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VISIBILITIES AND SETS OF SHORTEST PATHS IN A CONNECTED GRAPH

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By a graph we mean here an undirected (not necessarily finite) graph without loops and multiple edges. Thus if G is a graph with a vertex set $V(G)$ and an edge set $E(G)$, then $V(G)$ is a nonempty set and $E(G)$ is a subset of the set of all two-element subsets of $V(G)$; G is called finite if $V(G)$ is finite.

The letters h, i, j, k, m and n will be reserved for denoting integers.

Consider a graph G. We denote by $\mathscr{W}(G)$ the set of all finite sequences of vertices in G, including the empty sequence, which will be denoted by $*$. Thus $\mathscr{W}(G) - \{*\}$ is the set of all sequences

$$v_0, \ \ldots, \ v_j, \tag{0}$$

where $j \geqslant 0$ and $v_0, \ldots, v_j \in V(G)$. Similarly to [2], instead of (0) we will write $v_0 \ldots v_j$. Let $u_0, \ldots, u_i, w_0, \ldots, w_k \in V(G)$, where $i, k \geqslant 0$, and let $\alpha = u_0 \ldots u_i$ and $\beta = w_0 \ldots w_k$. Then we write

$$\alpha\beta = u_0 \ldots u_i w_0 \ldots w_k.$$

Moreover, we write $\gamma * = \gamma = * \gamma$ for every $\gamma \in \mathscr{W}(G)$. Let $x_0 \ldots, x_m \in V(G)$, where $m \geqslant 0$. Put $\delta = x_0 \ldots x_m$. We write

$$\|\delta\| = m, \ F\delta = x_0, \ L\delta = x_m, \ \text{ and } \ \bar{\delta} = x_m \ldots x_0.$$

Moreover, we define $\bar{*} = *$. Let $y_0, \ldots, y_n \in V(G)$, where $n \geqslant 0$. We say that $y_0 \ldots y_n$ is a path in G if the vertices $y_0, \ldots, y_n$ are mutually distinct and $\{y_i, y_{i+1}\} \in E(G)$ for every integer i such that $0 \leqslant i < n$. Let $\mathscr{P}(G)$ denote the set of all paths in G. Obviously, $\mathscr{P}(G) \subseteq \mathscr{W}(G) - \{*\}$. If $\alpha \in \mathscr{P}(G)$, then the number $\|\alpha\|$ is called the length of α. Consider $\mathscr{R} \subseteq \mathscr{P}(G)$ and $u, v \in V(G)$. Define

$$\mathscr{R}_{(u,v)} = \{\alpha \in \mathscr{R};\ F\alpha = u \text{ and } L\alpha = v\}.$$

We say that G is connected if $\mathscr{P}_{(t,z)} \neq \emptyset$ for every pair of $t, z \in V(G)$, where $\mathscr{P} = \mathscr{P}(G)$.

Consider a connected graph G. We define the distance $d_G(x,y)$ of vertices x and y in G as

$$d_G(x,y) = \min(\|\alpha\|\,;\ \alpha \in \mathscr{P}(G), F\alpha = x \text{ and } L\alpha = y).$$

Let $\xi \in \mathscr{W}(G) - \{*\}$; we say that ξ is a *shortest path* in G if $\xi \in \mathscr{P}(G)$ and $\|\xi\| = d_G(F\xi, L\xi)$. Let $\mathscr{S}(G)$ denote the set of all shortest paths in G.

The set $\mathscr{S}(G)$ was characterized by the present author in [2] (under the condition that G is finite); his characterization is "almost non-metric" in the sense that the lengths of paths greater than one are neither considered nor compared in it. In the present paper a more general result will be proved. We will obtain an "almost non-metric" necessary and sufficient condition for a set of paths in a connected graph G to be an element of a certain set of subsets of $\mathscr{S}(G)$. To describe such a set of subsets of $\mathscr{S}(G)$ we introduce the notion of visibility in G.

