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TWO HEURISTICS FOR THE ABSOLUTE p-CENTER PROBLEM IN GRAPHS

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1. Introduction

Given a connected graph G (finite, undirected, without loops and multiple edges), we denote by $V(G)$ and $E(G)$ the vertex and edge sets, respectively; also we put $n := |V(G)|$ and $m := |E(G)|$. It is supposed that each vertex $v \in V(G)$ is assigned a nonnegative real number $w(v)$, called the *weight* of v, and each edge $e \in E(G)$ is assigned a positive real number $a(e)$, called the *length* of e. For any two vertices $u, v \in V(G)$, $d(u, v)$ is the minimal sum of the edge lengths of a $u - v$ path and is called the distance between u and v. This definition can be extended also to the case when u and v are any two points of a geometric representation of G (the edges are considered as simple geometric curves with the corresponding lengths). The distance between a vertex $v \in V(G)$ and a point set X of G is $d(v, X) := \min\{d(v, x) | x \in X\}$. A p-set is a set of cardinality p.

Given G and p, the *absolute p-center problem* is to find a p-set X of G such that the objective function, the *weighted eccentricity* of X,

$$\eta(X) := \max_{v \in V(G)} \{d(v, X)\, w(v)\}$$

is minimized. An *absolute p-center* is any optimal p-set X. The optimal value of $\eta(X)$ is called the *absolute p-radius*. If the stronger constraint $X \subset V(G)$ is required, then the problem is referred to as the *p-center* (or vertex p-center) *problem*. The corresponding notions are a *p-center* and the *p-radius*.

We can suppose that $d(u, v) = a(uv)$ for any edge uv, because otherwise the edge uv could be deleted without affecting the optimal weighted eccentricity of a p-set. Further, it will be assumed that the distance matrix (with entries $d(u, v)$ for all $u, v \in V(G)$) is available.

Since the appearance of Hakimi's seminal paper [4] in 1964, the literature on network location problems has grown rapidly. At present, there are about one hundred papers concerning p-centers or absolute p-centers (e.g. see [1, 8, 9, 12, 13, 14].

While both problems are polynomially solvable if p is fixed (see e.g. [8]), they are NP-hard in general, even in very special cases [3, 7, 8, 10]. Moreover, the corresponding ϱ-approximation problems are NP-hard whenever $\varrho < 2$ [7, 10] (ϱ means a worst-case error ratio). On the other hand, there are 2-approximation polynomial algorithms for these problems and clearly, they are best possible, unless P = NP. For special cases of the p-center problem see [2, 5, 6] and for the general case see [11] where we developed a 2-approximation $O(n^2 \log n)$ algorithm, called CENTER, for the p-center problem and a 2-approximation $O(mn^2 \log n)$ algorithm, called ABCENTER, for the absolute p-center problem.

The aim of this paper is to give two faster heuristics for the absolute p-center problem. In Section 2 we approximate an absolute p-center by a p-center in a graph obtained by introducing $k - 1$ new vertices into each edge. This yields a $(2 + 2/k)$-approximation $O(kmn \log kmn)$ algorithm. In Section 3 we modify CENTER (from [11]) which results in a 2-approximation $O(n^2 \log n)$ algorithm for the absolute p-center problem. This paper strongly depends on our previous paper [11] and the reader should consult it.

2. A subdivision approach

Let $k \geqslant 1$ be a given integer. To approximate an absolute p-center of a graph G, each edge $e \in E(G)$ is subdivided into k new edges of length $a(e)/k$ by inserting $k - 1$ new vertices of weight zero, where $a(e)$ is the length of e. The resulting graph is denoted by $G^{(k)}$. Our heuristic is based on the following result; the special case $k = 1$ was proved in [11].

Theorem 1. *For any absolute p-center A of G and any p-center C of $G^{(k)}$, we have*

$$\eta(A) \leqslant \eta(C) \leqslant \left(1 + \frac{1}{k}\right) \eta(A).$$

Moreover, these bounds are best possible.

Proof. The left inequality and its tightness are trivial. The right inequality becomes equality e.g. if G has only one edge uv with length k, $w(u) = w(v) = 1$ and $p = 1$. If k is an odd integer, then $\eta(A) = k/2$ while $\eta(C) = (k + 1)/2$. Thus it remains to prove the right inequality.

