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ON SMALL SYSTEMS AND COMPACT FAMILIES OF BOREL FUNCTIONS

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The main purpose of the paper is to generalize Kisyński's result of [6] and to prove that a family **F** of Borel functions defined on a compact perfectly normal space is compact in the sense of the convergence with respect to an upper semicontinuous small system $(\mathscr{S}_n)$ of Borel sets if and only if, for each positive integer n, there exists a uniformly compact family $\mathbf{F}^*$ of continuous functions having the property that, for any $f \in \mathbf{F}$, there is an $f^* \in \mathbf{F}^*$ such that $\{x : f(x) \neq f^*(x)\} \in \mathscr{S}_n$.

To begin with, let us recall the most important definitions and establish some useful facts.

In what follows, X denotes a compact perfectly normal space. The symbol $\mathscr{B}(X)$ stands for the σ-field of Borel subsets of X (i.e. the smallest σ-field containing all open sets). By *a small system* on $\mathscr{B}(X)$ we mean a sequence $(\mathscr{S}_n)$ of non-empty subfamilies of $\mathscr{B}(X)$, satisfying the following conditions:

(I) for any $n \in N$, there exists a sequence (k_i) of positive integers such that if

$$A_i \in \mathscr{S}_{k_i} \text{ for } i \in N, \text{ then } \bigcup_{i=1}^{\infty} A_i \in \mathscr{S}_n ;$$

(II) for any $n \in N$, $A \in \mathscr{S}_n$ and $B \in \mathscr{B}(X)$ such that $B \subset A$, we have $B \in \mathscr{S}_n$;

(III) for any $n \in N$, $A \in \mathscr{S}_n$ and $B \in \bigcap_{i=1}^{\infty} \mathscr{S}_i$, we have $A \cup B \in \mathscr{S}_n$;

(IV) $\mathscr{S}_n \supset \mathscr{S}_{n+1}$ for each $n \in N$

(cf. [2, 5, 7, 8, 9]). If, in addition, $(\mathscr{S}_n)$ has the following property:

(V) if (A_n) is a non-increasing sequence of Borel sets for which there exists $i \in N$

$$\text{such that } A_n \notin \mathscr{S}_i \text{ for any } n \in N, \text{ then } \bigcap_{n=1}^{\infty} A_n \notin \bigcap_{m=1}^{\infty} \mathscr{S}_m,$$

then it is called *an upper semicontinuous small system* (cf. [7; Definition 18.29]). Now, let us give some serviceable characterization of upper semicontinuous small systems on $\mathscr{B}(X)$.

Proposition. *A small system $(\mathscr{S}_n)$ on $\mathscr{B}(X)$ is upper semicontinuous if and only if each Borel subset A of X has the following property:*

(R) *for any* $n \in N$, *there exist a closed subset* D *of* X *and an open subset* U *of* X, *such that* $D \subset A \subset U$ *and* $U \setminus D \in \mathcal{S}_n$.

Proof. Necessity. Without any difficulties one can check that if $(\mathcal{S}_n)$ is upper semicontinuous, then, since each open set in X is of type F_σ, the family of these subsets of X which have the property (R) forms a σ — field containing all open sets (cf. [9; proof of Theorem 2]).

Sufficiency. Suppose that $(\mathcal{S}_n)$ is not upper semicontinuous but every Borel set has the property (R). By virtue of (III) and (V), there exist a positive integer i and a non-increasing sequence (A_n) of Borel sets, such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$ but $A_n \notin \mathcal{S}_i$ for any $n \in N$. Take a sequence (k_n) of positive integers such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{S}_i$ whenever $E_n \in \mathcal{S}_{k_n}$ for $n \in N$. There exists a closed set $D_1 \subset A_1$ such that $A_1 \setminus D_1 \in \mathcal{S}_{k_1}$. We can inductively define a sequence (D_n) of closed sets such that $D_{n+1} \subset D_n \cap A_{n+1}$ and $(D_n \cap A_{n+1}) \setminus D_{n+1} \in \mathcal{S}_{k_{n+1}}$ for $n \in N$. Then $A_{n+1} \subset (D_n \cap \cap A_{n+1}) \cup \bigcup_{m=1}^{n} [(D_m \cap A_{m+1}) \setminus D_{m+1}] \cup (A_1 \setminus D_1)$, so $D_n \cap A_{n+1} \notin \mathcal{S}_{k_{n+2}}$ for any $n \in N$ (otherwise, A_{n+1} would belong to $\mathcal{S}_i$). In this way, we have obtained a non-increasing sequence (D_n) of non-empty closed subsets of X such that $D_n \subset A_n$ for any $n \in N$. The compactness of X yields $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$, which contradicts the fact that $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

The above proposition points out that the notions of upper semicontinuity and regularity (cf. [7; Definition 18.35]) are equivalent for small systems of Borel sets in perfectly normal compact spaces.

