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QUASICOMMUTATIVE WEAKLY PRIMARY SEMIGROUPS

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Let S be a quasicommutative semigroup, i.e. a semigroup with $ab = b^r a$ for all elements a, b of S and for some positive integer $r = r(a, b)$.

Quasicommutative semigroup have been studied by N. P. Mukherjee in [5]. M. Satyanarayana in [6] studied primary ideals in commutative semigroups and commutative primary semigroups.

An ideal A of a commutative semigroup is called primary if for $x, y \in S$, $xy \in A$ and $x \notin A$ imply $y^n \in A$ for some positive integer n. A commutative semigroup is called primary if every ideal of the semigroup is primary.

An ideal P of an arbitrary semigroup is called prime (completely prime) if for any two ideals A, B (elements a, b) of the semigroup, $AB \subseteq P$ ($ab \in P$) implies that either $A \subseteq P$ or $B \subseteq P$ ($a \in P$ or $b \in P$).

In the present paper we shall study quasicommutative weakly primary ideals and weakly primary semigroups.

An ideal of a quasicommutative semigroup S is said to be weakly primary if x, $y \in S$, $xy \in Q$ implies that either $x^m \in Q$ or $y^n \in Q$ for some positive integers m, n. An analogous notion can be found in [4, 247] (for subsemigroups). A quasicommutative semigroup S is said to be weakly primary if every ideal of S is weakly primary.

Evidently every completely prime ideal is weakly primary. Also every primary ideal is weakly primary. Conversely, a weakly primary ideal need not be primary. This is shown on the following example.

Let $S_1 = \{0, 4, 6\}$ be the semigroup with the multiplication by mod 12. The all ideals of S_1 are (0), (4), (6), S_1. The primary ideals of S_1 are (4), (6) and S_1. The ideal (0) is not primary since $6.4 \in (0)$, $6 \notin (0)$ but for any positive integer n we have $4^n = 4 \notin (0)$. However, the ideal (0) is weakly primary.

It is known (H. Lal [3, Theorem]) that in a quasicommutative semigroup S with an ideal J the set of all nilpotent elements of S with respect to J (i.e. the set of all $x \in S$ such that $x^n \in J$ for some positive integer n) is equal to the radicals $R(J)$ of Schwarz, $M(J)$ of McCoy, $L(J)$ of Ševrin, $C(J)$ of Luh, $R^*(J)$ of Clifford, respectively. For example the Schwarz radical $R(J)$ is the union of all nilpotent ideals of S with respect to J (i.e. the union of all ideals I of S with $I^n \subseteq J$ for some positive integer n). The McCoy radical $M(J)$ (Luh radical $C(J)$) is the intersection of all prime (completely prime) ideals of S containing J (for definitions see [1] or [8]).

We denote this radical by $R(J)$.

1. Weakly primary ideals

Lemma 1 (N. P. Mukherjee [5, Proposition]). *If for any two elements a, b of a semigroup S $ab=b^r a$ for some positive integer r, then for every positive integer m the following are true: (1) $ab^m=b^{mr}a$. (2) $a^m b=b^{r^m}a^m$. (3) $(ab)^m=b^{r+r^2+\dots+r^m}a^m$. (4) $(ba)^m=b^{1+r+r^2+\dots+r^{m-1}}a^m$.*

Lemma 2 (H. Lal [3, Lemma 1]). *Let S be a quasicommutative semigroup. Then an ideal of S is prime if and only if it is completely prime.*

Lemma 3. *In a quasicommutative semigroup S every left (right) ideal is two-sided.*

Proof. Let $L\subseteq S$ be a left ideal of S, i.e. $SL\subseteq L$. We show that L is also a right ideal of S. Let $a\in L$, $x\in S$ be arbitrary elements. Then for some positive integer r we have $ax=x^r a\in Sa\subseteq SL\subseteq L$, which means that L is a right ideal and therefore a two-sided ideal of S.

Similarly it can be proved for the right ideals of S.

Lemma 4. *If Q is a weakly primary primary ideal of a quasicommutative semigroup S, then the radical $R(Q)$ is a minimal prime ideal containing Q.*

Proof. If $Q\subseteq S$ is a weakly primary ideal of S and $R(Q)=S$, then S is the unique prime ideal containing Q. Let $Q\subset S$ be a weakly primary ideal of S and let $R(Q)\subset S$. It is sufficient to show that $R(Q)$ is a prime ideal of S. Evidently then $R(Q)$ is the minimal prime ideal containing Q.