Let G be a connected graph, and let $Q \subseteq V(G) \times V(G)$. We say that Q is a *visibility* in G if Q fulfils the following Axioms I–IV (for arbitrary $u, v, x, y \in V(G)$):

I if $(u,v) \in Q$, then $(v,u) \in Q$;

II if $(u,v) \in Q$ and $d_G(u,x) + d_G(x,v) = d_G(u,v)$, then $(u,x) \in Q$;

III if $(u,v) \in Q$, $\{u,x\}, \{v,y\} \in E(G)$ and $d_G(x,v) = d_G(u,v) - 1 = d_G(x,y)$, then $(u,y) \in Q$;

IV if $(u,v) \in Q$, $\{u,x\}, \{v,y\} \in E(G)$ and $d_G(x,v) = d_G(u,v) - 1 \geqslant 1$, then $(x,y) \in Q$.

We are now prepared to formulate the main result of the present paper.

Theorem. *Let G be a connected graph, and let $\mathscr{R} \subseteq \mathscr{P}(G)$. Denote $\mathscr{S} = \mathscr{S}(G)$. Then the following statements (1) and (2) are equivalent:*

(1) *there exists a visibility Q in G such that*

$$\mathscr{R}_{(t,z)} = \mathscr{S}_{(t,z)} \quad \text{if} \quad (t,z) \in Q \quad \text{and}$$
$$\mathscr{R}_{(t,z)} = \emptyset \quad \text{if} \quad (t,z) \notin Q,$$

for every pair of vertices t and z of G;

(2) *$\mathscr{R}$ fulfils the following Axioms A_1–A_4 and B_1–B_3 (for arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta, \gamma, \delta \in \mathscr{W}(G)$);*

A_1 if $\alpha \in \mathscr{R}$, then $\bar{\alpha} \in \mathscr{R}$;

A_2 if $\alpha uv \in \mathscr{R}$, then $\alpha u \in \mathscr{R}$;

A_3 if $ux\alpha v \in \mathscr{R}$, $\{v,y\} \in E(G)$, $u\varphi yv \notin \mathscr{R}$ for any $\varphi \in \mathscr{W}(G)$ and $ux\psi y \notin \mathscr{R}$ for any $\psi \in \mathscr{W}(G)$, then $x\alpha vy \in \mathscr{R}$;

A_4 *if* $ux\alpha v, u\beta yv \in \mathscr{R}$, *then* $\mathscr{R}_{(x,y)} \neq \emptyset$;

B_1 *if* $\alpha u\beta v\gamma, u\delta v \in \mathscr{R}$, *then* $\alpha u\delta v\gamma \in \mathscr{R}$;

B_2 *if* $ux\alpha v, u\beta yv, xu\beta y \in \mathscr{R}$, *then* $x\alpha vy \in \mathscr{R}$;

B_3 *if* $ux\alpha v \in \mathscr{R}$, *then* $\{u, v\} \notin E(G)$.

Proof. Instead of $d_G(t, z)$, where $t, z \in V(G)$, we will write $d(t, z)$.

Part One: (1) $\Rightarrow$ (2). Let (1) hold. We want to prove that $\mathscr{R}$ fulfils Axioms A_1–A_4 and B_1–B_3.

Consider arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta \in \mathscr{W}(G)$.

(Verification of Axiom A_1). Suppose $\alpha \in \mathscr{R}$. There exist $t, z \in V(G)$ such that $\alpha \in \mathscr{R}_{(t,z)}$. Hence $\mathscr{R}_{(t,z)} \neq \emptyset$. It follows from (1) that $(t, z) \in Q$ and therefore, $\mathscr{R}_{(t,z)} = \mathscr{S}_{(t,z)}$. We get $\alpha \in \mathscr{S}_{(t,z)}$. This means that $\bar{\alpha} \in \mathscr{S}_{(z,t)}$. Axiom I implies that $(z, t) \in Q$. According to (1), $\mathscr{R}_{(z,t)} = \mathscr{S}_{(z,t)}$. Thus $\bar{\alpha} \in \mathscr{R}$.

