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ON RADICAL CLASSES OF ABELIAN LINEARLY ORDERED GROUPS

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Radical classes of linearly ordered groups were introduced and investigated by C. G. Chehata and R. Wiegandt [1]; these authors developped a radical theory for linearly ordered groups in the sense of the Kuroš—Amitsur radical theory for groups and rings. Recall that a non-empty class C of linearly ordered groups is a radical class iff it is closed with respect to homomorphisms and with respect to transfinite extensions (cf. [1], Thm. 1).

Let $\mathcal{G}_a$ be the class of all abelian linearly ordered groups. In this note we modify the definition of the radical class of linearly ordered groups given in [1] in such a way that this modification enables one to work with the notion of a radical and a radical class in the class $\mathcal{G}_a$. Namely, a nonempty subclass C of $\mathcal{G}_a$ will be said to be a radical class (in $\mathcal{G}_a$) if it fulfils the following conditions: (i) C is closed with respect to homomorphisms, and (ii) if $G \in \mathcal{G}_a$ is a transfinite extension of linearly ordered groups belonging to C, then G belongs to C as well.

We denote by $\mathcal{R}_a$ the lattice of all radical classes in $\mathcal{G}_a$. There will be examined some questions concerning the lattice $\mathcal{R}_a$ analogous to those that were studied for the lattice $\mathcal{T}$ of torsion classes of lattice ordered groups (Martinez [5], [6]) and for the lattice $\mathcal{R}$ of radical classes of lattice ordered groups [4].

It turns out that the results concerning $\mathcal{R}_a$ essentially differ from the corresponding results on $\mathcal{T}$ and $\mathcal{R}$. E. g., both $\mathcal{T}$ and $\mathcal{R}$ are distributive; it will be shown below that $\mathcal{R}_a$ fails to be modular. If A is any principal element of $\mathcal{R}$, then there are infinitely many principal elements of $\mathcal{R}$ covering A. On the other hand, if A is a principal element of $\mathcal{R}_a$ generated by an archimedean linearly ordered group, then no element of $\mathcal{R}_a$ covers A; in particular, there are no atoms in the lattice $\mathcal{R}_a$. If A_1, A_2, $B \in \mathcal{R}$ such that A_1, A_2 are principal and $B \leqq A_1$, then (i) $A_1 \vee A_2$ is principal, and (ii) B is principal. Neither (i) nor (ii) remains valid in the lattice $\mathcal{R}_a$.

1. Preliminaries

For the basic definitions concerning lattice ordered groups and linearly ordered groups we refer to L. Fuchs [2] and P. Conrad [3]. The group operation in a linearly ordered group will be written additively.

All linearly ordered groups dealt with in this paper are assumed to be abelian; the expression 'linearly ordered group' below will always mean 'abelian linearly ordered group'.

Let $\mathcal{G}_a$ be the class of all linearly ordered groups. By considering a subclass X of $\mathcal{G}_a$ we always suppose that X is closed with respect to isomorphisms and that $\{0\} \in X$.

A subclass X of $\mathcal{G}_a$ is said to have the transfinite extension property if, whenever $G \in \mathcal{G}_a$ and

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots (\alpha < \delta)$$

is an ascending chain of convex subgroups of G such that

$$G_\beta / \bigcup_{\gamma<\beta} G_\gamma \in X \text{ for each } \beta < \delta,$$

then $\bigcup_{\alpha<\delta} G_\alpha$ belongs to X. We express this fact also by saying that X is closed with respect to transfinite extensions.

Under the above denotations, the linearly ordered group $\bigcup_{\alpha<\delta} G_\alpha$ is said to be a transfinite extension of linearly ordered groups $G'_\beta (\beta < \delta)$, where G'_β is isomorphic to $G_\beta / \bigcup_{\gamma<\beta} G_\gamma$ for each $\beta < \delta$.

1.1. Definition. *A class X of linearly ordered groups is called a radical class if*

(a) *X is closed under homomorphisms, and*

(b) *X is closed with respect to transfinite extensions.*

Let $X \subseteq \mathcal{G}_a$ and $G \in \mathcal{G}_a$. Further let N_α be the set of all convex subgroups of G belonging to X. We put $X(G) = \bigcup_\alpha N_\alpha$. Next we denote by UX the class of all linearly ordered groups H such that H has no non-trivial homomorphic image in X.

Let us consider the following condition for X:

(S1) If $G_1 \in X$, then every non-trivial convex subgroup G_2 of G_1 has a non-trivial homomorphic image in X.

The proofs of the following propositions 1.2 and 1.3 are analogous to the proofs of the corresponding propositions in [1].

