Mathematica Slovaca

Svatoslav Staněk

Multi-point boundary value problem for a class of functional-differential equations with parameter

Mathematica Slovaca, Vol. 42 (1992), No. 1, 85--96

Persistent URL: `http://dml.cz/dmlcz/128751`

Math. Slovaca, 42 (1992), No. 1, 85–96

Mathematica Slovaca

MULTI–POINT BOUNDARY VALUE PROBLEM FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PARAMETER

SVATOSLAV STANĚK

ABSTRACT. The existence and uniqueness of solutions of the problem $y''(t) - q(t)y(t) = f(t, y(t), y(h_0(t)), y'(t), y'(h_1(t)), \mu)$, $\sum_{i=1}^{m} \alpha_i y(t_i) = 0$, $y(c) = 0$, $\sum_{j=1}^{n} \beta_j y(x_j) = 0$ are studied.

1. Introduction

Consider the one-parameter functional differential equation

$$y''(t) - q(t)y(t) = f\big(t, y(t), y(h_0(t)), y'(t), y'(h_1(t)), \mu\big) \tag{1}$$

in which $f \in C^0(J \times \mathbb{R}^4 \times I; \mathbb{R})$, $h_0, h_1 \in C^0(J; J)$, $q \in C^0(J; \mathbb{R})$, $q(t) > 0$ for $t \in J$, where $J = \langle a, b\rangle$, $I = \langle k_1, k_2\rangle$, $-\infty < a < b < \infty$, $-\infty < k_1 < k_2 < \infty$.

Suppose m, n are positive integers, $c \in (a, b)$, $a = t_1 < t_2 < \cdots < t_m < c < x_n < \cdots < x_2 < x_1 = b$ and α_i, β_j ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) are positive constants, $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = 1$, $\alpha_1 \geqq \sum_{i=2}^{m} \alpha_i$ (provided $m \geqq 2$), $\beta_1 \geqq \sum_{j=2}^{n} \beta_j$ (provided $n \geqq 2$).

Our aim is to give sufficient conditions on the functions q and f for the existence and uniqueness of solutions of (1) satisfying the boundary conditions

$$\sum_{i=1}^{m} \alpha_i y(t_i) = 0, \quad y(c) = 0, \quad \sum_{j=1}^{n} \beta_j(x_j) = 0. \tag{2}$$

AMS Subject Classification (1991): Primary 34K10.
Key words: One-parameter functional differential equation, Multi-point boundary value problem, Schauder fixed point theorem.

The results presented in the paper may be formulated without difficulty for the equation

$$y''(t) - q(t)y(t) \\ = g\big(t, y(t), y(h_{00}(t)), \dots, y(h_{0r}(t)), y'(t), y'(h_{10}(t)), \dots, y'(h_{1s}(t)), \mu\big)$$

with $g \in C^0(J \times \mathbb{R}^{r+s+4} \times I; \mathbb{R})$, $h_{ij} \in C^0(J; J)$.

The boundary value problem $y'' - q(t)y = h(t, y, y', \mu)$, $y(a) = y(c) = y(b) = 0$ was studied by the author in [1].

2. Notation, lemmas

Let u, v be the solutions of the equation

$$y'' = q(t)y, \ q \in C^0(J; \mathbb{R}), \ q(t) > 0 \qquad \text{for} \quad t \in J, \tag{q}$$

$u(c) = 0$, $u'(c) = 1$, $v(c) = 1$, $v'(c) = 0$. Setting

$$r(t, s) = u(t)v(s) - u(s)v(t) \qquad (= -r(s,t)),$$
$$r_1'(t, s) = u'(t)v(s) - u(s)v'(t) \qquad \Big(= \frac{\partial r}{\partial t}(t, s)\Big),$$

for $(t, s) \in J^2$ then $r(t, s) > 0$ for $a \leqq s < t \leqq b$, $r(t, s) < 0$ for $a \leqq t < s \leqq b$ and $r_1'(t, s) > 1$ for $(t, s) \in J^2$, $t \neq s$ (see [1]).

Let K, L, Q, τ denote the positive constants defined by

$$K = \Big(\sum_{i=1}^{m} \alpha_i r(c, t_i)\Big)^{-1}, \quad L = \sum_{j=1}^{n} \beta_j r(x_j, c), \quad Q = \max\{q(t);\ t \in J\},$$
$$\tau = \max\{c - a, b - c\}.$$

