Mathematica Slovaca

Ladislav, Jr. Mišík

On the problem of preserving the class of continuous real functions

*Mathematica Slovaca*, Vol. 35 (1985), No. 1, 91--97

Persistent URL: http://dml.cz/dmlcz/128682

Math. Slovaca 35, 1985, No. 1, 91—97

# ON THE PROBLEM OF PRESERVING THE CLASS OF CONTINUOUS REAL FUNCTIONS

LADISLAV MIŠÍK Jr.

Let $(X, \mathcal{T})$ be a topological space, where $X$ is a set and $\mathcal{T}$ is a system of open sets. Let $C(X, \mathcal{T})$ or shortly $C(\mathcal{T})$ be the set of all continuous real functions on $(X, \mathcal{T})$, $B(X, \mathcal{T})$ or shortly $B(\mathcal{T})$ be the set of all bounded continuous real functions on $(X, \mathcal{T})$. Let $C(X)$, resp. $B(X)$, be the set of all real, resp. of all bounded real functions on the set $X$ and $D(\mathcal{T}) = C(X) - C(\mathcal{T})$. For $A \subset X$ let us denote by $\mathrm{cl}_{\mathcal{T}} A$ the closure of $A$ with respect to the topology $\mathcal{T}$ and $(A, \mathcal{T}/A)$ will mean the subspace $A$ of $X$ with the topology induced by $\mathcal{T}$.

In [3] the author proved that if $(X, \mathcal{T})$ is a compact metric space, then the following holds:

$$\text{if } \mathcal{T}' \supset \mathcal{T}, \text{then } C(X, \mathcal{T}) = C(X, \mathcal{T}') \text{ iff } C(X, \mathcal{T}') \subset B(X) \tag{I}$$

In [5] the author analyses the conditions on the space $(X, \mathcal{T})$ under which the condition (I) holds. She found the sufficient condition which in general is not necessary although in the class of linearly ordered topological spaces it is also necessary. On the other hand, she found the necessary condition which is not sufficient. She proved the following:

**Theorem** (Nonas): *Let $(X, \mathcal{T})$ be a pseudocompact space in which any one-point set is of the type $G\delta$, then $(X, \mathcal{T})$ fulfils* (I). *Let $(X, \mathcal{T})$ fulfil* (I), *then there are no subsets $A, B$ of $X$, such that $A \cup B = X$, $A \cap B = \emptyset$, $A \notin \mathcal{T}$ and $(A, \mathcal{T}|_A)$, $(B, T|_B)$ are star-compact spaces.* (See the definition below).

In [4] the authors raised the question to find a necessary and sufficient condition for the compact space $(X, \mathcal{T})$ to fulfil (I).

In the present paper we shall give some characterizations of comletely regular spaces fulfilling (I). We give the necessary and sufficient conditions in terms of lattices and functions and some sufficient conditions (Theorem 5) which are really stronger than the above sufficient condition of Nonas because they include also the Example 3 of her paper.

First, let us recall some definitions.

**Definition:** *Let $(X, \mathcal{T})$ be a topological space. $A \subset X$ is called a zero-set if there is some $f \in C(\mathcal{T})$ such that $f^{-1}(0) = A$. $(X, \mathcal{T})$ is a $T_{3\frac{1}{2}}$ space if the system of all its zero-sets forms a base for $\mathcal{T}$-closed sets. The topology is $T_1$ if any one-point set is closed. It is completely regular if it is both $T_1$ and $T_{3\frac{1}{2}}$. The topology is star-compact if $C(\mathcal{T}) = B(\mathcal{T})$. It is pseudocompact if it is both completely regular and star-compact.*

The following characterization of pseudocompact spaces is well known [1] and we shall use it in the following.