1.2. Proposition. (Cf. [1], Proposition 3.) *Let R be a radical class. Then $R(G)$ belongs to R.*

1.3. Proposition. (Cf. [1], Propos. 6.) *If X fulfils the condition (S1), then UX is a radical class.*

2. Basic properties of $\mathcal{R}_a$

Let $\mathcal{R}_a$ be the collection of all radical classes of linearly ordered groups. Then $\mathcal{G}_a$ is the greatest element of $\mathcal{R}_a$ and the class $R_0 = \{\{0\}\}$ is the least element of $\mathcal{R}_a$. If $\emptyset \neq R_1 \subseteq R_a$ and if R is the intersection of all radical classes belonging to R_1, then R fulfils the conditions (a) and (b) from 1.1; hence we have

2.1. Proposition. *$\mathcal{R}_a$ is a complete lattice.*

Let X be a subclass of $\mathcal{G}_a$. Let us denote by

TX — the intersection of all radical classes R with $X \subseteq R$;

Hom X — the class of all homomorphic images of linearly ordered groups belonging to X;

Ext X — the class of all linearly ordered groups G that have an ascending chain of convex subgroups

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (\alpha < \delta)$$

such that (i) $\bigcup_{\alpha<\delta} G_\alpha = G$, and (ii) for a each $\beta < \delta$, $G_\gamma / \bigcup_{\gamma<\beta} G_\gamma$ belongs to X.

In view of 2.1, TX is a radical class; it is said to be the radical class generated by X. If there is $G \in X$ such that for each $G_1 \in X$ either G_1 is isomorphic with G or $G_1 = \{0\}$, then we also say that TX is the radical class generated by G and we write $TX = T(G)$; the radical class $T(G)$ is called principal. Let $\mathcal{R}_p$ be the collection of all principal radical classes.

2.2. Proposition. *Let X be a subclass of $\mathcal{G}_a$. Then $TX = \text{Ext Hom} X$.*

Proof. Put $R = \text{Ext Hom} X$. Since TX is a radical class with $X \subseteq TX$, it follows from 1.1 that $R \subseteq TX$. Next, the class R has obviously the transfinite extension property. Hence it suffices to verify that R is closed under homomorphisms.

Let $G \in R$. There exists an ascending chain $\{G_\alpha\}$ $(\alpha < \delta)$ of convex subgroups of G with $G_1 = \{0\}$ such that

$$\bigcup_{\alpha<\delta} G_\alpha = G,$$

and for each $\beta < \delta$, $G_\beta / \bigcup_{\gamma<\beta} G_\gamma$ belongs to Hom X. Let H_1 be a convex subgroup of G, $H_1 \neq G$. Hence there exists the least $\beta < \delta$ with $H_1 \subset G_\beta$. We have

$$G = \bigcup_{\beta \leq \gamma < \delta} G_\gamma,$$

thus

$$G/H_1 = \bigcup_{\beta \leq \gamma < \delta} (G_\gamma / H_1).$$

Moreover, $\bigcup_{\gamma<\beta} G_\gamma \subseteq H_1$, hence G_β / H_1 is a homomorphic image of $G_\beta / \bigcup_{\gamma<\beta} G_\gamma$ and therefore G_β/H_1 belongs to Hom X. Denote $B_0=\{0\}$, $B_\gamma=G_\gamma/H_1$ for each γ with $\beta\leqq\gamma<\delta$, and consider the ascending chain of convex subgroups

$$B_0\subseteq B_\beta\subseteq B_{\beta+1}\subseteq\ldots\subseteq B_\gamma\subseteq\ldots\quad(\beta\leqq\gamma<\delta)$$

of G/H_1. We have already verified that B_β belongs to Hom X. If $\beta<\gamma<\delta$, then the linearly ordered group

$$B_\gamma/\bigcup_{\varkappa<\gamma}B_\varkappa=(G_\gamma/H_1)/\bigcup_{\varkappa<\gamma}(G_\varkappa/H_1)=$$
$$=(G_\gamma/H_1)/\Big(\big(\bigcup_{\varkappa<\gamma}G_\varkappa\big)/H_1\Big)$$

is isomorphic with $G_\gamma/\bigcup_{\varkappa<\gamma}G_\varkappa$, hence

$$B_\gamma/\bigcup_{\varkappa<\gamma}B_\varkappa\in\operatorname{Hom}X.$$

Therefore A/H_1 belongs to Ext Hom X, completing the proof.

Let us denote by $\wedge$ and $\vee$ the lattice operations in the complete lattice $\mathcal{R}_a$. We already noticed that $\wedge$ coincides with $\cap$ (= the intersection of classes). From 2.2 and 1.1 we obtain immediately:

2.3. Corollary. *Let $J\neq\emptyset$ be a class and for each $j\in J$ let X_j be a radical class. Then* $\bigvee_{j\in J}x_j=\operatorname{Ext}\bigcup_{j\in J}X_j$.

2.4. Lemma. *Let R_1, $R_2\in R_a$. Then $R_1\leqq R_2$ if and only if $R_1(G)\subseteq R_2(G)$ is valid for each $G\in\mathcal{G}_a$.*

Proof. If $R_1\leqq R_2$, then clearly $R_1(G)\subseteq R_2(G)$ for each $G\in\mathcal{G}_a$. Suppose that $R_1(G)\subseteq R_2(G)$ holds for each $G\in\mathcal{G}_a$. Let $H\in R_1$. Then $R_1(H)=H$, hence $R_2(H)=H$. Thus in view of 1.2, H belongs to R_2, hence $R_1\leqq R_2$.