Lemma 1. *Let $h \in C^0(J; \mathbb{R})$. The function*

$$y(t) = \int_c^t r(t, s)h(s)\,ds + Kr(t, c)\sum_{i=1}^{m} \alpha_i \int_c^{t_i} r(t_i, s)h(s)\,ds, \qquad t \in J, \tag{3}$$

is the unique solution of the equation

$$y'' - q(t)y = h(t) \tag{4}$$

satisfying the boundary conditions

$$\sum_{i=1}^{m} \alpha_i y(t_i) = 0, \qquad y(c) = 0. \tag{5}$$

Proof. One can easily check that the function y defined by (3) is a solution of (4) satisfying (5). Let z be a solution of (q), $z(c) = 0$. Since (q) is a disconjugate equation on J without loss of generality we may assume $z(t) \geqq 0$ for $t \in \langle a, c\rangle$. Then $\sum_{i=1}^{m} \alpha_i z(t_i) = 0$ if and only if $z(t) \equiv 0$ on J. Consequently the boundary value problem (q), (5) has only the trivial solution and therefore the boundary value problem (4), (5) has the unique solution.

Lemma 2. *Assume that $h \in C^0(J \times I; \mathbb{R})$, $h(t, \cdot)$ is an increasing function on I for every fixed $t \in J$ and*

$$h(t, k_1)h(t, k_2) \leqq 0 \qquad \text{for} \quad t \in J. \tag{6}$$

Then there exists the unique $\mu_0 \in I$ such that the equation

$$y'' - q(t)y = h(t, \mu) \tag{7}$$

with $\mu = \mu_0$ has a (and then the unique) solution y satisfying (2).

Proof. Let $y(t, \mu)$ be the solution of (7), $\sum_{i=1}^{m} \alpha_i y(t_i, \mu) = 0$, $y(c, \mu) = 0$. Then by Lemma 1

$$y(t, \mu) = \int_c^t r(t, s)h(s, \mu)\,\mathrm{d}s + Kr(t, c)\sum_{i=1}^{m} \alpha_i \int_c^{t_i} r(t_i, s)h(s, \mu)\,\mathrm{d}s, \qquad (t, \mu) \in J \times I,$$

and thus

$$\sum_{j=1}^{n} \beta_j y(x_j, \mu) = \sum_{j=1}^{n} \beta_j \int_c^{x_j} r(x_j, s)h(s, \mu)\,\mathrm{d}s + KL\sum_{i=1}^{m} \alpha_i \int_c^{t_i} r(t_i, s)h(s, \mu)\,\mathrm{d}s.$$

Since $r(x_j, s) > 0$ for $c \leqq s < x_j$, $j = 1, 2, \dots, n$ and $r(t_i, s) < 0$ for $t_i < s \leqq c$, $i = 1, 2, \dots, m$, we see that $\sum_{j=1}^{n} \beta_j y(x_j, \cdot)$ is a continuous increasing function on I and

$$\sum_{j=1}^{n} \beta_j y(x_j, k_1) \leqq 0, \qquad \sum_{j=1}^{n} \beta_j y(x_j, k_2) \geqq 0,$$

by assumption (6). Consequently $\sum_{j=1}^{n} \beta_j y(x_j, \mu_0) = 0$ for the unique $\mu_0 \in I$. This proves that the problem (7), (2) has a solution y if and only if $\mu = \mu_0$ and by Lemma 1 this solution y is unique.

Next we shall assume that the functions q, f satisfy for positive constants r_0, r_1 the following assumptions:

(8) $|f(t,y,z,w,s,\mu)| \leqq q(t)r_0$ for $(t,y,z,w,s,\mu) \in D \times I$, where $D = J \times \langle -r_0, r_0 \rangle \times \langle -r_0, r_0 \rangle \times \langle -r_1, r_1 \rangle \times \langle -r_1, r_1 \rangle$;

(9) $f(t,y,z,w,s,\cdot)$ is an increasing function on I for every fixed $(t,y,z,w,s) \in D$;

(10) $f(t,y,z,w,s,k_1) f(t,y,z,w,s,k_2) \leqq 0$ for $(t,y,z,w,s) \in D$;

(11) $\min\left\{(A + r_0 Q)\tau, 2\sqrt{r_0}\sqrt{A + r_0 Q}\right\} \leqq r_1$, where $A = \sup\left\{|f(t,y,z,w,s,\mu)|;\ (t,y,z,w,s,\mu) \in D \times I\right\}$.

LEMMA 3. *Suppose that assumptions* (8)–(11) *are satisfied for positive constants* r_0, r_1. *Then to every* $\varphi \in C^1(J;\mathbb{R})$, $|\varphi^{(i)}(t)| \leqq r_i$ *for* $t \in J$, $i = 0, 1$, *there exists the unique* $\mu_0 \in I$ *such that the equation*

$$y'' - q(t)y = f\big(t, \varphi(t), \varphi(h_0(t)), \varphi'(t), \varphi'(h_1(t)), \mu\big) \tag{12}$$

with $\mu = \mu_0$ *has a (and then the unique) solution* y *satisfying* (2). *For this* y *the inequalities*

$$|y^{(i)}(t)| \leqq r_i, \qquad t \in J,\ i = 0, 1, \tag{13}$$

hold.