Assume that for x, $y\in S$ we have $xy\in R(Q)$, $xy=y^r x$ for some positive integer r. Then $(xy)^m\in Q$ for some positive integer m. By Lemma 1 we have that $(xy)^m=y^{r+r^2+\dots+r^m}x^m=y^s x^m\in Q$. Since Q is a weakly primary ideal, from $y^s x^m\in Q$ it follows that either $(y^s)^n=y^{ns}\in Q$ or $(x^m)^t=x^{mt}\in Q$ for some positive integer ns, mt. Therefore $x\in R(Q)$ or $y\in R(Q)$, which means that $R(Q)$ is a prime ideal of S.

For weakly primary ideals the following Lemma 5 holds, which has no analogy for the primary ideals.

Lemma 5. *Let the radical $R(A)$ of an ideal A of a quasicommutative semigroup S be prime. Then A is weakly primary ideal of S.*

Proof. If $A\subseteq R(A)=S$, then the statement holds. If namely for x, $y\in S$, $xy\in A$, then x^m and also y^n belongs to A for some positive integer m, n.

Further, let $A\subseteq R(A)\subset S$, where $R(A)$ is a prime ideal of S. We prove that A is a weakly primary ideal of S. Indirectly:

Assume that $xy \in A$, but $x^m \notin A$, $y^n \notin A$ for all positive integers m, n. Then $x \notin R(A)$, $Y \notin R(A)$ and so $x \in S-R(A)$, $y \in S-R(A)$. By the assumption $R(A)$ is a prime ideal, by Lemma 2 also completely prime and its complement $S-R(A)$ in S is a filter. Then from x, $y \in S-R(A)$ it follows that $xy \in S-R(A)$, a contradiction with the assumption $xy \in A$.

We summarize Lemma 4 and Lemma 5 in the next

Theorem 6. *An ideal A of a quasicommutative semigroup S is weakly primary if and only if its radical $R(A)$ is a prime ideal of S.*

Corollary 7. *Let A and B be weakly primary ideals in a quasicommutative semigroup S. Then also $A \cup B$ is a weakly primary ideal of S.*

Proof. The statement follows immediately from the definition of a weakly primary ideal of S.

Lemma 8. *Let A be a weakly primary ideal, B an ideal of a quasicommutative semigroup S and let $R(A)=R(B)$. Then the ideals AB, BA, $A \cap B$ are weakly primary ideals of S.*

Proof. For every two ideals I, J of an arbitrary semigroup $R(IJ) = R(JI) = R(I \cap J) = R(I) \cap R(J)$ (I. Abrhan [1, Theorem 3]) hold. In our case we have

$$R(AB)=R(BA)=R(A \cap B)=R(A)=R(B).$$

By Theorem 6 we obtain that AB, BA, $A \cap B$ are weakly primary ideals of S.

Lemma 9. *Let P_1 and P_2 be prime ideals of an arbitrary semigroup S. Then $P_1 \cap P_2$ is a prime ideal of S if and only if $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.*

Proof. If $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, then $P_1 \cap P_2 = P_1$ or $P_1 \cap P_2 = P_2$ is a prime ideal of S.

Conversely, suppose that $P_1 \cap P_2$ is a prime ideal of S. Then $P_1P_2 \subseteq P_1 \cap P_2$ implies that either $P_1 \subseteq P_1 \cap P_2$ or $P_2 \subseteq P_1 \cap P_2$. However, $P_1 \cap P_2 \subseteq P_1$ and $P_1 \cap P_2 \subseteq P_2$, hence either $P_1 \cap P_2 = P_1$ or $P_1 \cap P_2 = P_2$. Therefore either $P_1 = P_1 \cap P_2 \subseteq P_2$ or $P_2 = P_1 \cap P_2 \subseteq P_1$ holds.