For $G\in\mathcal{G}_a$ we denote by $c(G)$ the set of all convex subgroups of G; the set $c(G)$ is partially ordered by inclusion. It is easy to verify that under this partial order, $c(G)$ is a complete chain. From 2.4 we infer that the relations

$$(R_1\wedge R_2)(G)\subseteq R_1(G)\wedge R_2(G),\tag{2.1}$$

$$R_1(G)\vee R_2(G)\subseteq(R_1\vee R_2)(G)\tag{2.2}$$

are valid for each $G\in\mathcal{G}_a$ and for each R_1, $R_2\in R_a$. (Analogous relations for torsion classes were examined in [5].) The following examples 2.5 and 2.6 show that the relation $\subset$ can occur in (2.1) and (2.2).

At first let us recall the notion of a lexicographic product of linearly ordered groups. Let I be a linearly ordered set and for each $i\in I$ let G_i be a linearly ordered group. The symbol $H=\Gamma_{i\in I}G_i$ denotes the lexicographic product of the system

$\{G_i\}_{i\in I}$; thus H is the set of all functions $f\colon I\to\bigcup_{i\in I} G_i$ such that (i) $f(i)\in G_i$ for each $i\in I$, and (ii) the set $\{i\in I\colon f(i)\neq 0\}$ is either empty or is dually well-ordered. The operation $+$ in H is defined coordinate-wise and for $f_1, f_2\in H$ with $f_1\neq f_2$ we put $f_1<f_2$ if there exists $i\in I$ such that $f_1(i)<f_2(i)$ and $f_1(j)=f_2(j)$ for each $j\in I$ with $j>i$. If $I=\{1, 2, \ldots, n\}$, then we write also $H=G_1\circ G_2\circ\ldots\circ G_n$.

2.5. Example. Let G_1, G_2, G_3 be the additive group of all integers, all rational numbers and all reals, respectively (with the natural linear order). Put

$$G=G_1\circ G_2\circ G_3,\quad R_1=T(G_1, G_3),\quad R_2=T(G_2, G_3).$$

(The above symbols have an obvious meaning; e.g., $T(G_1, G_3)=TX$, where X is the class of all $G'\in\mathcal{G}_a$ such that, whenever $G'\neq\{0\}$, then G' is isomorphic either to G_1 or to G_3. The meaning of the symbol $\mathrm{Ext}(G_1, G_3)$ which will be used below is analogous.)

From 2.2 we obtain

$$R_1(G)=G_1,\quad R_2(G)=\{0\}.$$

In view of 2.2 and 2.3, $(R_1\vee R_2)(G)=G$. Hence $R_1(G)\vee R_2(G)\neq(R_1\vee R_2)(G)$.

2.6. Example. Let G_1, G_2, G_3, G be as above. Put $R_1=T(G_1\circ G_2)$, $R_2=T(G)$. Then from 2.2 we infer that $R_1(G)=G_1\circ G_2$. Hence

$$R_1(G)\wedge R_2(G)=G_1\circ G_2$$

and $(R_1\wedge R_2)(G)\subseteq G_1\circ G_2$. Since $R_1\wedge R_2=R_1\cap R_2$, we infer that $(R_1\wedge R_2)(G)$ is the join of those convex subgroups of $G_1\circ G_2$ that belong to $R_1\cap R_2$. Therefore

$$(R_1\wedge R_2)(G)=R_2(G_1\circ G_2)=\{0\}.$$

Thus $R_1(G)\wedge R_2(G)\neq(R_1\wedge R_2)(G)$.

The lattice of all torsion classes of lattice ordered groups is distributive (Martinez [5]) and so is the lattice of all radical classes of lattice ordered groups (cf. [4]). The following example shows that the lattice $\mathcal{R}_a$ fails to be modular.

2.7. Example. Let G_1 and G_2 be as in 2.5. Put

$$R_1=T(G_1\circ G_2),\quad R_2=T(G_2),\quad R_3=T(G_1),\quad R=R_1\vee R_3.$$

In view of 2.2 we have $R_0<R_2<R_1$ and

$$R_0=R_1\wedge R_3,\quad R=R_2\vee R_3.$$

Hence $\{R_0, R_1, R_2, R_3, R\}$ is a sublattice of $\mathcal{R}_a$ isomorphic to the pentagon. Thus $\mathcal{R}_a$ is not modular.

3. Principal radical classes

The system of all principal radical classes of lattice ordered groups is an ideal in the lattice of all radical classes of lattice ordered groups. For the lattice $\mathcal{R}_a$ we have a different situation:

(i) if $Q_1, Q_2 \in \mathcal{R}_p$, then $Q_1 \vee Q_2$ need not belong to $\mathcal{R}_p$, and

(ii) if $Q_1 \in \mathcal{R}_p$, $Q_2 \in \mathcal{R}_a$ and $Q_2 < Q_1$, then Q_2 need not belong to $\mathcal{R}_p$.

The assertions (i) and (ii) are consequences of Propos. 3.1 below.

We introduce the following denotation. Let α be an infinite cardinal. Let us denote by $\omega(\alpha)$ the least ordinal having the property that the power of the set of all ordinals less than $\omega(\alpha)$ is α. For each $G \in \mathcal{G}_a$ we put

$$G_\alpha = \Gamma_{i \in I(\alpha)} G_i,$$

where $I(\alpha)$ is a linearly ordered set isomorphic with $\omega(\alpha)$ and G_i is a linearly ordered group isomorphic with G for each $i \in I(\alpha)$.