**Theorem:** *Let $(X, \mathcal{T})$ be completely regular. Then it is pseudocompact iff for any sequence $U_1 \supset U_2 \supset \ldots$ of non-empty $\mathcal{T}$-open sets there is $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathcal{T}} U_i \neq \emptyset$.*

Notation: Let $\mathcal{F} \subset C(X)$. Let us denote by $\mathcal{T}_{\mathcal{F}}$ the smallest topology on $X$ such that all the functions in $\mathcal{F}$ are $\mathcal{T}_{\mathcal{F}}$-continuous. This topology is necessarily $T_{3\frac{1}{2}}$. We shall write $\mathcal{T}_f$ instead of $\mathcal{T}_{\{f\}}$ for $f \in C(X)$. The position of $T_{3\frac{1}{2}}$ topologies is given by:

**Proposition:** *To any topology $T$ on $X$ there is a $T_{3\frac{1}{2}}$ topology $\mathcal{T}'$ such that $C(\mathcal{T}) = C(\mathcal{T}')$, $\mathcal{T}' \subset \mathcal{T}$. This topology is exactly the $\mathcal{T}_{C(\mathcal{T})}$. Any two different $T_{3\frac{1}{2}}$ topologies have different classes of continuous real functions.*

*The lattice of topologies.* The set of all topologies on the set $X$ forms the lattice which is denoted by $\Sigma(X)$. The lattice operations are given by: $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2$, $\mathcal{T}_1 \vee \mathcal{T}_2$ is the topology with the base consisting of sets of the form $U_1 \cap U_2$ where $U_1 \in \mathcal{T}_1$, $U_2 \in \mathcal{T}_2$. Let $P$ be a topological property. We say that the topology $\mathcal{T}$ is $P$-maximal if $\mathcal{T}$ has the property $P$ and if $\mathcal{T}' \supsetneqq \mathcal{T}$, then $\mathcal{T}'$ has not the property $P$.

Now we may characterize the $T_{3\frac{1}{2}}$ topologies fulfilling (I) by their position in $\Sigma(X)$.

**Theorem 1:** *Let $(X, \mathcal{T})$ be a $T_{3\frac{1}{2}}$ topology. Then it fulfils* (I) *iff it is a maximal pseudocompact topology.*

Proof: The sufficiency: Let $(X, \mathcal{T})$ be maximal pseudocompact and $\mathcal{T}'$ be a topology on $X$ (not necessarily $T_{3\frac{1}{2}}$) such that $\mathcal{T}' \supset \mathcal{T}$. Now if $C(\mathcal{T}') = C(\mathcal{T})$, then $C(\mathcal{T}') \subset B(\mathcal{T})$ since $\mathcal{T}$ is pseudocompact. On the other hand if $C(\mathcal{T}') \supsetneqq C(\mathcal{T})$, then $\mathcal{T}_{C(\mathcal{T}')}$ is $T_{3\frac{1}{2}}$ topology and there holds $\mathcal{T}_{C(T')} \supsetneqq \mathcal{T}$, and so $\mathcal{T}_{C(\mathcal{T}')}$ is not pseudocompact. Since $\mathcal{T}_{C(\mathcal{T}')}$ is $T_1$ and $T_{3\frac{1}{2}}$ it may not be star-compact and so $C(\mathcal{T}') = C(\mathcal{T}_{C(\mathcal{T}')}) \not\subset B(X)$. Then $(X, \mathcal{T})$ fulfils (I). The necessity: Let $\mathcal{T}$ be $T_{3\frac{1}{2}}$ on $X$ fulfilling (I). Then $\mathcal{T}$ is star-compact. Further $\mathcal{T}$ is maximal $T_{3\frac{1}{2}}$, star-compact, for if $\mathcal{T}'$ is $T_{3\frac{1}{2}}$, star-compact and $\mathcal{T}' \supsetneqq \mathcal{T}$, then $C(\mathcal{T}') \supsetneqq C(\mathcal{T})$ but $C(\mathcal{T}') \subset B(X)$, which would be a contradiction to (I). Now it suffices to show that $\mathcal{T}$ is a $T_1$ topology to end the proof. This shows the following lemma.