Proof. Setting $h(t,\mu) = f\Big(t,\ \varphi(t),\ \varphi(h_0(t)),\ \varphi'(t),\ \varphi'(h_1(t)),\ \mu\Big)$ for $(t,\mu) \in J \times I$, then $|h(t,\mu)| \leqq A$ on $J \times I$, $h(t,\cdot)$ is an increasing function on I for every fixed $t \in J$ (by (9)) and $h(t,k_1)h(t,k_2) \leqq 0$ on J (by (10)). Therefore by Lemma 2 there exi ts the unique $\mu_0 \in I$ such that equation (12) with $\mu = \mu_0$ has a (and then the unique) solution y satisfying (2).

Now we prove inequalities (13). From (8) follows $y''(t) > 0$ ($y''(t) < 0$) for every $t \in J$ where $y(t) > r_0$ ($y(t) < -r_0$). Consequently y does not achieve its local maximum (minimum) at any point $t = \xi$ where $y(\xi) > r_0$ ($y(\xi) < -r_0$). Next if $y(a) > r_0$ ($y(a) < -r_0$), then y is a decreasing (increasing) function in every right neighbourhood of the point a where $y(t) > r_0$ ($y(t) < -r_0$ and if $y(b) > r_0$ ($y(b) < -r_0$), then y is an increasing (de reasing) function in every left neighbourhood of the point b, where $y(t) > r_0$ ($y(t) < -r_0$). From this follows $|y(t)| \leq r_0$ on J if and only if $|y(a)| \leq r_0$, $|y(b)| \leq r_0$.

Suppose $|y(a)| > r_0$. Then $m > 1$ and let for example $y(a) < -r_0$. Since $-y(a) > y(t_i)$ for $i = 2,3,\dots,m$, we have $\sum_{i=1}^{m} \alpha_i y(t_i) < \alpha_1 y(a) - \sum_{i=2}^{m} \alpha_i y(a) = \left(\alpha_1 - \sum_{i=2}^{m} \alpha_i\right) y(a) \leqq 0$ contradicting $\sum_{i=1}^{m} \alpha_i y(t_i) = 0$. Suppose $|y(b)| > r_0$. Then $n > 1$ and let for example $y(b) > r_0$. Since $y(b) > -y(x_j)$ for $j = 2,3,\dots,n$, we have $\sum_{j=1}^{n} \beta_j y(x_j) > \beta_1 y(b) - \sum_{j=2}^{n} \beta_j y(b) = \left(\beta_1 - \sum_{j=2}^{n} \beta_j\right) y(b) \geqq 0$ contradicting $\sum_{j=1}^{n} \beta_j y(x_j) = 0$.

Next there exists $\xi_1 \in (a,c)$ ($\xi_2 \in (c,b)$) such that $y'(\xi_1) = 0$ ($y'(\xi_2) = 0$). In the opposite case we have $\sum_{i=1}^{m} \alpha_i y(t_i) \neq 0$ ($\sum_{j=1}^{n} \beta_j y(x_j) \neq 0$). Integrating the equality $y''(t) = q(t)y(t) + h(t,\mu_0)$ for $t \in J$ from ξ_i to t ($\in J$), we obtain

$$y'(t) = \int_{\xi_i}^{t} \big(q(s)y(s) + h(s,\mu_0)\big)\,\mathrm{d}s, \quad i = 1,2,$$

and thus

$$|y'(t)| \leqq (A + Qr_0)\tau, \quad t \in J. \tag{14}$$

Let $|y'(t)| > 0$ for $t \in (s_1,s_2) \subset J$ and let $y'(s_i) = 0$ for some $i \in \{1,2\}$. Then integrating the equality $2y''(t)y'(t) = 2q(t)y(t)y'(t) + 2h(t,\mu_0)y'(t)$ from s_i to t ($\in (s_1,s_2)$) we get

$$[y']^2(t) = 2\int_{s_i}^{t} q(s)y'(s)y(s)\,\mathrm{d}s + 2\int_{s_i}^{t} h(s,\mu_0)y'(s)\,\mathrm{d}s,$$

consequently

$$[y']^2(t) \leqq 2Qr_0|y(t) - y(s_i)| + 2A|y(t) - y(s_i)| \leqq 4r_0(A + Qr_0).$$

This proves

$$|y'(t)| \leqq 2\sqrt{r_0}\sqrt{A + Qr_0}, \qquad t \in J. \tag{15}$$

From (14) and (15) we conclude $|y'(t)| \leqq r_1$ for $t \in J$.