(Verification of Axiom A_2). Suppose $\alpha uv \in \mathscr{R}$. First, let $\alpha = *$. According to (1), $uv \in \mathscr{S}$ and $(u, v) \in Q$. Axiom II implies that $(u, u) \in Q$. As follows from (1), $\alpha u = u \in \mathscr{R}$. Let now $\alpha \neq *$. There exist $t \in V(G)$ and $\varphi \in \mathscr{W}(G)$ such that $\alpha = t\varphi$. Then $t\varphi uv \in \mathscr{R}_{(t,v)}$. According to (1), $t\varphi uv \in \mathscr{S}$ and $(t, v) \in Q$. Obviously, $t\varphi u \in \mathscr{S}$. We have $d(t, v) = d(t, u) + d(u, v)$. Axiom II implies that $(t, u) \in Q$. According to (1), $\mathscr{R}_{(t,u)} = \mathscr{S}_{(t,u)}$. We get $\alpha u = t\varphi u \in \mathscr{R}$.

(Verification of Axiom A_3). Suppose $ux\alpha v \in \mathscr{R}$, $\{v, y\} \in E(G)$, $u\varphi yv \notin \mathscr{R}$ for any $\varphi \in \mathscr{W}(G)$ and $ux\psi y \notin \mathscr{R}$ for any $\psi \in \mathscr{W}(G)$. Clearly, $\{u, x\} \in E(G)$. Since $\mathscr{R}_{(u,v)} \neq \emptyset$, it follows from (1) that $\mathscr{R}_{(u,v)} = \mathscr{S}_{(u,v)}$ and $(u, v) \in Q$. This implies that $ux\alpha v \in \mathscr{S}$ and $u\varphi yv \notin \mathscr{S}$ for any $\varphi \in \mathscr{W}(G)$. Thus $d(x, v) = d(u, v) - 1 \geqslant 1$ and $d(u, v) \leqslant d(u, y)$.

Obviously, $d(x, y) \geqslant d(u, y) - 1$. This means that $d(u, v) - 1 \leqslant d(x, y) \leqslant d(u, v)$. Assume that $d(x, y) = d(u, v) - 1$. Axiom III implies that $(u, y) \in Q$. As follows from (1), $\mathscr{R}_{(u,y)} = \mathscr{S}_{(u,y)}$. This means that $ux\psi y \notin \mathscr{S}$ for any $\psi \in \mathscr{W}(G)$. Thus $d(u, y) \leqslant d(x, y)$. Clearly, $d(u, v) \leqslant d(u, y) \leqslant d(x, y) \leqslant d(u, v) - 1$, which is a contradiction. Hence $d(x, y) = d(u, v)$. We see that $x\alpha vy \in \mathscr{S}$.

Recall that $d(x, v) = d(u, v) - 1 \geqslant 1$. Axiom IV implies that $(x, y) \in Q$. According to (1), $\mathscr{R}_{(x,y)} = \mathscr{S}_{(x,y)}$. We get $x\alpha vy \in \mathscr{R}$.

(Verification of Axiom A_4). Suppose $ux\alpha v, u\beta yv \in \mathscr{R}$. Then $\{u, x\}, \{v, y\} \in E(G)$. According to (1), $ux\alpha v \in \mathscr{S}$ and $(u, v) \in Q$. Since $d(x, v) = d(u, v) - 1 \geqslant 1$, it follows from Axiom IV that $(x, y) \in Q$. According to (1), $\mathscr{R}_{(x,y)} \neq \emptyset$.

Thus $\mathscr{R}$ fulfils Axioms A_1–A_4. Axioms B_1–B_3 follows from (1) and simple properties of $\mathscr{S}$. Hence (2) holds.