From now on, $(\mathcal{S}_n)$ *will denote a fixed upper semicontinuous small system on* $\mathcal{B}(X)$.

Let $\mathcal{J} = \bigcap_{n=1}^{\infty} \mathcal{S}_n$. Obviously, $\mathcal{J}$ forms a σ-ideal on $\mathcal{B}(X)$. One says that a property holds $\mathcal{J}$-*almost everywhere* (abbr. $\mathcal{J}$-a.e.) *on* X if the set of points not having this property belongs to $\mathcal{J}$. Denote by $\mathbf{M}(\mathcal{J})$ the family of all $\mathcal{J}$-a.e. finite $\mathcal{B}(X)$-measurable real functions defined on X.

Definition 1 (cf. [8]). *A sequence* (f_n) *of functions from* $\mathbf{M}(\mathcal{J})$ *converges with respect to the small system* $(\mathcal{S}_n)$ *to a function* $f \in \mathbf{M}(\mathcal{J})$ *if, for any* $\varepsilon > 0$ *and any* $m \in N$, *there exists* $n_0 \in N$ *such that* $\{x \in X: |f_n(x) - f(x)| > \varepsilon\} \in \mathcal{S}_m$ *whenever* $n \geqslant n_0$.

Definition 2 (cf. [5]). *A family* $\mathbf{F} \subset \mathbf{M}(\mathcal{J})$ *is compact in the sense of the convergence with respect to the small system* $(\mathcal{S}_n)$ *(abbr.* $(\mathcal{S}_n)$*-compact) if each* *sequence of functions from* $\mathbf{F}$ *contains a subsequence converging with respect to* $(\mathscr{S}_n)$ *to some function from* $\mathbf{M}(\mathscr{J})$.

By *a partition of* X is meant a finite family $\mathscr{P}$ of Borel sets such that $\bigcup\{P\colon P\in\mathscr{P}\}=X$.

Definition 3 (cf. [5]). *A family* $\mathbf{F}\subset\mathbf{M}(\mathscr{J})$ *is called*:

(a) $(\mathscr{S}_n)$*-equibounded if, for any* $n\in N$, *there exists a positive integer* t *such that* $\{x\in X\colon |f(x)>t\}\in\mathscr{S}_n$ *whenever* $f\in\mathbf{F}$.

(b) $(\mathscr{S}_n)$*-equimeasurable if, for any* $\varepsilon>0$ *and* $n\in N$, *there exist a partition* $\mathscr{P}$ *of* X *and a collection* $\{A_f\colon f\in\mathbf{F}\}\subset\mathscr{S}_n$, *such that, for each* $P\in\mathscr{P}$ *and* $f\in\mathbf{F}$, *we have* $|f(x)-f(y)|\leqslant\varepsilon$ *whenever* $x,y\in P\setminus A_f$.

In [5] we obtained the following abstract version of Fréchet's theorem characterizing compactness in the sense of the convergence with respect to a finite measure (cf. [1, 3, 4]):

Theorem 0. *A family* $\mathbf{F}\subset\mathbf{M}(\mathscr{J})$ *is* $(\mathscr{S}_n)$*-compact if and only if it is* $(\mathscr{S}_n)$*-equibounded and* $(\mathscr{S}_n)$*-equimeasurable* (cf. [5; Proposition 1 and Theorem 1]).

J. Kisyński gave in [6; Theorem 1] an elegant characterization of compact families of measurable real functions defined on a compact interval of the real line by approximating them to uniformly compact families of continuous functions. Here we shall extend the above mentioned result of Kisyński to $(\mathscr{S}_n)$-compact subfamilies of $\mathbf{M}(\mathscr{J})$. To do this, we need some lemma.

Denote by $\mathbf{C}(X)$ the space of all continuous real functions defined on X with the topology of uniform convergence.