Assume that the function $f(t,y,z,w,s,\mu) = g(t,y,z,\mu)$ in equation (1) is independent on w,s. Consider the equation

$$y''(t) - q(t)y(t) = g\big(t, y(t), y(h_0(t)), \mu\big) \tag{16}$$

with $g \in C^0(J \times \mathbb{R}^2 \times I; \mathbb{R})$, which is a special case of (1). Suppose that q, g satisfy for a positive constant r_0 the following assumptions:

(17) $|g(t,y,z,\mu)| \leqq q(t) r_0$ for $(t,y,z,\mu) \in H \times I$, where $H = J \times \langle -r_0, r_0 \rangle \times \langle -r_0, r_0 \rangle$;

(18) $g(t,y,z,.)$ is an increasing function on I for every fixed $(t,y,z) \in H$;

(19) $g(t,y,z,k_1) g(t,y,z,k_2) \leqq 0$ for $(t,y,z) \in H$.

LEMMA 4. *Suppose that assumptions* (17)–(19) *are satisfied for a positive constant* r_0. *Then to every* $\varphi \in C^0(J;\mathbb{R})$, $|\varphi(t)| \leqq r_0$ *for* $t \in J$ *there exists the unique* $\mu_0 \in I$ *such that the equation*

$$y'' - q(t)y = g\big(t, \varphi(t), \varphi(h_0(t)), \mu\big) \tag{20}$$

with $\mu = \mu_0$ *has a (and then the unique) solution* y. *For this* y

$$|y(t)| \leqq r_0, \quad |y'(t)| \leqq (B + Q r_0)\tau \qquad \text{for} \quad t \in J, \tag{21}$$

where $B = \max\{|g(t,y,z,\mu)|;\ (t,y,z,\mu) \in H \times I\}$, *hold.*

Proof. Setting $h(t,\mu) = g\big(t, \varphi(t), \varphi(h_0(t)), \mu\big)$ for $(t,\mu) \in J \times I$, then by Lemma 2 there exists the unique $\mu = \mu_0$ such that equation (20) with $\mu = \mu_0$ has a (and then the unique) solution y and $|y(t)| \leqq r_0$ for $t \in J$. Since $|h(t,\mu)| \leqq B$ for $(t,\mu) \in J \times I$ and $y'(\xi_1) = y'(\xi_2) = 0$, where $a < \xi_1 < c < \xi_2 < b$ (see the proof of Lemma 3), it follows from $|y''(t)| \leqq B + Q r_0$ and $y'(t) = \int_{\xi_i}^{t} y''(s)\,\mathrm{d}s$ for $t \in J$, $i = 1,2$, that $|y'(t)| \leqq (B + Q r_0)\tau$ for $t \in J$.

3. Existence theorems

THEOREM 1. *Assume that assumptions* (8)–(11) *are satisfied for positive constants* r_0, r_1. *Then there exists* $\mu_0 \in I$ *such that equation* (1) *with* $\mu = \mu_0$ *has a solution* y *satisfying* (2) *and* (13).

Proof. Let $X = \{y;\ y \in C^1(J;\mathbb{R})\}$ be the Banach space with the norm $\|y\| = \max\{|y(t)| + |y'(t)|;\ t \in J\}$ and let $\mathcal{K} = \{y;\ y \in X,\ |y^{(i)}(t)| \leqq r_i$ for $t \in J,\ i = 0,1\}$. $\mathcal{K}$ is a bounded convex closed subset of X. By Lemma 3 to every $\varphi \in \mathcal{K}$ there exists the unique $\mu_0 \in I$ such that equation (12) with $\mu = \mu_0$ has a (and then the unique) solution $y \in \mathcal{K}$ satisfying (2). Setting $T(\varphi) = y$ we obtain an operator $T \colon \mathcal{K} \to \mathcal{K}$. We prove T is a completely continuous operator. Let $\{y_n\}$, $y_n \in \mathcal{K}$ be a convergent sequence, $\lim_{n\to\infty} y_n = y$ and let

$z_n = T(y_n)$, $z = T(y)$. Then there exists a sequence $\{\mu_n\}$, $\mu_n \in I$ and $\mu_0 \in I$ such that

$$z_n(t) = \int_c^t r(t,s)h_n(s,\mu_n)\,\mathrm{d}s + Kr(t,c)\sum_{i=1}^m \alpha_i \int_c^{t_i} r(t_i,s)h_n(s,\mu_n)\,\mathrm{d}s, \qquad t \in J, \quad n \in \mathbb{N},$$

and

$$z(t) = \int_c^t r(t,s)h(s,\mu_0)\,\mathrm{d}s + Kr(t,c)\sum_{i=1}^m \alpha_i \int_c^{t_i} r(t_i,s)h(s,\mu_0)\,\mathrm{d}s, \qquad t \in J,$$

where

$$h_n(t,\mu) = f\big(t, y_n(t), y_n\big(h_0(t)\big), y'_n(t), y'_n\big(h_1(t)\big), \mu\big),$$
$$h(t,\mu) = f\big(t, y(t), y\big(h_0(t)\big), y'(t), y'\big(h_1(t)\big), \mu\big)$$
$$\text{for} \quad (t,\mu) \in J \times I, \ n = 1,2,\dots .$$