Let $x_1, \ldots, x_p$ be the points of A. We will show that any point $x \in A$ can be replaced by a suitable vertex of $G^{(k)}$ without changing weighted eccentricity $\eta(A)$ too much. We can assume that in every edge $uv \in E(G)$ there is at most one point $x \in A$ lying strictly between u and v (otherwise the closest points to u or v can be replaced by u or v, respectively, and the other points can be deleted without increasing $\eta(A)$) and if u, or v, or both belong to A, then there is no other point of A lying on uv (otherwise, such a point can be replaced by v, or u, or deleted, respectively, without increasing $\eta(A)$).

Now we are going to show that if every point $x \in A$, which is an internal point of an edge $u'v'$ of $G^{(k)}$, is replaced by u' or v' (properly chosen), then $\eta(A)$ can increase at most $1 + 1/k$ times. All the vertices of G are contained in p subsets, "regions", $S_{x_1}, \ldots, S_{x_p}$ such that for S_{x_i} the point x_i is an absolute 1-center with weighted eccentricity at most $\eta(A)$ (i.e., $S_{x_i} := \{v \in V(G) | d(x_i, v)\, w(v) \leqslant \eta(A)\}$). Clearly, any two distinct points of A can be handled separately and thus we can confine ourselves to one point $x \in A$. Let x be an internal point of an edge $u'v'$ in G such that its ux section contains u'. The region S_x can be decomposed into two sets T_u and T_v, where a vertex $y \in V(G)$ belongs to T_u iff a shortest $x - y$ path contains u; the other vertices of S_x belong to T_v. If the region S_x cannot be covered in $G^{(k)}$ by either u' or v' without exceeding weighted eccentricity $(1 + 1/k)\, \eta(A)$, then there are vertices $u_1 \in T_u$ and $v_1 \in T_v$ such that

$$d(u', v_1)\, w(v_1) > (1 + 1/k)\, \eta(A) \tag{1}$$

$$d(v', u_2)\, w(u_1) > (1 + 1/k)\, \eta(A) \tag{2}$$

(because all the new vertices have weight zero). Since the triangle inequality holds, inequality (1) yields

$$[d(u', x) + d(x, v_1)]\, w(v_1) > (1 + 1/k)\, \eta(A) \geqslant$$
$$\geqslant (1 + 1/k)\, d(x, v_1)\, w(v_1).$$

Thus

$$d(u', x) > d(x, v_1)/k. \tag{3}$$

Fully analogously, (2) yields

$$d(v', x) > d(x, u_1)/k. \tag{4}$$

Summing up (3) and (4), we obtain

$$kd(u', v') > d(u_1, x) + d(x, v_1). \tag{5}$$

Clearly, $kd(u', v') = d(u, v) = a(uv)$ but $u_1 \in T_u$ and $v_1 \in T_v$. Therefore $d(u_1, x) + d(x, v_1) \geqslant d(u, v)$ and (5) gives a contradiction. ■

Now, given G and k, we can suggest the following approximation algorithm for the absolute p-center problem.

Heuristic SUBDIVISION

Step 1. Construct the $n[n - 1 + (k - 1)m]$-multiset D of non-null weighted distances in $G^{(k)}$.

Step 2. Apply the heuristic CENTER [11] to $G^{(k)}$ (to the multiset D) and output the obtained p-set B of vertices of $G^{(k)}$ as a p-set of points of G and end.

Since it is assumed that the distance matrix of G is available and that $n \leqslant O(m)$, Step 1 can be performed in time $O(kmn)$. Thus (see [11]) Step 2 is of complexity $O(kmn \log kmn)$, which is the overall complexity of SUBDIVISION.

As CENTER is a 2-approximation algorithm, Theorem 1 implies that for any p-center C of $G^{(k)}$ and any absolute p-center A of G, we have $\eta(B) \leqslant \leqslant 2\eta(C) \leqslant (2 + 2/k)\,\eta(A)$. Thus SUBDIVISION is a $(2 + 2/k)$-approximation $O(kmn \log kmn)$ algorithm for the absolute p-center problem.

Clearly, for $k \to \infty$ SUBDIVISION runs to a 2-approximation algorithm but then the complexity of SUBDIVISION will be rather large when compared to $O(mn^2 \log n)$ of ABCENTER [11]. Thus SUBDIVISION is recommended to use for small k and sparse graphs (e.g. if $m \leqslant O(n)$).