Lemma. *If a family* $\mathbf{F}\subset\mathbf{M}(\mathscr{J})$ *is* $(\mathscr{S}_n)$*-equimeasurable, then, for any* $\varepsilon>0$ *and* $n\in N$, *there exist a closed subset* D *of* X, *a family* $\{A_f\colon f\in\mathbf{F}\}$ *of Borel sets, a continuous function* $h\colon X\to[0,1]$ *and a real number* $\delta>0$, *such that the following conditions are satisfied:*

(a) $(X\setminus D)\cup A_f\in\mathscr{S}_n$ *for any* $f\in\mathbf{F}$;

(b) *for any* $f\in\mathbf{F}$ *and* $x,y\in D\setminus A_f$, *we have*

$$|f(x)-f(y)|\leqslant\varepsilon\quad\text{whenever}\quad |h(x)-h(y)|\leqslant\delta.$$

Proof. Let us fix $\varepsilon>0$ and $n_0\in N$. Take a sequence (k_i) of positive integers such that if $A_i\in\mathscr{S}_{k_i}$ for $i\in N$, then $\bigcup_{i=1}^{\infty}A_i\in\mathscr{S}_{n_0}$. Since $\mathbf{F}$ is $(\mathscr{S}_n)$-equimeasurable, there exist a family $\{P_1,P_2,\ldots,P_m\}$ of pairwise disjoint Borel subsets of X and a family $\{A_f\colon f\in\mathbf{F}\}\subset\mathscr{S}_{k_1}$ such that $\bigcup_{i=1}^{m}P_i=X$ and, moreover, for any $f\in\mathbf{F}$ and $i=1,2,\ldots,m$, we have $|f(x)-f(y)|\leqslant\varepsilon$ whenever $x,y\in P_i\setminus A_f$. By virtue of our Proposition we can find closed subsets $D_1,D_2,\ldots,D_m$ of X such that $D_i\subset P_i$ and $P_i\setminus D_i\in\mathscr{S}_{k_{i+1}}$ for $i=1,2,\ldots,m$. It follows from the normality of X

that there exists a continuous function $h\colon X \to [0, 1]$ such that $h(D_i) = \left\{\frac{1}{i}\right\}$ for $i = 1, 2, \ldots, m$. Let us put $D = \bigcup_{i=1}^{m} D_i$ and $\delta = \frac{1}{2m(m-1)}$. Then, for any $f \in \mathbf{F}$, the set $(X \setminus D) \cup A_f \subset A_f \cup \bigcup_{i=1}^{m} (P_i \setminus D_i)$ is a member of $\mathcal{S}_{n_0}$. To complete the proof, it suffices to observe that if $x, y \in D$ and $|h(x) - h(y)| \leqslant \delta$, then $x, y \in D_i$ for some $i \in \{1, 2, \ldots, m\}$.

Now we are in a position to prove the main theorem of the paper.

Theorem 1. *A family $\mathbf{F} \subset \mathbf{M}(\mathcal{J})$ is $(\mathcal{S}_n)$-compact if and only if, for any $n \in N$, there exists a compact subset $\mathbf{F}^*$ of $\mathbf{C}(X)$ having the property that, for any $f \in \mathbf{F}$, there is an $f^* \in \mathbf{F}^*$ such that $\{x \in X\colon f(x) \neq f^*(x)\} \in \mathcal{S}_n$.*

Proof. Necessity. Let us fix $n_0 \in N$. There exists $m \in N$ such that $A \cup B \in \mathcal{S}_{n_0}$ whenever $A, B \in \mathcal{S}_m$. Take a sequence (k_i) of positive integers such that if $A_i \in \mathcal{S}_{k_i}$ for $i \in N$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}_m$. By Theorem 0, the family $\mathbf{F}$ is $(\mathcal{S}_n)$-equibounded, so there exists $t \in N$ such that $\{x \in X\colon |f(x)| > t\} \in \mathcal{S}_m$ whenever $f \in \mathbf{F}$. The Lemma, along with Theorem 0, implies that, for $i \in N$, there exist closed sets $D_i \subset X$, collections $\{A_f^i\colon f \in \mathbf{F}\} \subset \mathcal{B}(X)$, continuous functions $h_i\colon X \to [0, 1]$ and real numbers $\delta_i > 0$, such that, for any $f \in \mathbf{F}$, the following conditions are satisfied:

(a) $(X \setminus D_i) \cup A_f^i \in \mathcal{S}_{k_i}$;

(b) $|f(x) - f(y)| \leqslant \frac{1}{i}$ whenever $x, y \in D_i \setminus A_f^i$ and $|h_i(x) - h_i(y)| \leqslant \delta_i$.