To prove that $\{\mu_n\}$ is a convergent sequence, suppose that there exist subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$, $\lim_{n\to\infty} \mu_{k_n} = \lambda_1$, $\lim_{n\to\infty} \mu_{r_n} = \lambda_2$ and $\lambda_1 < \lambda_2$. Then

$$(w_1(t) =) \quad \lim_{n\to\infty} z_{k_n}(t) = \int_c^t r(t,s)h(s,\lambda_1)\,\mathrm{d}s + Kr(t,c)\sum_{i=1}^m \alpha_i \int_c^{t_i} r(t_i,s)h(s,\lambda_1)\,\mathrm{d}s,$$

$$(w_2(t) =) \quad \lim_{n\to\infty} z_{r_n}(t) = \int_c^t r(t,s)h(s,\lambda_2)\,\mathrm{d}s + Kr(t,c)\sum_{i=1}^m \alpha_i \int_c^{t_i} r(t_i,s)h(s,\lambda_2)\,\mathrm{d}s$$

uniformly on J. Since $h(t,\lambda_1) < h(t,\lambda_2)$ (by (9)), we have $\sum_{j=1}^n \beta_j w_1(x_j) < \sum_{j=1}^n \beta_j w_2(x_j)$ contradicting $\sum_{j=1}^n \beta_j z_n(x_j) = 0$ for $n \in \mathbb{N}$, consequently $\{\mu_n\}$ is convergent and let $\lim_{n\to\infty} \mu_n = \mu^*$. Then

$$\begin{aligned}(\,w(t) =) \quad \lim_{n\to\infty} z_n(t) \\ &= \int_c^t r(t,s)h(s,\mu^*)\,\mathrm{d}s + Kr(t,c)\sum_{i=1}^m \alpha_i \int_c^{t_i} r(t_i,s)h(s,\mu^*)\,\mathrm{d}s\end{aligned}$$

uniformly on J and therefore the function w is a solution of the equation

$$w'' - q(t)w = h(t,\mu^*)$$

satisfying (2). Hence by Lemma 3 $w = z$ and $\mu_0 = \mu^*$. Next

$$\begin{aligned}\lim_{n\to\infty} z_n'(t) = \int_c^t r_1'(t,s)h(s,\mu_0)\,\mathrm{d}s + Kr_1'(t,c)\sum_{i=1}^m \alpha_i \int_c^{t_i} r(t_i,s)h(s,\mu_0)\,\mathrm{d}s \\ (\, = z'(t)\,)\end{aligned}$$

uniformly on J, consequently $\lim_{n\to\infty} T(y_n) = T(y)$. This proves T is a continuous operator.

Let $y \in \mathcal{K}$ and let $z = T(y)$. From the equality

$$z''(t) = q(t)z(t) + f\Big(t, y(t), y(h_0(t)), y'(t), y'\big(h_1(t)\big), \mu_0\Big), \qquad t \in J,$$

where $\mu_0 \in I$ is an appropriate number, we conclude

$$|z''(t)| \leqq Qr_0 + A \ (= S) \qquad \text{for} \quad t \in J.$$

Since $T(\mathcal{K}) \subset \mathcal{L} = \big\{y;\ y \in C^2(J;\mathbb{R}),\ |y^{(i)}(t)| \leqq r_i,\ |y''(t)| \leqq S$ for $t \in J$, $i = 0,1\big\}$ and $\mathcal{L}$ is a compact subset of X, $T(\mathcal{K})$ is a compact subset of X, too. Using the Schauder fixed point theorem there exists a fixed point y of T. This y has the required properties in the assertion of Theorem 1.

Example 1. Assume that ν is a positive integer, $J = \langle 1, 10\rangle$, $I = \langle -(1+5\pi), 1+5\pi\rangle$, $h_0, h_1 \in C^0(J;J)$, $q \in C^0(J;\mathbb{R})$, $q(t) \geqq 3(1+5\pi)$ for $t \in J$. Let $c \in (1,10)$. Consider the equation

$$y''(t) - q(t)y(t) = \frac{\cos y^\nu(t)}{1 + \big(y'(h_1(t))\big)^2} + t\cdot \operatorname{arctg}\big(\sinh y'(t)\big) + \mu \ln\Big(\mathrm{e} + \big|y\big(h_0(t)\big)\big|\Big). \tag{22}$$

The assumptions of Theorem 1 hold with $r_0 = 3$, $r_1 = 6\sqrt{1+5\pi+Q}$, where $Q = \max\{q(t);\ t \in J\}$, and therefore there exists $\mu_0 \in I$ such that equation (22) with $\mu = \mu_0$ has a solution y satisfying (2) and $|y(t)| \leqq 3$, $|y'(t)| \leqq 6\sqrt{1+5\pi+Q}$ for $t \in J$.