**Lemma 1:** *Let $\mathcal{T}$ be maximal $T_{3\frac{1}{2}}$, star-compact topology. Then it is $T_1$ and so maximal pseudocompact.*

Proof: Let $\mathcal{T}$ be $T_{3\frac{1}{2}}$, star-compact but not $T_1$. Let $x \in X$ be such that $\{x\}$ is not $\mathscr{T}$-closed, and so the set $A = \mathrm{cl}_{\mathscr{T}}\{x\} - \{x\}$ is non-empty. Since $\mathscr{T}$ is $T_{3\frac{1}{2}}$, $\{x\}$ is not open because in $T_{3\frac{1}{2}}$ spaces any open set is the union of cozero-sets (i.e. complements of zero-sets) and assuming $\{x\}$ to be a cozero-set we get its closedness. Now put $Y = X - \{x\}$. Let $a$, $b \in A$ and $U_a$, $U_b$ be their $\mathscr{T}$-open neighbourhoods in $X$. Then $U_a \cap U_b$ is a non-empty $\mathscr{T}$-open set and so $U_a \cap U_b \cap Y \neq \emptyset$ and $a$ and $b$ have no disjoint neighbourhoods in $(Y, \mathscr{T}_{/Y})$. Therefore any $f \in C(Y, \mathscr{T}_{/Y})$ is constant on $A$ and it may be extended to $\bar{f} \in C(X, \mathscr{T})$ by putting $\bar{f}(x) = f(a)$ for $a \in A$. Thus $(Y, \mathscr{T}_{/Y})$ is star-compact. Now put $\mathscr{T}' = \mathscr{T} \vee \mathscr{T}_g$ where $g \in C(X)$ and $g(x) = 0$, $g(Y) = \{1\}$. Then $\mathscr{T}'$ is $T_{3\frac{1}{2}}$, further $\mathscr{T}'_{/Y} = \mathscr{T}_{/Y}$, hence the space $(X, \mathscr{T}')$ is the sum of two star-compact spaces $(Y, \mathscr{T}'_{/Y})$ and $(\{x\}, \mathscr{T}'_{/\{x\}})$ and is star-compact too. Finitely we have $\mathscr{T}' \supsetneqq \mathscr{T}$ and so $\mathscr{T}$ is not maximal $T_{3\frac{1}{2}}$, star-compact.

Remark. Theorem 1 gives no internal characterization (only by topological concepts) of spaces fulfilling (I). It should be noted that the internal characterization of maximal star-compact spaces (called pseudocompact although they need not fulfil any separation axiom) is an unsolved problem [2].

Theorem 1 has an interesting consequence. Combining Theorem 1 with the results in [6] we get:

**Theorem 2:** *Any $T_{3\frac{1}{2}}$ space $(X, \mathscr{T})$ fulfilling* (I) *(especially any pseudocompact space with any one-point set of the type $G_\delta$) has the following property: If $h: (X, \mathscr{T}) \to (X, \mathscr{T})$ is a continuous bijection, then $h$ is a homeomorphism.*

Now let $(X, \mathscr{T})$ be a completely regular space. Let us consider the following partial quasi-order on the set $D(\mathscr{T})$ of all $\mathscr{T}$-discontinuous functions: For $f, g \in D(\mathscr{T})$ define $f \underset{\mathscr{T}}{\leqq} g$ iff $\mathscr{T} \vee \mathscr{T}_f \subset \mathscr{T} \vee \mathscr{T}_g$. Call the subset $A$ of the partially quasi-ordered set $(P, \leqq)$ down-cofinal if for any $p \in P$ there is some $a \in A$ such that $a \leqq p$.