We may assume that $\delta_{i+1} < \delta_i$ for $i \in N$. Denote $D = \bigcap_{i=1}^{\infty} D_i$ and $A_f = \{x \in X\colon |f(x)| > t\} \cup \bigcup_{i=1}^{\infty} A_f^i$ for $f \in \mathbf{F}$. Clearly, $(X \setminus D) \cup A_f \in \mathcal{S}_{n_0}$ for any $f \in \mathbf{F}$. Let us consider the pseudometric $h(x, y) = \sum_{i=1}^{\infty} \frac{|h_i(x) - h_i(y)|}{2^i}$. For any $f \in \mathbf{F}$ and $i \in N$, we have $|f(x) - f(y)| \leqslant \frac{1}{i}$ whenever $x, y \in D \setminus A_f$ and $h(x, y) \leqslant \frac{\delta_i}{2^i}$. It is not difficult to construct a non-decreasing bounded uniformly continuous function $g\colon [0, +\infty) \to R$ having the properties that $g(0) = 0$, $g(\varepsilon) \geqslant \frac{1}{i}$ for $\varepsilon \in \left(\frac{\delta_{i+1}}{2^{i+1}}, \frac{\delta_i}{2^i}\right]$ and $g(\varepsilon) \geqslant 2t$ for $\varepsilon > \frac{\delta_1}{2}$. Then, for any $f \in \mathbf{F}$, we have

$$|f(x) - f(y)| \leqslant g(h(x,y)) \quad \text{whenever} \quad x, y \in D \setminus A_f.$$

Following [6; proof of Lemma 1], we define

$$f^*(x) = \sup\{f(y) - g(h(x,y)) \colon y \in D \setminus A_f\} \text{ for } f \in \mathbf{F} \text{ and } x \in X.$$

If $x, y \in D \setminus A_f$, then $f^*(x) \geqslant f(x) \geqslant f(y) - g(h(x,y))$, so $\{x \in X \colon f(x) \neq f^*(x)\} \subset \subset (X \setminus D) \cup A_f$; hence $\{x \in X \colon f(x) \neq f^*(x)\} \in \mathscr{S}_{n_0}$ for any $f \in \mathbf{F}$. Obviously, the family $\{f^* \colon f \in \mathbf{F}\}$ is equibounded. In view of Ascoli's theorem, it suffices to show that $\{f^* \colon f \in \mathbf{F}\}$ is evenly continuous. Bearing this in mind, let us define

$$G(\varepsilon) = \sup\{|g(\varepsilon_1) - g(\varepsilon_2)| \colon \varepsilon_1, \varepsilon_2 \geqslant 0 \quad \text{and} \quad |\varepsilon_1 - \varepsilon_2| \leqslant \varepsilon\}.$$

Consider any $\delta > 0$ and $x \in X$. There exists $\varepsilon_0 > 0$ such that $G(\varepsilon) < \delta$ for $0 \leqslant \varepsilon < \varepsilon_0$. We can find a neighbourhood U of x such that $h(x,y) < \varepsilon_0$ for any $y \in U$. Arguing similarly as in the proof of Lemma 1 in [6], one checks that

$$|f^*(x) - f^*(y)| \leqslant G(h(x,y)) \quad \text{for any} \quad f \in \mathbf{F} \quad \text{and} \quad y \in X.$$

All this implies that $|f^*(x) - f^*(y)| < \delta$ for any $f \in \mathbf{F}$ and $y \in U$; therefore the family $\{f^* \colon f \in \mathbf{F}\}$ is evenly continuous.

Sufficiency. Let $n_0 \in N$ and $\varepsilon > 0$ be fixed. Take a compact set $\mathbf{F}^* \subset \mathbf{C}(X)$ having the property that to each $f \in \mathbf{F}$ one can assign some $f^* \in \mathbf{F}^*$ such that the set $B_f = \{x \in X \colon f(x) \neq f^*(x)\}$ is a member of $\mathscr{S}_{n_0}$. The equiboundedness of $\mathbf{F}^*$ implies the $(\mathscr{S}_n)$-equiboundedness of $\mathbf{F}$. Since $\mathbf{F}^*$ is evenly continuous, there exists, for any $x \in X$, an open neighbourhood U_x of x such that $|f^*(x) - f^*(y)| \leqslant \varepsilon$ whenever $f \in \mathbf{F}$ and $y \in U_x$. If $\mathscr{P}$ is a finite subcover of the cover $\{U_x \colon x \in X\}$ of X, then, for any $f \in \mathbf{F}$ and $P \in \mathscr{P}$, we have $|f(x) - f(y)| \leqslant \varepsilon$ whenever $x, y \in P \setminus B_f$; hence $\mathbf{F}$ is $(\mathscr{S}_n)$-equimeasurable. Theorem 0 completes the proof.

An immediate consequence of Theorem 1 is the following

Corollary. *A family* $\mathbf{F} \subset \mathbf{M}(\mathscr{J})$ *is* $(\mathscr{S}_n)$*-compact if and only if, for any* $n \in N$ *and* $\varepsilon > 0$, *there exists a finite set* $\mathbf{F}^* \subset \mathbf{C}(X)$ *with the property that, for any* $f \in \mathbf{F}$, *there is an* $f^* \in \mathbf{F}^*$ *such that* $\{x \in X \colon |f(x) - f^*(x)| > \varepsilon\} \in \mathscr{S}_n$.