3.1. Proposition. *Let $G_1, G_2 \in \mathcal{G}_a$, $G_1 \neq \{0\} \neq G_2$. Assume that G_1 and G_2 are non-isomorphic and archimedean. Let α be a cardinal, $\alpha > \operatorname{card}(G_1 \circ G_2)$. Then*

(i) $T((G_1 \circ G_2)_\alpha) \vee T((G_2)_\alpha) < T(G_1 \circ G_2)$,

(ii) *the radical class $T((G_1 \circ G_2)_\alpha) \vee T((G_2)_\alpha)$ fails to be principal.*

Proof. We have obviously

$$T((G_1 \circ G_2)_\alpha) \bigvee T((G_2)_\alpha) \leqq T(G_1 \circ G_2).$$

If $H \neq \{0\}$ is a homomorphic image of $(G_1 \circ G_2)_\alpha$ or of $(G_2)_\alpha$, then $\operatorname{card} H \geqq \alpha$, hence $G_1 \circ G_2 \notin \operatorname{Ext} \operatorname{Hom} \{T((G_1 \circ G_2)_\alpha) \cup T((G_2)_\alpha)\}$; therefore (in view of 2.2) (i) is valid.

For proving (ii) let us assume (by way of contradiction) that there is $K \in \mathcal{G}_a$ with

$$T((G_1 \circ G_2)_\alpha) \vee T((G_2)_\alpha) = T(K).$$

According to (i) we have $T(K) < T(G_1 \circ G_2)$, thus $K \in T(G_1 \circ G_2)$. Hence in view of 2.2 and 2.3 there are convex subgroups $K_\varkappa$ $(\varkappa < \delta)$ of K such that

$$\{0\} = K_1 \subseteq K_2 \ldots \subseteq K_\varkappa \ldots \quad (\varkappa < \delta),$$
$$\bigcup_{\varkappa < \delta} K_\varkappa = K,$$

and for each $\beta < \delta$, the linearly ordered group

$$\bar{K}_\beta = K_\beta / \bigcup_{\gamma < \beta} K_\gamma$$

is isomorphic to some of the linearly ordered groups $\{0\}$, G_2, $G_1 \circ G_2$. (In fact, since G_1 and G_2 are archimedean, the only homomorphic images of $G_1 \circ G_2$ are $\{0\}$, G_2

and $G_1 \circ G_2$.) Without loss of generality we can suppose that for each $\beta_0 < \delta$ there exists $\beta < \delta$ such that $\beta \leqq \beta_0$ and $\bar{K}_\beta \neq \{0\}$.

Let us distinguish the following cases:

(a) Assume that there is $\beta_0 < \delta$ such for each $\beta \geqq \beta_0$, either $\bar{K}_\beta = \{0\}$ or $\bar{K}_\beta$ is isomorphic to G_2. In this case $(G_1 \circ G_2)_\alpha \notin \text{Ext Hom}\{K\} = T(K)$, which is a contradiction.

(b) Assume that there is $\beta_0 < \delta$ such that for each $\beta \geqq \beta_0$ either $\bar{K}_\beta = \{0\}$ or $\bar{K}_\beta$ is isomorphic to $G_1 \circ G_2$. Hence because of (i), δ must be a limit ordinal. Therefore $G_2 \notin \text{Ext Hom}\{K\} = T(K)$, a contradiction.

(c) Assume that neither (a) nor (b) is valid. Hence δ must be a limit ordinal. Therefore neither G_2 nor $G_1 \circ G_2$ belongs to Ext Hom$\{K\}$, which is a contradiction. Therefore (ii) holds.

3.2. Remark. The following question remains open:

Characterize linearly ordered groups G having the property that there exist $G_1, G_2 \in \mathscr{G}_a$ such that (i) $T(G_1) \vee T(G_2) < T(G)$, and (ii) $T(G_1) \vee T(G_2)$ is not principal.

The following proposition implies that there exist radical classes $A \neq \mathscr{G}_a$ such that no principal radical class is larger than A.

3.3. Proposition. *Let G_1, $G_2 \in \mathscr{G}_a$, $G_1 \neq \{0\} \neq G_2$. Assume that G_1 is not isomorphic to G_2 and that both G_1 and G_2 are archimedean. Then $\{T(G_1), T(G_2)\}$ is not upper bounded in $\mathscr{R}_p$ and $T(G_1) \vee T(G_2) \neq \mathscr{G}_a$.*

Proof. By way of contradiction, assume that there is $H \in \mathscr{G}_a$ with $T(H) \geqq T(G_1)$, $T(H) \geqq T(G_2)$. From this and from 2.2 we infer that $G_1 \in \text{Ext Hom}\{H\}$ and thus, because G_1 is archimedean, we get $G_1 \in \text{Hom}\{H\}$. Hence there is a convex subgroup H_1 of H with $H_1 \neq H$ such that H/H_1 is isomorphic with G_1. Similarly, there exists a convex subgroup H_2 of H with $H_2 \neq H$ such that H/H_2 is isomorphic with G_2. Since G_1 and G_2 are not isomorphic, $H_1 \neq H_2$. Thus without loss of generality we can assume that $H_1 \subset H_2$ is valid. But in this case H/H_1 is not o-simple, thus it is not archimedean, which is a contradiction. There exist archimedean linearly ordered groups G having the property that G is isomorphic neither to G_1 nor to G_2; then $G \notin \text{Ext}\{G_1, G_2\} = T(G_1) \vee T(G_2)$, hence $T(G_1) \vee T(G_2) \neq \mathscr{G}_a$.