It may be easily seen that if $f \in C(\mathscr{T})$, $g \in D(\mathscr{T})$, then $f + g \underset{\mathscr{T}}{\leqq} g$, if $f \cdot g \in D(\mathscr{T})$, then $f \cdot g \underset{\mathscr{T}}{\leqq} g$ and if $h$ is a continuous real function of a real variable and $h \circ g \in D(\mathscr{T})$, then $h \circ g \underset{\mathscr{T}}{\leqq} g$. This implies that if $\mathscr{U}$ is any open covering of $X$, then the set $\mathscr{F}(\mathscr{U}) = \{f \in D(\mathscr{T}),\ f(X) \subset \langle 0, 1 \rangle\}$ and there is some $U \in \mathscr{U}$ such that $f(X - U) = \{0\}$ is an example of a down-cofinal subset of the partially ordered set $(D(\mathscr{T}), \underset{\mathscr{T}}{\leqq})$.

The following theorem is an immediate consequence of the fact that a smaller topology than a star-compact topology is star-compact too.

**Theorem 3:** *Let $(X, \mathcal{T})$ be pseudocompact. Then $(X, \mathcal{T})$ is maximal pseudocompact iff there is some down-cofinal subset $\mathcal{F}$ of $(D(\mathcal{T}), \underset{\mathcal{T}}{\leqq})$ such that for any $f \in F$ $\mathcal{T} \vee \mathcal{T}_f$ is not pseudocompact.*

The following lemma restrict the set of spaces which may be maximal pseudocompact.

**Lemma 2:** *Let $(X, \mathcal{T})$ be maximal pseudocompact. Then it fulfils the following condition (C): For any $x \in X$ there is some sequence $U_1 \supset U_2 \supset \ldots$ of non-empty $\mathcal{T}$-open sets with $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathcal{T}} U_i = \{x\}$.*

Proof: Let $(X, \mathcal{T})$ be maximal pseudocompact and $x \in X$. There is some unbounded $f \in C(X - \{x\}/X - \{x\})$ for otherwise $X$ would be the union of two disjoint pseudocompact subsets, which is impossible by the necessary condition of Nonas [5]. Put $U_n = (X - \{x\}) - f^{-1}\langle -n, n\rangle$. Then for each $n \in U_n$ is a non-empty $\mathcal{T}$-open set in $X$, so we have $\bigcap_{n=1}^{\infty} \mathrm{cl}_{\mathcal{T}} U_n \neq \emptyset$. But this intersection is empty in $X - \{x\}$, and so $\{x\} = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\mathcal{T}} U_n$.

Now let us consider what it means that $\mathcal{T} \vee \mathcal{T}_f$ is not pseudocompact for a pseudocompact $\mathcal{T}$ which fulfils (C) and $f \in D(\mathcal{T}) \cap B(X)$. Let us denote by $\bar{A}$ the closure of the subset $A$ of the set of real numbers with the usual topology.

**Lemma 3:** *Let $(X, \mathcal{T})$ be a pseudocompact space in which (C) holds and $f \in D(\mathcal{T}) \cap B(X)$. Then $\mathcal{T} \vee \mathcal{T}_f$ is not pseudocompact iff there is a decreasing sequence $U_1 \supset U_2 \supset \ldots$ of non-empty $\mathcal{T}$-open sets with $\bigcap_{i=1}^{\infty} \overline{f(U_i)} \not\subset f\left(\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathcal{T}} U_i\right)$.*

Proof: $\Rightarrow$ We shall distinguish two cases.

a) Let $f$ be quasicontinuous, i.e. for any $x \in X$ and any $\varepsilon > 0$ $x \in \mathrm{cl}_{\mathcal{T}}(\mathrm{int}_{\mathcal{T}} f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon))$. Let $A_1 \supset A_2 \supset \ldots$ be a sequence of non-empty $\mathcal{T} \vee \mathcal{T}_f$-open sets with $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathcal{T} \vee \mathcal{T}_f} A_i = \emptyset$. We may suppose that $A_i$-s are of the form $A_i = V_i \cap f^{-1}(a_i, b_i)$ where $V_1 \supset V_2 \supset \ldots$ are $\mathcal{T}$-open and $a_1 \leqq a_2 \leqq \ldots \leqq b_2 \leqq b_1$ are real numbers. Now we wish to make the sets $A_i$ smaller. Choose some sequence $x_i \in A_i$ such that the sequence $f(x_1)$, $f(x_2)$, ... converges to some real number $r \in \bigcap_{i=1}^{\infty} \langle a_i, b_i \rangle$. Let $n_1 < n_2 < \ldots$ be some sequence of natural numbers such that for any $i$ there is