THEOREM 2. *Let assumptions* (17)-(19) *be satisfied for a positive constant* r_0. *Then there exists* $\mu_0 \in I$ *such that equation* (16) *with* $\mu = \mu_0$ *has a solution* y *satisfying* (2) *and* (21), *where* B *is defined as in Lemma* 4.

Proof. Let $Y = C^0(J;\mathbb{R})$ be the Banach space with the norm $\|y\| = \max\{|y(t)|; t \in J\}$. Setting $\mathcal{K} = \{y;\ \|y\| \leqq r_0\}$ and $\mathcal{L} = \{y;\ y \in C^1(J;\mathbb{R}), \|y\| \leqq r_0,\ \|y'\| \leqq (Qr_0 + B)\tau\}$, then $\mathcal{K}$ is a bounded convex closed subset of Y and $\mathcal{L}$ is a precompact set of Y. By Lemma 4 to every $\varphi \in \mathcal{K}$ there exists the unique $\mu_0 \in I$ such that equation (20) with $\mu = \mu_0$ has the unique solution $y \in \mathcal{K}$ satisfying (2). Setting $T(\varphi) = y$ we obtain an operator $T\colon \mathcal{K} \to \mathcal{L}$. Analogous to the proof of Theorem 1 we can prove T is a completely continuous operator and using the Schauder fixed point theorem a fixed point y of T is a solution of (16) with some $\mu = \mu_0 \in I$ satisfying (2) and (21).

Example 2. Let ξ, ν, ϱ be positive integers. Consider the equation

$$y''(t) - q(t)y(t) = t^{\xi} \exp\{\, y^{\nu}(t)\big[y\big(h_0(t)\big)\big]^{\varrho}\} + \mu \tag{23}$$

where $Q \geqq q(t) \geqq 2\mathrm{e}\cdot\max\{|a|^{\xi}, |b|^{\xi}\}$ for $t \in J$. For equation (23) are satisfied assumptions of Theorem 2 with $r_0 = 1$, $I = \langle k_1, k_2\rangle$, where $k_2 = -k_1 = \mathrm{e}\cdot\max\{|a|^{\xi}, |b|^{\xi}\}$. Consequently there exists $\mu_0 \in I$ such that equation (23) with $\mu = \mu_0$ has a solution y satisfying (2) and $|y(t)| \leqq 1$, $|y'(t)| \leqq \big(Q + 2e.\max\{|a|^{\xi},\ |b|^{\xi}\}\big)\tau$ for $t \in J$.

4. Uniqueness theorem

THEOREM 3. *Assume that assumptions* (8)-(11) *are satisfied for positive constants* r_0, r_1. *Let* $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \omega}, \frac{\partial f}{\partial s} \in C^0(D \times I;\mathbb{R})$ *and let*

$$\begin{aligned} &\frac{\partial f}{\partial y}(t,y,z,w,s,\mu) + q(t) \geqq 0, \qquad \frac{\partial f}{\partial z}(t,y,z,w,s,\mu) \geqq 0,\\ &(t-c)\frac{\partial f}{\partial s}(t,y,z,w,s,\mu) \geqq 0 \qquad \text{for} \quad (t,y,z,w,s,\mu) \in D \times I. \end{aligned} \tag{24}$$

If $t \leqq h_i(t) \leqq c$ *for* $t \in \langle a,c\rangle$ *and* $c \leqq h_i(t) \leqq t$ *for* $t \in \langle c,b\rangle$ $(i = 0,1)$, *then there exists the unique* $\mu_0 \in I$ *such that equation* (1) *with* $\mu = \mu_0$ *has a solution* y *satisfying* (2) *and* (13). *Furthermore this solution* y *is unique.*

Proof. By Theorem 1 there exists $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ has a solution y satisfying (2) and (13). Suppose there exists $\mu_1 \in I$, $\mu_0 \leqq \mu_1$ such that equation (1) with $\mu = \mu_1$ has a solution y_1 satisfying (2) and (13), where in place of y we put y_1 and let $y \neq y_1$. Setting $w = y - y_1$ we have

$$\sum_{i=1}^{m} \alpha_i w(t_i) = 0, \qquad w(c) = 0, \qquad \sum_{j=1}^{n} \beta_j w(x_j) = 0$$

and

$$w''(t) = \big(q(t) + p_1(t)\big)w(t) + p_2(t)w\big(h_0(t)\big) + p_3(t)w'(t) \\ + p_4(t)w'\big(h_1(t)\big) + p(t) \qquad \text{for} \quad t \in J, \tag{25}$$

where $p_1, p_2, p_3, p_4, p \in C^0(J;\mathbb{R})$, $p_1(t) + q(t) \geqq 0$, $p_2(t) \geqq 0$, $(t-c)p_4(t) \geqq 0$ for $t \in J$ (by (24)) and if $\mu_0 < \mu_1$ ($\mu_0 = \mu_1$), then $p(t) < 0$ ($p(t) = 0$) for $t \in J$.