The following proposition says that for each principal radical class A there exists a radical class $B \neq \mathscr{G}_a$ with $A < B$ such that no principal radical class is larger than B.

If we have a lexicographic product $H = {}_{i \in I} G_i$, then we denote by $\Gamma'_{i \in I} G_i$ the subgroup of H consisting of all $g \in H$ such that the set $\{i \in I\colon g(i) \neq 0\}$ is finite.

Let α be an infinite cardinal and let $I(\alpha)$ be as above. Let $J(\alpha)$ be the linearly ordered set dual to $I(\alpha)$. Let $G \in \mathscr{G}_a$. Put

$$G_\alpha^1 = \Gamma_{i \in J(\alpha)} G_i, \quad G_\alpha^2 = \Gamma'_{i \in J(\alpha)} G_i,$$

where each G_i is isomorphic to G.

Recall that for $G \in \mathscr{G}_a$, $c(G)$ is the set of all convex subgroups of G (cf. 2). From the construction of G_α^i $(i=1, 2)$ we obtain by routine calculations:

3.4. Lemma. *Let $G \in \mathscr{G}_a$, $G \neq \{0\}$. Let α be a cardinal, $\alpha > \operatorname{card} G$. Let $H_i \neq \{0\}$ be a convex subgroup of G_α^i $(i=1, 2)$ and let $K_1 \neq \{0\}$ be a convex subgroup of H_1. Then*

(i) $\operatorname{card} H_1 = 2^\alpha$, $\operatorname{card} H_2 = \alpha$,

(ii) $\operatorname{card} c(H_2) = \alpha$, $\operatorname{card} c(H_1/K_1) < \alpha$.

3.5. Proposition. *Let $G \in \mathscr{G}_a$, $G \neq \{0\}$. Let α be a cardinal, $\alpha > \operatorname{card} G$. Then*

(i) *$T(G) < T(G_\alpha^i)$ $(i=1, 2)$, $T(G_\alpha^1) \vee T(G_\alpha^2) < \mathscr{G}_a$, and*

(ii) *if $A \in \mathscr{R}_a$, $T(G_\alpha^1) \vee T(G_\alpha^2) \leqq A$,*

then A fails to be principal.

Proof. Let $i \in \{1, 2\}$. From the definition of G_α^i it follows that if K is a convex subgroup of G_α^i with $K \neq \{0\}$, then $\operatorname{card} K \geqq \alpha$; hence K cannot be isomorphic to any homomorphic image of G. Therefore $G_\alpha^i \notin \operatorname{Ext} \operatorname{Hom}\{G\} = T(G)$, whence $T(G_\alpha^i) \not\leqq T(G)$. On the other hand, $G \in \operatorname{Hom}\{G_\alpha^i\}$ and thus $T(G) < T(G_\alpha^i)$. Moreover, from 3.4 (i) we conclude that for each $\beta > 2^\alpha$ and each $i \in \{1, 2\}$ the linearly ordered group G_β^i does not belong to $T(G_\alpha^1) \vee T(G_\alpha^2)$. Thus (i) is valid.

Assume that there is $H \in \mathscr{G}_a$ such that $T(G_\alpha^1) \leqq T(H)$ and $T(G_\alpha^2) \leqq T(H)$. Thus there are convex subgroups H_i $(i=1, 2)$ of G_α^i such that $H_i \neq \{0\}$ and $H_i \in \operatorname{Hom}\{H\}$. Hence either

(a) H_1 is a homomorphic image of H_2, or

(b) H_2 is a homomorphic image of H_1.

In view of 3.4 (i), the condition (a) cannot hold. From this and from 3.4 (ii) it follows that (b) cannot be valid. Therefore $\{T(G_\alpha^1), T(G_\alpha^2)\}$ is not upper bounded in $\mathscr{R}_p$; hence (ii) holds.

From 3.5 (i) we obtain immediately:

3.6. Corollary. *The partially ordered class $\mathscr{R}_p$ has no maximal element.*

Similarly as in the case of 3.4, the following lemma is a direct consequence of the definition of G_α^1 and G_α^2.

3.7. Lemma. *Let G and α be as in 3.5. Assume that G is archimedean. Let $K_i \neq \{0\}$ be a homomorphic image of G_α^i $(i=1, 2)$. Suppose that there are $j, k \in \{1, 2\}$, $j \neq k$ such that K_j is isomorphic with a convex subgroup of K_i. Then $K_j \in T(G)$.*