$$\left(f(x_i) - \frac{1}{n_i}, f(x_i) + \frac{1}{n_i}\right) \subset (a_i, b_i)$$

and put

$$U_i = V_i \cap \mathrm{int}_{\mathcal{T}} f^{-1}\left(f(x_i) - \frac{1}{n_i}, f(x_i) + \frac{1}{n_i}\right).$$

Since $f$ is quasicontinuous for any $i \in U_i$ is a non-empty $\mathscr{T}$-open set and $U_i \subset A_i$. This implies that $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i \neq \emptyset$ and $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T} \vee \mathscr{T}_f} U_i = \emptyset$. Now suppose that there is some $p \in \bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i$ with $f(p) = r$. Let $N_p \cap f^{-1}(r - \varepsilon, r + \varepsilon)$ be an arbitrary $\mathscr{T} \vee \mathscr{T}_f$-neighbourhood of $p$ where $N_p$ is some $\mathscr{T}$-neighbourhood of $p$ and $\varepsilon > 0$. Then $N_p \cap f^{-1}(r - \varepsilon, r + \varepsilon) \cap U_i \neq \emptyset$ for any $i$ and this implies that $p \in \bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T} \vee \mathscr{T}_f} U_i$, which is impossible. Thus

$$\{r\} = \bigcap_{i=1}^{\infty} \overline{f(U_i)} \not\subset f\left(\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i\right).$$

b) Let $f$ be not quasicontinuous and $x \in X$ and $\varepsilon > 0$ are such that $x \notin \mathrm{cl}_t(\mathrm{int}_{\mathscr{T}} f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon))$. Let $U_1 \supset U_2 \supset \dots$ be the sequence of $\mathscr{T}$-open sets with $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i = \{x\}$. Then there is some sequence $x_1, x_2, \dots$ of points with $x_i \in U_i$ and $f(x_i) \notin (f(x) - \varepsilon, f(x) + \varepsilon)$. Let $r$ be some accumulation point of the sequence $f(x_1), f(x_2), \dots$. Then $r \in \bigcap_{i=1}^{\infty} \overline{f(U_i)}$ and $r \notin f\left(\bigcap_{i=1}^{\infty} \mathrm{cl}_t U_i\right) = \{f(x)\}$.

$\Leftarrow$ Choose some real $r \in \bigcap_{i=1}^{\infty} \overline{f(U_i)} - f\left(\bigcap_{i=1}^{\infty} \mathrm{cl}_t U_i\right)$ and put $V_i = U_i \cap f^{-1}\left(r - \frac{1}{i}, r + \frac{1}{i}\right)$. Then for any $i \in V_i$ is a non-empty $\mathscr{T} \vee \mathscr{T}_f$-open set and $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T} \vee \mathscr{T}_f} V_i = \emptyset$, so that $\mathscr{T} \vee \mathscr{T}_f$ is not pseudocompact.

**Theorem 4:** *Let $(X, \mathscr{T})$ be pseudocompact and $\mathscr{F}$ some downcofinal subset of $(D(\mathscr{T}), \underset{\mathscr{T}}{\leqq})$. Then $(X, \mathscr{T})$ fulfils* (I) *iff it fulfils* (C) *and if for any $f \in \mathscr{F}$ there is some sequence $U_1 \supset U_2 \supset \dots$ of non-empty $\mathscr{T}$-open sets with $\bigcap_{i=1}^{\infty} \overline{f(U_i)} \not\subset f\left(\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i\right)$.*

Proof: It is immediate consequence of Theorem 1, Theorem 3 and Lemma 3.