Part Two: (2) $\Rightarrow$ (1). Let $\mathscr{R}$ fulfil Axioms A_1–A_4 and B_1–B_3. Combining Axioms A_1 and A_2, we get

(3) if $u \in V(G)$, $\alpha, \beta \in \mathscr{W}(G)$ and $\alpha u\beta \in \mathscr{R}$, then $\alpha u, u\beta, u\bar{\alpha}, \bar{\beta}u \in \mathscr{R}$.

Combining Axioms A_2 and A_3, we get

(4) if $u, v, x, y \in V(G), \alpha \in \mathscr{W}(G), ux\alpha v \in \mathscr{R}, \{v, y\} \in E(G)$ and $x\alpha vy \notin \mathscr{R}$, then $\mathscr{R}(u, y) \neq \emptyset$.

This part of the proof will be divided into Sections 1 and 2. In Section 1 we will prove that

(5) if $\mathscr{R}_{(u,v)} \neq \emptyset$, then $\mathscr{R}_{(u,v)} = \mathscr{S}_{(u,v)}$ for every pair of vertices u and v of G.

In Section 2 we will prove that

$$\{(u, v)\,;\, u, v \in V(G) \text{ such that } \mathscr{R}_{(u,v)} \neq \emptyset\}$$

is a visibility in G.

Section 1. We denote by M the set of all integers k such that there exist $t, z \in V(G)$ with the property that $d(t, z) = k$. Obviously, either M is the set of all non-negative integers or there exists $h \geqslant 0$ such that $M = \{0, \ldots, h\}$. For each $m \in M$ we will prove that

(6_m) if $\mathscr{R}_{(u,v)} \neq \emptyset$, then $\mathscr{S}_{(u,v)} \subseteq \mathscr{R}_{(u,v)}$ for every pair of vertices u and v of G such that $d(u, v) \leqslant m$,

and

(7_m) $\mathscr{R}_{(u,v)} \subseteq \mathscr{S}_{(u,v)}$ for every pair of vertices u and v of G such that $d(u, v) \leqslant m$.

We proceed by induction on m. First, let $m = 0$. Since $\mathscr{R} \subseteq \mathscr{P}(G)$, we get $\mathscr{R}_{(w,w)} \subseteq \{w\}$ for each $w \in V(G)$. Hence (6_0) and (7_0) follow. Next, let $m = 1$. Consider arbitrary $t, z \in V(G)$ such that $d(t, z) = 1$. Axiom B_3 implies that $\mathscr{R}_{(t,z)} \subseteq \{t, z\}$. Hence, ($6_1$) and ($7_1$) follow.

Now, let $m \geqslant 2$. Suppose (6_{m-1}) and (7_{m-1}) hold. This section of the proof will be divided into two subsections. In 1.1, combining (6_{m-1}) and (7_{m-1}) we will prove that (6_m) holds. In 1.2, combining (6_m) and (7_{m-1}) we will prove that (7_m) holds.

1.1. If $\mathscr{R}_{(t,z)} = \emptyset$ for every pair of vertices t and z of G such that $d(t, z) = m$, then (6_{m-1}) implies that (6_m) holds. Assume that there exist $t, z \in V(G)$ such that $\mathscr{R}_{(t,z)} \neq \emptyset$ and $d(t, z) = m$.

Consider arbitrary $u, v \in V(G)$ such that $\mathscr{R}_{(u,v)} \neq \emptyset$ and $d(u, v) = m$. Consider an arbitrary $\xi \in \mathscr{S}_{(u,v)}$. We want to prove that $\xi \in \mathscr{R}$. Since $\mathscr{R}_{(u,v)} \neq \emptyset$, there exists $\zeta \in \mathscr{R}_{(u,v)}$.

We first assume that ξ and ζ have a common vertex w such that $u \neq w \neq v$. Then

(8) there exist $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathscr{W}(G) - \{*\}$ such that $\xi = \varphi_1 w \varphi_2$ and $\zeta = \psi_1 w \psi_2$.