Let $\mu_0 < \mu_1$. If $w'(t) < 0$ for $t \in \langle c, c_2) \subset \langle c, b\rangle$ and $w'(c_2) = 0$ (such c_2 always exists), then $w(t) < 0$, $w\big(h_0(t)\big) < 0$, $w'\big(h_1(t)\big) < 0$ for $t \in (c, c_2)$ and from (25) it follows $w''(c_2) \leqq p(c_2) < 0$ contradicting $w'(c_2) = 0$. If $w'(t) > 0$ for $t \in (c_1, c) \subset \langle a, c\rangle$ and $w'(c_1) = 0$ (such c_1 always exists), then $w(t) < 0$, $w\big(h_0(t)\big) < 0$, $w'\big(h_1(t)\big) > 0$ for $t \in (c_1, c)$ and from (25) it follows $w''(c_1) \leqq p(c_1) < 0$ contradicting $w'(c_1) = 0$. If $w'(c) = 0$, then using (25) we have $w''(c) = p(c) < 0$ and proceeding as in the case $w'(t) < 0$ for $t \in \langle c, c_2)$ we obtain once again a contradiction. Consequently $\mu_0 = \mu_1$ and then from (25) we get

$$w'(t) = \exp\Big(\int_c^t p_3(s)\,\mathrm{d}s\Big)\Big[w'(c) + \int_c^t \exp\Big(-\int_c^s p_3(\tau)\,\mathrm{d}\tau\Big) \cdot \\ \cdot \Big(\big(q(s) + p_1(s)\big)w(s) + p_2(s)w\big(h_0(s)\big) + p_4(s)w'\big(h_1(s)\big)\Big)\,\mathrm{d}s\Big], \quad t \in J.$$

If $w'(c) > 0$ ($w'(c) < 0$), then necessarily $w'(t) > 0$, $w(t) < 0$ for $t \in \langle a, c)$ ($w'(t) < 0$, $w(t) < 0$ for $t \in (c, b\rangle$) contradicting $\sum_{i=1}^{m} \alpha_i w(t_i) = 0$ ($\sum_{j=1}^{n} \beta_j w(x_j) = 0$). If $w'(c) = 0$, then

$$w'(t) = \int_c^t \exp\Big(\int_s^t p_3(\tau)\,\mathrm{d}\tau\Big)\Big[\big(q(s) + p_1(s)\big)\int_c^s w'(\tau)\,\mathrm{d}\tau \\ + p_2(s)\int_c^{h_0(s)} w'(\tau)\,\mathrm{d}\tau + p_4(s)w'\big(h_1(s)\big)\Big]\,\mathrm{d}s, \quad t \in J. \tag{26}$$

Let $X(t) = \max\{|w'(s)|;\ t \leqq s \leqq c\}$ for $t \in \langle a, c\rangle$ and let $Y(t) = \max\{|w'(s)|;\ c \leqq s \leqq t\}$ for $t \in \langle c, b\rangle$. To prove $X(a) = Y(b) = 0$ let $X(a) > 0$ ($Y(b) > 0$). Then $X(t) > 0$ for $t \in \langle a, a_1)$ and $X(t) = 0$ for $t \in \langle a_1, c\rangle$ ($Y(t) > 0$ for $t \in (b_1, b\rangle$ and $Y(t) = 0$ for $t \in \langle c, b_1\rangle$) and from (26) we get

$$|w'(t)| \leqq X(t) \int_t^{a_1} \exp\Big(\int_t^s |p_3(\tau)|\,\mathrm{d}\tau\Big)\Big[\big(q(s) + p_1(s)\big)(a_1 - s) + p_2(s)\big(a_1 - h_0(s)\big) - p_4(s)\Big]\,\mathrm{d}s, \qquad t \in \langle a, a_1\rangle$$

$$\Big(|w'(t)| \leqq Y(t) \int_{b_1}^{t} \exp\Big(\int_s^t |p_3(\tau)|\,\mathrm{d}\tau\Big)\Big[\big(q(s) + p_1(s)\big)(s - b_1) + p_2(s)\big(h_0(s) - b_1\big) + p_4(s)\Big]\,\mathrm{d}s, \qquad t \in (b_1, b\rangle\Big),$$