Note that instead of CENTER one can use in SUBDIVISION also the heuristic PROXICENTER which will be developed in the next section.

2. A common 2-approximation algorithm

In this section we develop a heuristic like CENTER [11] which works for both the p-center problem and the absolute p-center one.

Theorem 2. *For any real number $r > 0$, if there exists a p-set X of points of G with $\eta(X) \leqslant r$, then there exists a weighted distance $R \leqslant 2r$ between two vertices of G such that the following procedure finds a set $S \subset V(G)$ with $|S| \leqslant p$ and $\eta(S) \leqslant R$.*

Procedure DISTRICT

Step 0. At first all vertices of G are unlabelled; $S := \emptyset$.

Step 1. If all vertices are labelled, then go to Step 2. Else choose an unlabelled vertex u of the maximum weight and put $S := S \cup \{u\}$; label the vertex u and every unlabelled vertex v such that $w(v)\,d(u, v) \leqslant R$; go to Step 1.

Step 2. Output S.

Proof. Let X consist of points $x_1, x_2, \ldots, x_p$ and let "the regions" corresponding to these points be $S_1, S_2, \ldots, S_p$, respectively (i.e. $S_1 \cup \ldots \cup S_p = V(G)$ and for every $i = 1, \ldots, p$, we have $w(v)\,d(x_i, v) \leqslant r$ whenever $v \in S_i$). Let

$$R := \max\{d(u, v)\,w(v) | d(u, v)\,w(v) \leqslant 2r;\ u, v \in V(G)\}.$$

By Step 1, we have $w(v)\,d(S, v) \leqslant R$ for any $v \in V(G)$ and hence $\eta(S) \leqslant R$. To prove that $|S| \leqslant p$ we will show that at most one vertex of each S_i belongs to S. Let us consider an iteration of Step 1. Let u be the chosen vertex and let $u \in S_i$ (posibly, there are several such sets). Then for every unlabelled vertex v of S_i we have $w(v) \leqslant w(u)$ and the triangle inequality gives

$$w(v)\, d(u, v) \leqslant w(v)\,[d(u, x_i) + d(x_i, v)] \leqslant$$
$$\leqslant w(u)\, d(u, x_i) + w(v)\, d(x_i, v) \leqslant 2r.$$

According to the definition of R, we see that $w(v)\, d(u, v) \leqslant R$. Therefore one must label all the unlabelled vertices of S_i and thus no other vertex than u will be added to S. ■

Now we can give the following heuristic for both the p-center problem and the absolute p-center one.

Heuristic PROXICENTER

Step 1. Arrange the $n(n-1)$-multiset of weighted distances $d(u, v)\, w(v)$ with $u, v \in V(G)$ into a non-decreasing sequence and deleting duplicates reduce it to an increasing sequence

$$f_1 < f_2 < \ldots < f_q. \tag{6}$$

Step 2. Find R^*, the least value of $R \in \{f_1, \ldots, f_q\}$ for which DISTRICT yields an output S with $|S| \leqslant p$.

Step 3. Augment S arbitrarily to a set S' of p vertices. Output S' and end.

Formally, PROXICENTER is the same as CENTER from [11]. Thus the complexity of PROXICENTER is $O(n^2 \log n)$.

According to Theorem 2 we have $\eta(S') \leqslant \eta(S) \leqslant R^* \leqslant 2r^*$, where r^* is the absolute p-radius of G. Hence PROXICENTER is a 2-approximation strongly polynomial algorithm for the absolute p-center problem (and simultaneously for the p-center problem).

Note that PROXICENTER is of a lower complexity than ABCENTER from [11] (its complexity is $O(mn^2 \log n)$). Although in a worst case, the error ratio of approximations is the same, one can see that in some cases PROXICENTER provides better results than ABCENTER or CENTER (because it may be that $R^* < 2r^*$).

We also note that PROXICENTER is a best polynomial heuristic as to the error ratio in a worst case because the ϱ-approximation absolute (or vertex) p-center problem is NP-hard whenever $\varrho < 2$ (see [10] or [7, 10], respectively). Nevertheless, we have the following result. First we need a definition.