3.8. Lemma. *Let G and α be as in 3.5. Suppose that G is archimedean. Then $T(G_\alpha^1) \wedge T(G_\alpha^2) = T(G)$.*

Proof. We have already verified that $T(G) < T(G_\alpha^i)$ $(i=1, 2)$ is valid. Hence $T(G) \leqq T(G_\alpha^1) \wedge T(G_\alpha^2)$. Let $H \in T(G_\alpha^1) \wedge T(G_\alpha^2)$. It suffices to prove that for $T(G) = R$ we have $R(H) = H$. By way of contradiction, assume that $R(H) \subset H$. Put $H_1 = H/R(H)$. Then (since R has the transfinite extension property) we must have $R(H_1) = \{0\}$. On the other hand, $H_1 \in T(G_\alpha^1) \wedge T(G_\alpha^2)$, hence there exist convex subgroups $K_1 \neq \{0\}$ and $K_2 \neq \{0\}$ of H_1 such that K_i is a homomorphic image of G_α^i for $i = 1, 2$. Put $K = K_1 \cap K_2$. Since $K = K_1$ or $K = K_2$, we have $K \neq \{0\}$ and in view of 3.7, $K \in T(G)$, hence $R(H_1) \supseteq K \neq \{0\}$, which is a contradiction. Therefore $T(G) = T(G_\alpha^1) \wedge T(G_\alpha^2)$.

3.9. Proposition. *Let $G \in \mathscr{G}_a$, $T(G) = A$. Assume that G is archimedean. Then there are principal radical classes B_1, B_2 with $B_i > A$ $(i=1, 2)$, $B_1 \wedge B_2 = A$.*

Proof. Let $A = T(G)$. If $G \neq \{0\}$, then our assertion follows from 3.8. If $G = \{0\}$, then it suffices to take any pair of non-isomorphic 0-simple linearly ordered groups B_1, B_2 with $B_1 \neq \{0\} \neq B_2$; we have clearly $T(B_1) \wedge T(B_2) = R_0$.

Let us remark that the assertion dual to 3.9 does not hold (cf. 4.7 below).

4. Covering relations in the lattice $\mathscr{R}_a$

Let $G \neq \{0\}$ be a linearly ordered group and let α be a cardinal. The lattice ordered group G_α was defined in § 3.

4.1. Lemma. *Let $\alpha > \operatorname{card} G$, $\beta > \alpha$. Then*

(i) *$R_0 < T(G_\alpha) < T(G)$, and*

(ii) *$T(G_\beta) < T(G_\alpha)$.*

Proof. The way of proving (i) is analogous to that used in the proof of 3.1. The relation (ii) follows from the fact that for each homomorphic image $H \neq \{0\}$ of G_β we have $\operatorname{card} H = \beta$, for each convex subgroup $K \neq \{0\}$ of G_α there holds $\operatorname{card} K \leqq \alpha$, and clearly $G_\beta \in T(G_\alpha)$.

4.2. Corollary. *Let R be a radical class of linearly ordered groups, $R \neq R_0$. Then there is a chain $C \subset [R_0, R]$ such that C is a proper class and $C \subset \mathscr{R}_p$.*

4.3. Corollary. *The lattice $\mathscr{R}_a$ does not contain any atoms.*

For $A, B \in \mathscr{R}_a$ we write $A \succ B$ or $B \prec A$ if $B < A$ and if there does not exist any $C \in \mathscr{R}_a$ with $B < C < A$; in such a case we say that A covers B.

Corollary 4.3 can be generalized as follows.

4.4. Lemma. *Let H be an archimedean linearly ordered group. Let $G \in \mathscr{G}_a$, $G \notin T(H)$ and let α be a cardinal with $\alpha > \operatorname{card} G$. Then $G_\alpha \notin T(H)$.*

Proof. By way of contradiction, assume that $G_\alpha \in T(H)$. Hence in view of 2.2 and because H is o-simple, $G_\alpha \in \operatorname{Ext}\{H\}$. Thus there are convex subgroups $H_{\alpha'}$ of G_α such that

$$\{0\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_{\alpha'} \ldots (\alpha' < \gamma), \bigcup_{\alpha'<\gamma} H_{\alpha'} = G_\alpha,$$

and for each $\alpha' < \gamma$, $H_{\alpha'} / \bigcup_{\beta<\alpha'} H_\beta$ is isomorphic either with $\{0\}$ or with H.

There exists a convex subgroup K of G_α such that K is isomorphic to G. Thus there is $\alpha' < \gamma$ with $K \subseteq H_{\alpha'}$; let α' be the first ordinal having this property. Hence

$$\bigcup_{\beta<\alpha'} H_\beta \subseteq K \subseteq H_{\alpha'}.$$

If $\bigcup_{\beta<\alpha'} H_\beta = K$ or $H_{\alpha'} = K$, then $K \in \operatorname{Ext}\{H\} = T(H)$, which is impossible. Hence

$$\bigcup_{\beta<\alpha'} H_\beta \subset K \subset H_{\alpha'},$$

but in this case $H_{\alpha'} / \bigcup_{\beta<\alpha'} H_\beta$ fails to be o-simple, thus it is not archimedean, which is a contradiction.