Remark: If A, B are weakly primary ideals of a quasicommutative semigroup S and if $R(A) \not\subseteq R(B)$ and $R(B) \not\subseteq R(A)$, then the ideals AB, BA, $A \cap B$ are not weakly primary. Namely, from Lemma 9 it follows that the ideal $R(AB) = R(BA) = R(A \cap B) = R(A) \cap R(B)$ is not prime and by Theorem 6 the ideals AB, BA, $A \cap B$ are not weakly primary.

2. Weakly primary semigroups

We recall that a quasicommutative semigroup S is weakly primary if each of its ideals is weakly primary.

The next Theorems 10, 11, 13 are analogous to Theorems 2,1 (i), (ii) and 2,4 of

[6]. For weakly primary quasicommutative semigroups the statement 2,1 (i) admits a converse statement.

Theorem 10. *A quasicommutative semigroup S is weakly primary if and only if the prime ideals of S form a chain under set inclusion.*

Proof. Let the prime ideals of S form a chain under set inclusion. Then for every ideal A of S we have $A \subseteq R(A)$, where $R(A)$ is the intersection of all prime ideals of S containing A. Then $R(A)$ is the minimal prime ideal containing A. By Theorem 6 this means that A is a weakly primary ideal of S.

Conversely, let S be a weakly primary semigroup. Suppose that P_1, P_2 are arbitrary prime ideals of S. The ideal $P_1 \cap P_2$ is weakly primary since S is a weakly primary semigroup. Then by Theorem 6 we have that $R(P_1 \cap P_2) = R(P_1) \cap R(P_2) = P_1 \cap P_2$ is a prime ideal of S. By Lemma 9 this implies that either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ holds. Therefore the prime ideals of S form a chain under set inclusion.

We note that the idempotents of a quasicommutative semigroup S commute, i.e. for any two idempotents e, f of S $ef = fe$. Moreover ef is an idempotent of S.

The natural ordering of the idempotents of S is defined as follows: $e \leqq f$ if $ef = fe = e$.

Theorem 11. *In a quasicommutative weakly primary semigroup S the idempotents form a chain under natural ordering.*

Proof. Every principal ideal $(e) = e \cup eS \cup Se \cup SeS$ of S generated by an idempotent e of S can be written in the form $(e) = eS$. This follows immediately from the fact that for every idempotent e and arbitrary x of S we have $ex = xe$.

For any two idempotents e, f of S the radicals $R(eS)$, $R(fS)$ are prime ideals of S. Then by Theorem 10 either $R(eS) \subseteq R(fS)$ or $R(fS) \subseteq R(eS)$. Let $R(eS) \subseteq R(fS)$. Then $e \in fS$ and so $e = fy$ for some $y \in S$ and from this $fe = f^2y = fy = e$. Since any two idempotents of S commute, we obtain $ef = fe = e$.

Analogously, from $R(fS) \subseteq R(eS)$ it follows that $ef = fe = f$.

Therefore we have either $e \leqq f$ or $f \leqq e$, i.e. the idempotents of S form a chain under natural ordering.

Lemma 12 (H. Lal [3, Lemma 2]). *Let S be a quasicommutative semigroup. Then for any* x, y *in* S, $(x)(y) = (xy)$.

We note that a regular quasicommutative semigroup is an inverse semigroup [2, Theorem 1.17].

Theorem 13. *Let S be a regular quasicommutative semigroup. Then the following statements are equivalent:*

(1) *Every ideal of S is prime.*
(2) *S is a weakly primary semigroup.*
(3) *The idempotents of S form a chain under natural ordering.*
(4) *The principal ideals of S form a chain under set inclusion.*
(5) *The ideals of S form a chain under set inclusion.*

Proof. Evidently from (1) there follows (2) and from (2) by Theorem 11 there follows (3).

We prove that (3) implies (4).

Let (a), (b) be two principal ideals of S. Since S is regular and the idempotents commute with the all elements of S we have $(a)=(e)=eS$, $(b)=(f)=fS$ for some idempotents e, f of S. By the assumption of either $e\leqq f$ or $f\leqq e$ we therefore have either $ef=fe=e$ or $fe=ef=f$.

If $ef=fe=e$, then $eS\subseteq fS$, since for arbitrary $x\in S$ it follows that $ex=fex\in fS$.

Analogously, if $fe=ef=f$, then $fS\subseteq eS$, since for arbitrary $y\in S$ it follows that $fy=efy\in eS$.