Obviously, $\varphi_1 w \in \mathscr{S}_{(u,w)}$ and $w\varphi_2 \in \mathscr{S}_{(w,v)}$. As follows from (3), $\psi_1 w \in \mathscr{R}_{(u,w)}$ and $w\psi_2 \in \mathscr{R}_{(w,v)}$. It is clear that $d(u, w) < m$ and $d(w, v) < m$. Since $\mathscr{R}_{(u,w)} \neq$

$\emptyset \neq \mathscr{R}_{(w,v)}$, (6_{m-1}) implies that $\varphi_1 w, w\varphi_2 \in \mathscr{R}$. Recall that $\psi_1 w \psi_2 \in \mathscr{R}_{(u,v)}$. Using Axiom B_1 we get $\psi_1 w \varphi_2 \in \mathscr{R}$ and $\xi = \varphi_1 w \varphi_2 \in \mathscr{R}$.

We now assume that ξ and ζ have no common vertex different from u and v. Put $n = \|\zeta\|$. Obviously, $n \geqslant m \geqslant 2$. There exist mutually distinct $x_0, \dots, x_{m+n-1} \in V(G)$ such that

(9) $\quad \xi = x_0 x_{m+n-1} \dots x_n$ and $\zeta = x_0 x_1 \dots x_n$.

Obviously, $x_0 = u$ and $x_n = v$. Put

(10) $\quad x_{k+m+n} = x_k$ for each $k \in \{0, \dots, m+n-1\}$.

Then $\xi = x_{m+n} x_{m+n-1} \dots x_n$. We define

(11) $\quad \xi_i = x_{i+m+n} x_{i+m+n-1} \dots x_{i+n}$ and $\zeta_i = x_i x_{i+1} \dots x_{i+n}$

for each $i \in \{0, \dots, m\}$. Obviously, $\xi_0 = \xi$ and $\zeta_0 = \zeta$. Recall that we want to prove that $\xi_0 \in \mathscr{R}$. Suppose, to the contrary, that $\xi_0 \notin \mathscr{R}$. It follows from (3) that $\zeta_m \notin \mathscr{R}$.

Since $\xi_0 \notin \mathscr{R}$, $\zeta_0 \in \mathscr{R}$ and $\zeta_m \notin \mathscr{R}$, there exists $j \in \{0, \dots, m-1\}$ such that

(a) $\xi_j \notin \mathscr{R}$, $\zeta_j \in \mathscr{R}$ and (b) either $\xi_{j+1} \in \mathscr{R}$ or $\zeta_{j+1} \notin \mathscr{R}$.

Let $\zeta_{j+1} \in \mathscr{R}$. According to (b), $\xi_{j+1} \in \mathscr{R}$. Since $\zeta_j \in \mathscr{R}$, Axiom B_2 implies that $\xi_j \in \mathscr{R}$, which is a contradiction. Thus $\zeta_{j+1} \notin \mathscr{R}$.

Clearly, $d(x_j, x_{j+n}) \leqslant \|\xi_j\| = m$. If $d(x_j, x_{j+n}) < m$, then—combining (7_{m-1}) with the fact that $\zeta_j \in \mathscr{R}$—we get $\zeta_j \in \mathscr{S}$ and therefore $n = \|\zeta_j\| = d(x_j, x_{j+n}) < m$, which is a contradiction. Thus $d(x_j, x_{j+n}) = m$. This means that $\xi_j \in \mathscr{S}$. Put

$$\sigma = x_j \dots x_0 x_{m+n-1} \dots x_{j+n+1}.$$

Then $\xi_j = \sigma x_{j+n}$. Clearly, $\sigma \in \mathscr{S}$. Recall that $\zeta_{j+1} \notin \mathscr{R}$. It follows from (4) that

$$\mathscr{R}_{(x_j, x_{j+n+1})} \neq \emptyset.$$

Since $\sigma \in \mathscr{S}$, it follows from (6_{m-1}) that $\sigma \in \mathscr{R}$. Since $\xi_j \notin \mathscr{R}$, Axiom B_1 implies that