Finally, let us formulate Theorem 1 in terms of the σ-ideal $\mathscr{J}$.

A family $\mathbf{F} \subset \mathbf{M}(\mathscr{J})$ is called *compact in the sense of the convergence with respect to the* σ*-ideal* $\mathscr{J}$ (abbr. $\mathscr{J}$*-compact*) provided each sequence of functions from $\mathbf{F}$ contains a subsequence converging $\mathscr{J}$-a.e. on X to some function $f \in \mathbf{M}(\mathscr{J})$ (cf. [5; Definition 2(b)]). Since $(\mathscr{S}_n)$ is upper semicontinuous, $\mathscr{J}$-compactness is equivalent to $(\mathscr{S}_n)$-compactness, as observed before in [5].

Theorem 2. *A family* $\mathbf{F} \subset \mathbf{M}(\mathscr{J})$ *is* $\mathscr{J}$*-compact if and only if there exists a sequence* $(\mathbf{F}_n^*)$ *of compact subsets of* $\mathbf{C}(X)$ *with the property that, for any sequence*

(f_n) of functions from $\mathbf{F}$*, there exists a sequence (f_n^*) of continuous functions such that* $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in X : f_n(x) \neq f_n^*(x)\} \in \mathscr{J}$ *and* $f_n^* \in \mathbf{F}_n^*$ *for any* $n \in N$.

Proof. Necessity. Lemma 1 of [8] implies the existence of a sequence (k_n) of positive integers such that if $A_n \in \mathscr{S}_{k_n}$ for $n \in N$, then $\bigcup_{n=m}^{\infty} A_n \in \mathscr{S}_m$ for any $m \in N$. In view of Theorem 1, there exists a sequence $(\mathbf{F}_n^*)$ of compact subsets of $\mathbf{C}(X)$ having the property that, for any $n \in N$ and $f \in \mathbf{F}$, there is an $f^* \in \mathbf{F}_n^*$ such that $\{x \in X : f(x) \neq f^*(x)\} \in \mathscr{S}_{k_n}$. It is evident that $(\mathbf{F}_n^*)$ is the required sequence.

Sufficiency. Using similar arguments as in the proof of Theorem 1, we find a sequence (t_n) of positive integers such that $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in X : |f_n(x)| > t_n\} \in \mathscr{J}$ for any sequence (f_n) of functions from $\mathbf{F}$. Therefore, by Proposition 2(b) of [5], $\mathbf{F}$ is $(\mathscr{S}_n)$-equibounded.

Let us fix $\varepsilon > 0$. According to the proof of Theorem 1, one can show without any difficulties that there exists a sequence $(\mathscr{P}_n)$ of partitions of X having the property that, for any sequence (f_n) of functions from $\mathbf{F}$, there exists a sequence (A_n) of Borel sets such that $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n \in \mathscr{J}$ and, furthermore, for any $n \in N$ and $P \in \mathscr{P}_n$, we have $|f_n(x) - f_n(y)| \leqslant \varepsilon$ whenever $x, y \in P \setminus A_n$. Following the proof of Proposition 4(b) in [5], we show that $\mathbf{F}$ is $(\mathscr{S}_n)$-equimeasurable. By virtue of Theorem 0, $\mathbf{F}$ is $\mathscr{J}$-compact.

Let us note that Theorems 1 and 2, together with the Corollary, remain true if we assume that X is a compact Hausdorff space (not necessarily perfectly normal) and $(\mathscr{S}_n)$ is a regular small system on $\mathscr{B}(X)$ (i.e. every Borel set has the property (R)).

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МАЛЫЕ СИСТЕМЫ И КОМПАКТНЫЕ МНОЖЕСТВА БОРЕЛЕВСКИХ ФУНКЦИЙ

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Резюме

Главная цель работы доказать, что семейство $\mathbf{F}$ борелевских функций, опрелеленных на компактном совершенно нормальном пространстве, является компактным по сходимости по непрерывной сверху малой системе $(\mathscr{S}_n)$ борелевских множеств в том и только в том случае, когда для произвольного натурального числа n существует такое компактное в топологии равномерной сходимости семейство $\mathbf{F}^*$ непрерывных функций, что для каждого $f \in \mathbf{F}$ найдется такое $f^* \in \mathbf{F}^*$, что $\{x: f(x) \neq f^*(x)\} \in \mathscr{S}_n$.