4.5. Proposition. *Let H be an archimedean lattice ordered group, $B = T(H)$, $A \in \mathcal{R}_a$, $B < A$. Then there is a chain $C \subset [B, A]$ such that C is a proper class.*

Proof. There is $G \in A \setminus B$. Let α, γ be cardinals with

$$\gamma > \alpha > \max\{\operatorname{card} G, \operatorname{card} H\}.$$

In view of 4.4, $G_\alpha \notin B$. Therefore $T(G_\alpha) \not\subseteq B$. On the other hand, from $\alpha > \operatorname{card} G$ it follows that $\operatorname{card} K = \alpha$ for each homomorphic image K of G_α with $K \neq \{0\}$, whence $H \notin \operatorname{Ext} \operatorname{Hom}\{G_\alpha\}$, implying $B \not\subseteq T(G_\alpha)$. Therefore B and $T(G_\alpha)$ are incomparable. Clearly $T(G_\alpha) < T(G) \leqq A$. Hence

$$B < B \vee T(G_\alpha) \leqq A.$$

Analogous relations are valid for G_γ. In view of 4.1 we have

$$B \vee T(G_\gamma) \leqq B \vee T(G_\alpha). \tag{4.1}$$

Now it suffices to verify that in this relation the equality cannot hold.

From 2.2 we obtain

$$B \vee T(G_\gamma) = \operatorname{Ext} \operatorname{Hom}\{H, G_\gamma\}.$$

If the equality holds in (4.1), then $G_\alpha \in \operatorname{Ext} \operatorname{Hom}\{H, G_\gamma\}$. Because of $\operatorname{card} G_\alpha = \alpha$ and $\operatorname{card} K_1 = \gamma$ for each homomorphic image $K_1 \neq \{0\}$ of G_γ, we must have $G_\alpha \in \operatorname{Ext} \operatorname{Hom}\{H\}$, thus $G_\alpha \in B$, which is a contradiction.

4.6. Corollary. *Let $B = T(H)$, where H is an archimedean linearly ordered group. Then there does not exist any radical class covering B.*

The existence of an infinite number of prime intervals in the lattice $\mathcal{R}_a$ is a consequence of the following proposition:

4.7. Proposition. *Let $G \in \mathcal{R}_a$, $G \neq \{0\}$. Assume that G is archimedean. Let B be the join of all radical classes B_1 with $B_1 < T(G)$. Then $B < T(G)$.*

Proof. Let us consider the class $\mathcal{H}$ of all linearly ordered groups $H_i \in T(G)$ with $T(H_i) \neq T(G)$. We have clearly

$$\bigvee T(H_i) = B \leqq T(G).$$

From the fact that G is archimedean and from 2.2 we obtain

$$T(G) = \operatorname{Ext}\{G\}.$$

Let H_i be as above, $H_i \neq \{0\}$. Then $H_i \in \operatorname{Ext}\{G\}$. Since $T(H_i) \neq T(G)$, no homomorphic image of H_i is isomorphic with G.

For proving that B is covered by $T(G)$ it suffices to verify that $B < T(G)$ is valid. By way of contradiction, assume that $B = T(G)$. Then $G \in B$, hence in view of 2.3, $G \in \operatorname{Ext}\operatorname{Hom}\{H_i\}$ $(H_i \in \mathcal{H})$. From this and from the o-simplicity of G we infer that $G \in \operatorname{Hom}\{H_i\}$ for some $H_i \in \mathcal{H}$, which is impossible.

4.8. Corollary. *For each archimedean linearly ordered group $G \neq \{0\}$ there exists exactly one radical class which is covered by $T(G)$.*

Again, let $\{0\} \neq G \in \mathcal{G}_a$ and let α be an infinite cardinal. Recall that the linearly ordered group G_α^2 was defined in § 3. A radical class A will be said to be $*$-closed if, whenever $\{0\} \neq G \in A$, then $G_\alpha^2 \in A$ for each cardinal α.

As an immediate consequence of 3.5 (i) we obtain:

4.9. Corollary. *Let A be a $*$-closed radical class, $A \neq R_0$. Then A is not principal. (In particular, $\mathcal{G}_a$ is not principal.)*

From the definition of G_α^2 it follows immediately:

4.10. Lemma. *Let $H_1 \neq \{0\}$ be a convex subgroup of G_α^2. Then there exists a convex subgroup H_2 of G_α^2 such that $H_2 \subset H_1$ and H_2 is isomorphic with G_α^2.*

4.11. Proposition. *Let A be a $*$-closed radical class. Then there is no radical class B with $B < A$.*

Proof. By way of contradiction, suppose that there is a radical class B with $B < A$. Hence for each $G_1 \in A \setminus B$ we have $B \vee T(G_1) = A$.

There exists $G \in A \setminus B$. Let α be a cardinal with $\alpha > \operatorname{card} G$. As A is $*$-closed, G_α^2 must belong to A, and in view of 3.5 (i), G_α^2 does not belong to B. From $G_\alpha^2 \in B \vee T(G)$ and from 2.2 and 2.3 it follows that there exists a convex subgroup H_1 of G_α^2 with $H_1 \neq \{0\}$ such that either (i) $H_1 \in B$, or (ii) H_1 is isomorphic to some homomorphic image of G. The case (ii) is impossible in view of 4.10 (with respect to $\operatorname{card} H_1 = \alpha > \operatorname{card} G$). Hence (i) is valid. Put $B(G_\alpha^2) = H$. Then $H \supseteq H_1$.