(12) $$x_j \varphi x_{j+n+1} x_{j+n} \notin \mathscr{R} \quad \text{for any} \quad \varphi \in \mathscr{W}(G).$$

Combining the fact that $\zeta_{j+1} \notin \mathscr{R}$ with (12) and Axiom A_3, we see that there exists $\psi \in \mathscr{W}(G)$ such that

$$x_j x_{j+1} \psi x_{j+n+1} \in \mathscr{R}.$$

Put $\omega = x_{j+1} \psi x_{j+n+1}$. Since $d(x_j, x_{j+n+1}) = m - 1$, (7_{m-1}) implies that $x_j \omega \in \mathscr{S}$. Since $\sigma x_{j+n} \in \mathscr{S}$, we get $x_j \omega x_{j+n} \in \mathscr{S}$. Hence $\omega x_{j+n} \in \mathscr{S}$ and $d(x_{j+1}, x_{j+n}) = \|\omega x_{j+n}\| = m - 1$.

Define

$$\varrho = x_{j+1} \dots x_{j+n}. \tag{13}$$

Since $\zeta_j \in \mathscr{R}$, (3) implies that $\varrho \in \mathscr{R}$. Since $F\varrho = x_{j+1}$, $L\varrho = x_{j+n}$ and $\omega x_{j+n} \in \mathscr{S}$, it follows from (6_{m-1}) that $\omega x_{j+n} \in \mathscr{R}$. Obviously, $x_j\varrho \in \mathscr{R}$. According to Axiom B_1, $x_j\omega x_{j+n} \in \mathscr{R}$. Since $L\omega = x_{j+n+1}$, we get a contradiction with (12).

We have proved that $\xi \in \mathscr{R}$. This means that (6_m) holds.

1.2. Consider arbitrary $u, v \in V(G)$ such that $d(u,v) = m$. If $\mathscr{R}_{(u,v)} = \emptyset$, then $\mathscr{R}_{(u,v)} \subseteq \mathscr{S}_{(u,v)}$. Let $\mathscr{R}_{(u,v)} \neq \emptyset$. Consider an arbitrary $\zeta \in \mathscr{R}_{(u,v)}$. We want to prove that $\zeta \in \mathscr{S}$. Obviously, there exists $\xi \in \mathscr{S}_{(u,v)}$.

We first assume that ξ and ζ have a common vertex w such that $u \neq w \neq v$. Then (8) holds. Clearly, $d(u,w) < m$ and $d(w,v) < m$. As follows from (7_{m-1}), $\psi_1 w \in \mathscr{S}_{(u,w)}$ and $w\psi_2 \in \mathscr{S}_{(w,v)}$. This implies that $\zeta \in \mathscr{S}$.

We now assume that ξ and ζ have no common vertex different from u and v. Put $n = \|\zeta\|$. Obviously, $n \geqslant m = d(u,v)$. Recall that we want to prove that $\zeta \in \mathscr{S}$. Suppose, to the contrary, that $\zeta \notin \mathscr{S}$. Then $n > m$. There exist mutually distinct $x_0, \dots, x_{m+n-1} \in V(G)$ such that (9) holds. We adopt the convention (10) and define ξ_i and ζ_i as in (11) for each $i \in \{0, \dots, m\}$. Recall that

$$\zeta_0 = \zeta = x_0 \dots x_m \dots x_n, \ \zeta_m = x_m \dots x_n \dots x_{m+n} \ \text{ and } \ x_{m+n} = x_0.$$

If $\zeta_m \in \mathscr{R}$, then Axioms A_1 and B_1 imply that

$$x_m \dots x_n \dots x_m \dots x_0 \in \mathscr{R},$$

which contradicts the fact that $\mathscr{R} \subseteq \mathscr{P}(G)$. Hence $\zeta_m \notin \mathscr{R}$.