Therefore either $(a)=(e)\subseteq(f)=(b)$ or $(b)=(f)\subseteq(e)=(a)$, i.e. the principal ideals of S form a chain under set inclusion.

From (4) there follows (5). This was proved by G. Szász [7, Lemma 3].

We prove that (5) implies (1).

Let A be an arbitrary ideal of S and for x, $y\in S$ let $xy\in A$. We show that either $x\in A$ or $y\in A$.

Since S is regular and the idempotents of S commute with all $x\in S$, we have $(x)=(e)=eS$, $(y)=(f)=fS$ for some idempotents e, $f\in S$. By Lemma 12 we have $(xy)=(x)(y)=(e)(f)=(ef)$. From $xy\in A$ we obtain $(xy)=(ef)\subseteq A$ and so $ef\in A$. The ideals of S by the assumption form a chain under set inclusion, therefore either $(e)=eS\subseteq fS=(f)$ or $(f)=fS\subseteq eS=(e)$. If $(e)\subseteq(f)$, then $e=fx$ for some $x\in S$, therefore $fe=fx=e$. If $(f)\subseteq(e)$, then $f=ey$ for some $y\in S$, therefore $ef=ey=f$. By the preceding we have either $ef=e\in A$ or $ef=f\in A$. This implies that either $(e)=(x)\subseteq A$ or $(f)=(y)\subseteq A$. From this it follows that either $x\in A$ or $y\in A$ holds.

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КВАЗИКОММУТАТИВНЫЕ СЛАБО ПРИМАРНЫЕ ПОЛУГРУППЫ

Франтишек Кметь

Резюме

В статье доказаны следующие утверждения:

1) В квазикоммутативной полугруппе идеал слабо примарный тогда и только тогда, когда его радикал простой идеал.

2) Квазикоммутативная полугруппа слабо примарная тогда и только тогда, когда простые идеалы полугруппы образуют цепь.

F. A. Szász: RADICALS OF RINGS, Akademiai Kiado, Budapest 1981, 287 pages

The ro e of the notion of the radical in the structural ring theory is well-known. The monograph under rev ew is an easily readable introduction to the many facets of this part of the theory of rings. Apart from the knowledge of the basic algebra and habitudes in mathematical reasoning, the book requires no special prerequisites and is therefore recommendable to the neophytes or to non-specialists seeking the orientation amongst various radical concepts or in the variety of their properties

The book consists of six chapters and contains bibliography on over 60 pages. Every chapter ends with a list of open problems. This English version is a slightly adapted translation of the original German one published as Vol. 6 of the series "Disquisitiones Mathematicae Hungaricae". The problems solved after the publication of the German version are marked and moreover listed in an Appendix.

The first chapter (52 pp.) is devoted to the general radical theory (some key words: Amitsur—Kurosh radical, independence of axioms for the Amitsur—Kurosh radical property, upper and lower radicals, hereditary radicals, preradical, Maranda—Michler quasiradical). The second chapter (30 pp.) entitled "Theory of the supernilpotent and special radicals" discusses the most important properties of these two classes of the radicals. Although the nil radicals, the Jacobson radical or the Brown McCoy radical are special instances of the class of special radicals, they are analysed in details in the next three chapters: Chapter III (34 pp.) "nil radicals", Chapter IV (37 pp) 'The Jacobson radical" and Chapter V (32 pp) "The Brown—McCoy radical". The last chapter (22 pp) "Further concrete radicals and zeroid-pseudoradicals" has the following subheadings: 1. The maximal von Neumann regular ideal as a radical; 2 The maximal biregular ideal of a ring, and 3. Zeroid pseudoradicals

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MATHEMATICAL MODELS IN COMPUTER SYSTEMS. Edited by M. Arato and L. Varga, Akadémiai Kiadó, Budapest 1981, 371 pages

The book contains 25 papers selected from lectures presented at the Third Hungarian Computer Science Conference, held in Budapest, Hungary, in January 26—28, 1981.

The aim of the conference was to present most of the results achieved in the field of computer science in Hungary and to provide an opportunity for the exchange of ideas among the researchers and practitioners from 16 countries.

This volume contains invited papers and selected lectures on the following discussed theoretical and