Although this theorem has not much practical application in general, it implies some sufficient conditions. We present here two of them. Recall that the regular closed set is the closure of some open set.

**Theorem 5:** *The pseudocompact space $(X, \mathscr{T})$ fulfils* (I) *if any of the following two conditions holds:*

a) *Let $A$ be an open subset of $X$ and $a \in \mathrm{cl}_{\mathscr{T}} A$. Then there is a sequence $A \supset U_1 \supset U_2 \supset \dots$ of non-empty $\mathscr{T}$-open sets with $i \bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i = \{a\}$.*

b) *Any point in X has some regular closed neighbourhood which is maximal pseudocompact.*

Proof: a) Let $f \in D(\mathscr{T}) \cap B(X)$ and $A \subset X$, $a \in X$ are such that $a \in \mathrm{cl}_{\mathscr{T}} A$ and $f(a) \notin \overline{f(A)}$. Then there is some $g \in D(\mathscr{T}) \cap B(X)$ with $g \underset{\mathscr{T}}{\leqq} f$ and $g(a) = 2$, $g(A) = \{0\}$ and $g(X) \subset \langle 0, 2 \rangle$. It suffices to show that $\mathscr{T} \vee \mathscr{T}_g$ is not pseudocompact. If there is some $q \in (0, 2)$ for which $a \in \mathrm{cl}_{\mathscr{T}}(\mathrm{int}_{\mathscr{T}} g^{-1} \langle 0, q))$, then there is some sequence $\mathrm{int}_{\mathscr{T}} g^{-1} \langle 0, q) \supset U_1 \supset U_2 \supset \ldots$ of $\mathscr{T}$-open sets with $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i = \{a\}$, which implies $\bigcap_{i=1}^{\infty} \overline{g(U_i)} \not\subset g\left(\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i\right) = \{g(a)\} = \{2\}$ and $\mathscr{T} \vee \mathscr{T}_g$ is not pseudocompact by Theorem 4. On the other hand, let there be for any $q \in (0, 2)$ some neighbourhood $N_q$ of $a$ such that $(\mathrm{int}_{\mathscr{T}} g^{-1} \langle 0, q)) \cap N_q = \emptyset$. Choose some $x \in A \cap N_1$ with $\{x\} \notin \mathscr{T}$. Then there is some sequence $N_1 - \{x\} \supset V_1 \supset V_2 \supset \ldots$ of $\mathscr{T}$-open sets with $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} V_i = \{x\}$. Since for any $i$ there is $g(V_i) \cap \langle 1, 2 \rangle \neq \emptyset$, we have $\bigcap_{i=1}^{\infty} \overline{g(V_i)} \not\subset g\left(\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} V_i\right) = \{g(x)\} = \{0\}$ so that $\mathscr{T} \vee \mathscr{T}_g$ is not pseudocompact by Theorem 4.

b) Let $f \in D(\mathscr{T}) \cap B(X)$, $x$ be some point of discontinuity of $f$, $\mathrm{cl}_{\mathscr{T}} U$ be the regular closed neighbourhood of $x$ which is maximal pseudocompact. Let $V$ be some open neighbourhood of $x$ with $\mathrm{cl}_{\mathscr{T}} V \subset U$ and $g \in C(\mathscr{T})$ be such that $g(x) = 1$ and $g(X - V) = \{0\}$. Now let us consider the space $\mathrm{cl}_{\mathscr{T}} U$ with the topology induced by $\mathscr{T} \vee \mathscr{T}_{f \cdot g}$. Since it is not pseudocompact there is some sequence $V \supset A_1 \supset A_2 \supset \ldots$ of sets open in this space and the intersection of their closures in this space is empty. But these sets are open also in $(X, \mathscr{T} \vee \mathscr{T}_{f \cdot g})$ and the intersection of their closures is empty in the last space too. Thus $(X, \mathscr{T} \vee \mathscr{T}_{f \cdot g})$ is not pseudocompact and since $f \cdot g \underset{\mathscr{T}}{\leqq} f$, applying Theorem 4, we get the maximal pseudocompactness of $(X, \mathscr{T})$.