Since $\xi_0 \in \mathscr{S}, \zeta_0 \in \mathscr{R}$ and $\zeta_m \notin \mathscr{R}$, there exists $j \in \{0, \dots, m-1\}$ such that

(a) $\xi_j \in \mathscr{S}$, $\zeta_j \in \mathscr{R}$ and (b) either $\xi_{j+1} \notin \mathscr{S}$ or $\zeta_{j+1} \notin \mathscr{R}$.

Since $\xi_j \in \mathscr{S}$, it follows from (6_m) that $\xi_j \in \mathscr{R}$. Axiom A_4 implies that

$$\mathscr{R}_{(x_{j+1}, x_{j+n+1})} \neq \emptyset.$$

Let $\xi_{j+1} \in \mathscr{S}$. According to (6_m), $\xi_{j+1} \in \mathscr{R}$. Recall that $\xi_j, \zeta_j \in \mathscr{R}$. Axiom B_2 implies that $\zeta_{j+1} \in \mathscr{R}$, which contradicts (b).

Thus $\xi_{j+1} \notin \mathscr{S}$. This means that $d(x_{j+1}, x_{j+n+1}) \leqslant m-1$. Hence $d(x_{j+1}, x_{j+n}) \leqslant m$. Define ϱ as in (13). Assume that $d(x_{j+1}, x_{j+n}) \leqslant m-1$; then (7_{m-1}) implies that

$\varrho \in \mathscr{S}$; therefore $n-1 \leqslant m-1$, which is a contradiction. Thus $d(x_{j+1}, x_{j+n}) = m$. This means that $d(x_{j+1}, x_{j+n+1}) = m-1$. There exists $\psi \in \mathscr{W}(G)$ such that

$$x_{j+1}\psi x_{j+n+1}x_{j+n} \in \mathscr{S}.$$

Similarly to 1.1, put $\omega = x_{j+1}\psi x_{j+n+1}$. Then $\|\omega\| = m-1$. It follows from (6_m) that $\omega x_{j+n} \in \mathscr{R}$. Since $\zeta_j \in \mathscr{R}$, Axiom B_1 implies that $x_j\omega x_{j+n} \in \mathscr{R}$. According to (3), $x_j\omega \in \mathscr{R}$. Since $d(x_j, x_{j+n+1}) = m-1$, (7_{m-1}) implies that $x_j\omega \in \mathscr{S}$. But $\|x_j\omega\| = m > d(x_j, x_{j+n+1})$, which is a contradiction.

We have proved that $\zeta \in \mathscr{S}$. This means that (7_m) holds.

Summarizing the results of 1.1 and 1.2, we see that (5) holds.

Section 2. Denote

$$Q = \{(t,z);\, t,z \in V(G) \quad \text{such that} \quad \mathscr{R}_{(t,z)} \neq \emptyset\}.$$

We want to prove that Q fulfils Axioms I–IV.

Consider arbitrary $u,v,x,y \in V(G)$. Suppose $(u,v) \in Q$. Then $\mathscr{R}_{(u,v)} \neq \emptyset$. According to (5), $\mathscr{R}_{(u,v)} = \mathscr{S}_{(u,v)}$.

(Verification of Axiom I) It follows from Axiom A_1 that $\mathscr{R}_{(v,u)} \neq \emptyset$. We get $(v,u) \in Q$.

(Verification of Axiom II) Suppose $d(u,v) = d(u,x) + d(x,v)$. If $x = v$, then it is obvious that $(u,x) \in Q$. Let $x \neq v$. Then there exist $\alpha, \beta \in \mathscr{W}(G)$ such that $\alpha x\beta v \in \mathscr{S}_{(u,v)}$. Hence $\alpha x\beta v \in \mathscr{R}_{(u,v)}$. It follows from (3) that $\alpha x \in \mathscr{R}_{(u,x)}$. Therefore, $\mathscr{R}_{(u,x)} \neq \emptyset$. We get $(u,x) \in Q$.