Let us write (as above in § 3)

$$G_\alpha^2 = \Gamma'_{i \in J(\alpha)} G_i.$$

Put $J_0=\{i\in J(\alpha): G_i\cap H\neq\{0\}\}$. (As usual, we denote by G_i also the set of all $g\in G^2_\alpha$ such that $g(j)=0$ for each $j\in J(\alpha)$, $j\neq i$.) Then $J_0\neq\emptyset$ and thus J_0 possesses the greatest element j. Hence $G_i\cap H=G_i$ for each $i<j$ and we have

$$H=(\Gamma'_{i<j}G_i)\circ(G_j\cap H).$$

Therefore $G_j\cap H$ is a homomorphic image of H and thus $G_j\cap H\in B$. From this it follows that $G_j\cap H\neq G_j$ (since G_j is isomorphic to G and $G\notin B$). According to 1.2, $B(G_j)\supseteq G_j\cap H$. Hence $B(G_j)\neq\{0\}$ and therefore

$$B(G)\neq\{0\}. \tag{4.2}$$

Denote $K=G_j/G_j\cap H$; then $K\neq\{0\}$. The linearly ordered group G^2_α/H is isomorphic with

$$K\circ\Gamma'_{i>j}G_i.$$

If $B(K)\neq\{0\}$, then $B(K\circ\Gamma'_{i>j}G_i)\neq\{0\}$ and hence (because B has the transfinite extension property) we would have

$$B(G^2_\alpha)\supset H,$$

which is a contradiction. Hence

$$B(K)=\{0\}. \tag{4.3}$$

Because of $G_i\in A$ and $K\in\operatorname{Hom}\{G_i\}$ we obtain $K\in A$. From (4.3) it follows that $K\notin B$. Hence in the above consideration we can replace G with K and in view of (4.2) we infer that the relation $B(K)\neq\{0\}$ is valid, contradicting (4.3).

4.12. Corollary. *$\mathcal{G}_a$ does not cover any radical class* (i.e., *the lattice $\mathcal{R}_a$ has no dual atoms*).

For $X\subseteq\mathcal{G}_a$ let UX be as in §1.

There are many $*$-closed radical classes; this is a consequence of the following Proposition:

4.13. Propostion. *Let $X\subseteq\mathcal{G}_a$. Assume that each $H\in X$ is archimedean. Then*

(i) *UX is a $*$-closed radical class;*

(ii) *if H_1 and H_2 are archimedean linearly ordered groups with $H_1\in X$, $H_2\notin X$, then $H_1\notin UX$ and $H_2\in UX$.*

Proof. From 1.3 it follows that UX is a radical class. According to the definition of UX, (ii) is valid. If $G\in UX$ and if α is a cardinal, then clearly $G^2_\alpha\in UX$; hence UX is $*$-closed.

4.14. Proposition. *Let B be a $*$-closed radical class, $B\neq R_0$. Then there is no $G\in\mathcal{G}_a$ with $B\leqq T(G)$.*

Proof. By way of contradiction, assume that $B\leqq T(G)$ for some $G\in\mathcal{G}_a$. Let α

be a cardinal, $\alpha > \operatorname{card} G$ and let $H \in B$, $H \neq \{0\}$. Then $H_\alpha^2 \in B$, hence $H_\alpha^2 \in T(G) = \operatorname{Ext} \operatorname{Hom}(G)$. Thus there exists a convex subgroup H_1 of H_α^2 with $H_1 \neq \{0\}$, $H_1 \in \operatorname{Hom}(G)$. Then $\operatorname{card} H_1 < \alpha$, which is impossible in view of the definition of H_α^2.

4.15. Proposition. *Let $G \neq \{0\}$ be an archimedean lattice ordered group. Denote $B = U\{G\}$, $A = B \vee T(G)$. Then*

(i) *$B < A$,*

(ii) *B is not principal, and*

(iii) *A is not principal.*

Proof. In view of 4.13, B is a $*$-closed radical class and $G \notin B$, hence $T(G) \nleqq B$. Thus $B < A$. Let C be a radical class with $B < V \leqq A$. There exists $G_1 \in C \setminus B$. According to the definition of $U\{G\}$, some homomorphic image of G_1 is isomorphic to G and therefore $G \in C$. Thus $T(G) \leqq C$, implying $C = A$; hence (i) is valid. From 4.14 it follows that (ii) and (iii) hold.

For $C \in \mathcal{R}_a$ we denote by $a(C)$ the class of all $D \in \mathcal{R}_a$ such that $C < D$. Let $a'(C)$ be defined dually. As we have shown above, there exist $C_1, C_2 \in \mathcal{R}_a$ distinct from R_0 and $\mathcal{G}_a$ such that $a(C_1) = \emptyset$ and $a'(C_2) = \emptyset$ (cf. 4.6 and 4.11). The following question remains open:

Give an internal characterization of radical classes C with $a(C) = \emptyset$ (and, analogously, with $a'(C) = \emptyset$).

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О РАДИКАЛЬНЫХ КЛАССАХ АБЕЛЕВЫХ ЛИНЕЙНО УПОРЯДОЧЕННЫХ ГРУПП

Ján Jakubík

Резюме

Класс $C \neq \emptyset$ абелевых линейно упорядоченных групп называется радикальным классом, если C замкнут относительно гомоморфизмов и относительно трансфинитных расширений. В статье рассматривается решетка всех радикальных классов абелевых линейно упорядоченных групп.