Given a real number b with $1 \leqslant b \leqslant 2$, $\mathscr{P}_b$ denotes the class of all instances of the p-center problem such that the (vertex) p-radius η_C and the absolute p-radius η_A fulfil the inequality

$$\eta_C \geqslant b\eta_A .$$

(It is well known [11] that always $\eta_A \leqslant \eta_C \leqslant 2\eta_A$.)

Theorem 3. *For any class $\mathscr{P}_b$ of p-center problems* PROXICENTER *is a $(2/b)$-approximation algorithm.*

Proof. Let us consider an instance of the p-center problem from $\mathscr{P}_b$. Let η_C and η_A be its p-radius and the absolute p-radius of the corresponding absolute p-center problem, respectively. PROXICENTER provides a p-set S' of vertices with $\eta(S') \leqslant 2\eta_A$. Since $\eta_A \leqslant \eta_C/b$, we have $\eta(S') \leqslant (2/b)\,\eta_C$, as desired. ■

Consequently, we see that in the class $\mathscr{P}_2$ PROXICENTER provides an exact solution of the p-center problem. We must admit, however, that we are unable to find out quickly whether or not a given instance belongs to a class $\mathscr{P}_b$. Therefore Theorem 3 seems to be interesting from the theoretical view-point only.

Remark. Although PROXICENTER seems to be a superior heuristic, ABCENTER [11] or SUBDIVISION can be combined with other heuristics (e.g. the interchange heuristic [12]) and thus can give better results because they can output also points different from vertices, while PROXICENTER always yields only vertex p-sets.

REFERENCES

[1] CHRISTOFIDES, N.: Graph Theory: an Algorithmic Approach. Academic Press, London, 1975.

[2] DYER, M. E.—FRIEZE, A. M.: A simple heuristic for the p-centre problem. J. Oper. Res. Soc. 35, 1984, 285—288.

[3] GAREY, M. R.—JOHNSON, D. S.: Computers and Intractability: a Guide to the Theory of NP-Completeness. Freeman, San Francisco, 1979.

[4] HAKIMI, S. L.: Optimum locations of switching centers and the absolute centers and medians of a graph. Operations Res. 12, 1964, 450—459.

[5] HOCHBAUM, D. S.—SHMOYS, D. B.: A best possible heuristic for the k-center problem. Math. Oper. Res. 10, 1985, 180—184.

[6] HOCHBAUM, D. S.—SHMOYS, D. B.: A unified approach to approximation algorithms for bottleneck problems. J. Assoc. Comput. Mach. 33, 1986, 533—550.

[7] HSU, W. L.—NEMHAUSER, G. L.: Easy and hard bottleneck location problems. Discrete Appl. Math. 1, 1979, 209—215.

[8] KARIV, O.—HAKIMI, S. L.: An algorithmic approach to network location problems I: the p-centers. SIAM J. Appl. Math. 37, 1979, 513—538.

[9] MINIEKA, E.: The centers and medians of a graph. Operations Res. 25, 1977, 641—650.

[10] PLESNÍK, J.: On the computational complexity of centers locating in a graph. Aplikace Mat. 25, 1980, 445—452.

[11] PLESNÍK, J.: A heuristic for the p-center problem in graphs. Discrete Appl. Math. 17, 1987, 263—268.
[12] PLESNÍK, J.: On the interchange heuristic for locating centers and medians in a graph. Math. Slovaca, 37, 1987, 209—216.
[13] TANSEL, B. C.—FRANCIS, R. L.—LOWE, T. J.: Location on networks: a survey; part I: the p-center and p-median problems. Management Sci. 29, 1983, 482—497.
[14] WONG, R. T.: Location and network design. In: Combinatorial Optimization: Annotated Bibliographies (M. O'hEigeartaigh, J. K. Lenstra and A. H. G. Rinnooy Kan, eds.), Wiley, New York, 1985, 129—147.

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ДВЕ ЭВРИСТИКИ ДЛЯ ЗАДАЧИ АБСОЛЮТНОГО p-ЦЕНТРА НА ГРАФАХ

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Резюме

Предлагаются два эвристических полиномиальных алгоритма для нахождения абсолютного p-центра графа с длинами ребер и весами вершин. Один из этих алгоритмов находит p-множество, стоимость которого в самом худшем случае не больше, чем вдвое оптимальной стоимости.