism.

REFERENCES

[1] FRASER, G. A.: *Tensor product of semilattices and distributive lattices*, Semigroup Forum **13** (1976), 178–184.

[2] FULP, R.: *Tensor and torsion products of semigroups*, Pacific J. Math. **32** (1970), 685–696.

[3] GALANOVÁ, J.: *The tensor product on a variety.* In: Universal and Applied Algebra, Proceedings of the V. Universal algebra symposium Turava 1988, World Scientific, 1989, pp. 92–98.

[4] GRILLET, P. A.: *The tensor product of semigroups*, Trans. Amer. Math. Soc. **138** (1969), 267–280.

[5] GRILLET, P. A.: *The tensor product of commutative semigroups*, Trans. Amer. Math. Soc. **138** (1969), 281–293.

[6] HEAD, T. J.: *Homomorphisms of commutative semigroups as tensor maps*, J. Natur. Sci. Math. **7** (1967), 39–49.

[7] SATKO, L.: *Tensor Products of Semigroups and Groupoids* (Slovak). Doctoral Thesis. EF SVŠT, Bratislava, 1977.

[8] SATKO, L.: *The greatest semigroup image of a groupoid*, Pure Math. Appl. Ser. A **3** (1992), 259–265.

Received November 17, 1993

Department of Mathematics
Faculty of Electrical Engineering
Slovak Technical University
Ilkovičova 3
SK-812 19 Bratislava
Slovakia