(Verification of Axiom III) Suppose $\{u,x\}, \{v,y\} \in E(G)$ and $d(x,v) = d(u,v) - 1 = d(x,y)$. Clearly, $x \neq v$. There exists $\alpha \in \mathscr{W}(G)$ such that $ux\alpha v \in \mathscr{S}$. Since $d(x,v) = d(x,y)$, we have $x\alpha vy \notin \mathscr{S}$. Since $ux\alpha v \in \mathscr{S}$, we have $ux\alpha v \in \mathscr{R}$. Since $x\alpha vy \notin \mathscr{S}$, (5) implies that $x\alpha vy \notin \mathscr{R}$. It follows from (4) that $\mathscr{R}_{(u,y)} \neq \emptyset$. We get $(u,y) \in Q$.

(Verification of Axiom IV) Suppose $\{u,x\}, \{v,y\} \in E(G)$ and $d(x,v) = d(u,v) - 1 \geqslant 1$. There exists $\alpha \in \mathscr{W}(G)$ such that $ux\alpha v \in \mathscr{S}$. Hence $ux\alpha v \in \mathscr{R}$. If $x\alpha vy \in \mathscr{R}$, then $\mathscr{R}_{(x,y)} \neq \emptyset$. Let $x\alpha vy \notin \mathscr{R}$. If there exists $\beta \in \mathscr{W}(G)$ such that $ux\beta y \in \mathscr{R}$, then (3) implies that $x\beta y \in \mathscr{R}$, and thus $\mathscr{R}_{(x,y)} \neq \emptyset$. Let $ux\varphi y \in \mathscr{R}$ for any $\varphi \in \mathscr{W}(G)$. Axiom A_3 implies that there exists $\gamma \in \mathscr{W}(G)$ such that $u\gamma yv \in \mathscr{R}$. Since $ux\alpha v \in \mathscr{R}$, Axiom A_4 implies that $\mathscr{R}_{(x,y)} \neq \emptyset$. We get $(x,y) \in Q$.

We have proved that Q is a visibility in G.

The proof of the theorem is complete. $\square$

The following corollary is similar to the result which was (under the condition that G is finite) originally proved in [2]:

Corollary. *Let G be a connected graph, and let $\mathscr{R} \subseteq \mathscr{P}(G)$. Then $\mathscr{R} = \mathscr{S}(G)$ if and only if $\mathscr{R}$ fulfils Axioms A_1–A_3, B_1–B_3 and the following Axiom A_0 (for arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta, \gamma, \delta \in \mathscr{W}(G)$):*

A_0 $\mathscr{R}_{(u,v)} \neq \emptyset$.

Proof. Let $\mathscr{R} = \mathscr{S}(G)$. Then $\mathscr{R}$ fulfils Axiom A_0. Our theorem implies that $\mathscr{R}$ fulfils Axioms A_1–A_3 and B_1–B_3.

Conversely, let $\mathscr{R}$ fulfil Axioms A_0–A_3 and B_1–B_3. Axiom A_0 implies that $\mathscr{R}$ fulfils Axiom A_4. According to our theorem, there exists a visibility Q in G such that (1) holds. Axiom A_0 states that $\mathscr{R}_{(u,v)} \neq \emptyset$ for every pair of vertices u, v of G. Combining this fact with (1), we get $\mathscr{R} = \mathscr{S}(G)$, which completes the proof. □

Remark. Let G be a finite connected graph. The set $\mathscr{S}(G)$ is closely related to the interval function of G in the sense of H.M. Mulder [1]. An "almost non-metric" characterization of the interval function of G was given in [3].

References

[1] *H.M. Mulder*: The Interval Function of a Graph. Mathematisch Centrum, Amsterdam, 1980.

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