consequently

$$1 \leqq \int_t^{a_1} \exp\Big(\int_t^s |p_3(\tau)|\,\mathrm{d}\tau\Big)\Big[\big(q(s) + p_1(s)\big)(a_1 - s) + p_2(s)\big(a_1 - h_0(s)\big) - p_4(s)\Big]\,\mathrm{d}s,\ t \in \langle a, a_1)$$

$$\Big(1 \leqq \int_{b_1}^{t} \exp\Big(\int_s^t |p_3(\tau)|\,\mathrm{d}\tau\Big)\Big[\big(q(s) + p_1(s)\big)(s - b_1) + p_2(s)\big(h_0(s) - b_1\big) + p_4(s)\Big]\,\mathrm{d}s,\ t \in (b_1, b\rangle\Big),$$

which is a contradiction. Thus $w(t)$ is a constant function on J and since $w(c) = 0$ we get $w = 0$ contradicting $w = y - y_1 \neq 0$.

COROLLARY 1. *Assume that assumptions* (17)–(19) *are satisfied for a positive constant* r_0. *Let* $\frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \in C^0(H \times I; \mathbb{R})$ *and let*

$$\frac{\partial g}{\partial y}(t, y, z, \mu) + q(t) \geqq 0 \qquad \textit{for} \quad (t, y, z, \mu) \in H \times I.$$

If $t \leqq h_0(t) \leqq c$ *for* $t \in \langle a, c\rangle$ *and* $c \leqq h_0(t) \leqq t$ *for* $t \in \langle c, b\rangle$, *then there exists the unique* $\mu_0 \in I$ *such that equation* (16) *with* $\mu = \mu_0$ *has a solution* y *satisfying* (2) *and* (21) *Furthermore this solution* y *is unique.*

Example 3. Let ν be a positive integer. Consider the equation

$$y''(t) - q(t)y(t) = \frac{\sin t}{12\,\mathrm{e}}\,\mathrm{e}^{y'(\frac{t}{2})} + \frac{\mathrm{e}^t}{12\,\mathrm{e}\cosh(1)}y^2(t)\cosh\bigl(y'(t)\bigr)\bigl(y(\sin t)\bigr)^{2\nu+1} + \mu. \quad (27)$$

Assumptions (8)–(11) are satisfied for $J = \langle -1, 1\rangle$, $c = 0$, $I = \left\langle -\frac{1}{6}, \frac{1}{6}\right\rangle$, $\frac{2}{3} \geqq q(t) \geqq \frac{1}{3}$ for $t \in J$ and $r_0 = r_1 = 1$. Setting $f(t, y, z, w, s, \mu) = \frac{\sin t}{12\,\mathrm{e}}\,\mathrm{e}^s + \frac{\mathrm{e}^t}{12\,\mathrm{e}\cosh(1)}y^2(\cosh w)z^{2\nu+1} + \mu$ for $(t, y, z, w, s, \mu) \in J \times \langle -1, 1\rangle \times \langle -1, 1\rangle \times \langle -1, 1\rangle \times \langle -1, 1\rangle \times I$ $(= S)$, then

$$\frac{\partial f}{\partial y} + q(t) = \frac{\mathrm{e}^t\, y\cosh(w)}{6\,\mathrm{e}\cosh(1)}z^{2\nu+1} + q(t) \geqq \frac{1}{6}, \quad \frac{\partial f}{\partial z} = \frac{(2\nu+1)\,\mathrm{e}^t\cosh(w)}{12\,\mathrm{e}\cosh(1)}y^2 z^{2\nu} \geqq 0,$$

$t\frac{\partial f}{\partial s} = \frac{t\,\mathrm{e}^s \sin t}{12\,\mathrm{e}} \geqq 0$ for $(t, y, z, w, s, \mu) \in S$ and since $t \leqq \frac{t}{2} \leqq 0$, $t \leqq \sin t \leqq 0$ for $t \in \langle -1, 0\rangle$ and $0 \leqq \frac{t}{2} \leqq t$, $0 \leqq \sin t \leqq t$ for $t \in \langle 0, 1\rangle$, there follows from Theorem 3 the existence of the unique $\mu_0 \in I$ such that equation (27) with $\mu = \mu_0$ has a solution y satisfying (2) and $|y(t)| \leqq 1$, $|y'(t)| \leqq 1$ for $t \in J$. Moreover this solution y is unique.

REFERENCES

[1] STANĚK, S.: *Three-point boundary value problem of nonlinear second-order differential equation with parameter*, Czechoslovak Math. J. (To appear.).

Received September 18, 1989

Katedra matematické analýzy
Přírodovědecké fakulty UP
tř. Svobody 26
772 00 Olomouc