Question: Are the conditions of Theorem 5 also necessary for maximal pseudocompactness? Notice that the necessity of a) is equivalent to the necessity of b).

The following theorem gives the partial answer to this question.

**Theorem 6:** *Let $(X, \mathscr{T})$ be maximal pseudocompact space, $A \subset X$ be open and $a \in \mathrm{cl}_{\mathscr{T}} A$. Let further $(X - \{a\}, \mathscr{T}/X - \{a\})$ be a normal space and there exists some $B \subset A$ with $\mathrm{cl}_{\mathscr{T}} B = B \cup \{a\}$. Then there is some sequence $A \supset U_1 \supset U_2 \supset \ldots$ of non-empty $\mathscr{T}$-open sets with $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i = \{a\}$.*

Proof: If $a \in A$, then apply Lemma 2. Let $a \in \mathrm{cl}_{\mathscr{T}} A - A$, then the normality of $(X - \{a\}, \mathscr{T}/X - \{a\})$ implies the existence of some $f \in C(X - \{a\}, \mathscr{T}/X - \{a\})$ with $f(B) = 1$ and $f((X - \{a\}) - A) = \{0\}$. Extend $f$ to $g: X \to R$ defining $g(a) = 0$. Then $a$ is the only point of discontinuity of $g$. Since $\mathscr{T} \vee \mathscr{T}_g$ is not pseudocompact, there is some sequence $V_1 \supset V_2 \supset \ldots$ of non-empty $\mathscr{T}$-open sets with $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T} \vee \mathscr{T}_g} V_i = \emptyset$ and for any $n$ $V_n \cap A \neq \emptyset$, since $g$ is continuous on $X - A$. Put $U_n = V_n \cap A$. Then $\bigcap_{i=1}^{\infty} \mathrm{cl}_{\mathscr{T}} U_i = \{a\}$.

## REFERENCES


[1] ENGELKING, R.: General Topology, Polish Scientific Publishers, Warszawa 1977.

[2] GRANT, D. L.: MR 80j 54017, review to Cameron, D. E.: A note on maximal pseudocompactnes-s. Bull. Malaysian Math. Soc. (2) 2, 1979, 45—46.

[3] KOCELA, E.: Properties of some generalizations of the notion continuity of a function. Fund. Math. 78, 1973, 133—139.

[4] KOSZELA, B., SWIATKOWSKI, T., WILCZYNSKI, W.: Classes of continuous functions. Real analysis exchange 4, 1978—1979, 139—157.

[5] NONAS, H.: Stronger topologies preserving the class of continuous functions. Fund. Math. 101, 1978, 121—127.

[6] RAJAGOPALAN, M., WILANSKY, A.: Reversible topological spaces. J. Austral. Math. Soc. 6, 1966, 129—138.



Received September 8, 1982



*Matematický ústav SAV*
*Obrancov mieru 49*
*814 73 Bratislava*


## О ПРОБЛЕМЕ СОХРАНЕНИЯ КЛАССА НЕПРЕРЫВНЫХ ВЕЩЕСТВЕННЫХ ФУНКЦИЙ


Ladislav Mišík Jr.


### Резюме


В этой работе автор занимается вполне регулярными пространствами $(X, \mathscr{T})$ имеющими следующее свойство: Если $\mathscr{T} \subset \mathscr{T}'$-топологии на множестве $X$, то $C(X, \mathscr{T}) = C(X, \mathscr{T}')$ тогда и только тогда, когда все функции из $C(X, \mathscr{T}')$ ограничены. Здесь $C(X, \mathscr{T})$ означает множество всех непрерывных функций на пространстве $X$, $\mathscr{T}$).

В работе показано, что это свойство эквивалентно максимальной псевдокомпактности этого пространства. Приводится необходимое и достаточное условие в терминах функций. Более того, приводится несколько новых достаточных условий.