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CONNECTIONS ON SOME FUNCTIONAL BUNDLES

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INTRODUCTION

Our starting point was the idea of the Schrödinger connection on a double fibered manifold by Jadczyk and Modugno, [4], [5]. We discuss the "pure case" of two classical fiber bundles E_1 and E_2 over the same base and define a connection Γ on the bundle $\mathscr{F}(E_1, E_2)$ of all smooth maps from a fiber of E_1 into the fiber of E_2 over the same base point. We study systematically the geometry of the iterated tangent bundle of the infinite dimensional space $\mathscr{F}(E_1, E_2)$ as well as the jet prolongations of $\mathscr{F}(E_1, E_2)$ by means of the ideas introduced by the second author in [9]. Since we deal with functional bundles, our vector fields and connections represent a kind of differential operators. That is why we pay special attention to the case of finite order operators, in which we are able to deduce a very concrete description of the objects and operations in question.

In such a situation we found the simplest way for introducing the curvature of Γ in a construction by Ehresmann, [2], which is based on the notion of semiholonomic 2-jets. In the new context we were obliged to rearrange some results, deduced in the finite dimension by direct evaluation, into a more geometrical setting, which could be generalized to our infinite dimensional case. Only then we study the bracket of two vector fields on $\mathscr{F}(E_1, E_2)$. This is a modification of the bracket of two vertical prolongation operators on a classical fibered manifold by Kosmann-Schwarzbach, [11], and the second author, [8]. In Proposition 14 we deduce a satisfactory bracket formula for the curvature of Γ. We also discuss the absolute differentiation with respect to Γ and the special case E_2 is a vector bundle.

If we deal with two finite dimensional manifolds and a map between them, we always assume they are of class C^∞, i.e. smooth in the classical sense. On the other

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hand, the idea of smoothness in the infinite dimension is taken from the theory of smooth structures by Frölicher, [3].

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1. The tangent bundle of $\mathscr{F}(E_1, E_2)$

Let $p_1 \colon E_1 \to M$ and $p_2 \colon E_2 \to M$ be two classical fiber bundles (i.e. locally trivial fibered manifolds) over the same base. Consider the set of all fiber maps

$$\mathscr{F}(E_1, E_2) = \bigcup_{x \in M} C^\infty(E_{1x}, E_{2x})$$

and denote by $p \colon \mathscr{F}(E_1, E_2) \to M$ the canonical projection. We define no topology on $\mathscr{F}(E_1, E_2)$, but we introduce the concept of a smooth map from a classical manifold Q into $\mathscr{F}(E_1, E_2)$.

Definition 1. A map $f \colon Q \to \mathscr{F}(E_1, E_2)$ is called smooth, if
(i) $p \circ f \colon Q \to M$ is smooth and
(ii) the induced map $\tilde{f} \colon (p \circ f)^* E_1 \to E_2$,

$$\tilde{f}(q, y) = f(q)(y), \qquad (q, y) \in (p \circ f)^* E_1$$

is also smooth.

As usual, $(p \circ f)^* E_1 \to Q$ denotes the bundle induced from E_1 by means of $p \circ f$, i.e.

$$(p \circ f)^* E_1 = \{(q, y) \in Q \times E_1 \mid (p \circ f)(q) = p_1(y)\}.$$

Thus, $\mathscr{F}(E_1, E_2)$ is endowed with a smooth structure in the sense of Frölicher, [3].

For every smooth curve $f \colon \mathbb{R} \to \mathscr{F}(E_1, E_2)$ we first construct the tangent vector $X = \frac{\partial}{\partial t}\big|_0 (p \circ f) \in TM$ of its base map at $t = 0$. Write

$$T_X E_1 = (Tp_1)^{-1}(X) \subset TE_1 \quad \text{or} \quad T_X E_2 = (Tp_2)^{-1}(X) \subset TE_2,$$

so that $T_X E_1$ or $T_X E_2$ is an affine bundle over E_{1x} or E_{2x}, $x = p(f(0))$, with the derived vector bundle $T(E_{1x}) := V_x E_1$ or $T(E_{2x}) := V_x E_2$, respectively. Then f defines a map $T_0 f \colon T_X E_1 \to T_X E_2$ by

$$T_0 f\Big(\frac{\partial}{\partial t}\Big|_0 h(t)\Big) = \frac{\partial}{\partial t}\Big|_0 f(t)\big(h(t)\big) \tag{1}$$

where we may assume that $h \colon \mathbb{R} \to E_1$ satisfies $p \circ f = p_1 \circ h$.

Definition 2. We say that two smooth curves $f, g\colon \mathbb{R} \to \mathscr{F}(E_1, E_2)$ satisfying $\frac{\partial}{\partial t}\big|_0 p \circ f = \frac{\partial}{\partial t}\big|_0 p \circ g = X$ determine the same tangent vector at $f(0) = g(0) = \varphi$, if

$$T_0 f = T_0 g\colon T_X E_1 \to T_X E_2.$$

The set $T\mathscr{F}(E_1, E_2)$ of all equivalence classes will be called the tangent bundle of $\mathscr{F}(E_1, E_2)$.

We write $\frac{\partial}{\partial t}\big|_0 f(t) \in T\mathscr{F}(E_1, E_2)$ for the tangent vector determined by f and $\pi\colon T\mathscr{F}(E_1, E_2) \to \mathscr{F}(E_1, E_2)$ and $Tp\colon T\mathscr{F}(E_1, E_2) \to TM$ for the canonical projections. If $A \in T\mathscr{F}(E_1, E_2)$, then we denote by $\tilde{A}\colon T_{Tp(A)}E_1 \to T_{Tp(A)}E_2$ the associated map (1).

Remark 1. Let $\varepsilon \subset \mathscr{F}(E_1, E_2)$ be any subset. Then we define $T\varepsilon \subset T\mathscr{F}(E_1, E_2)$ by restricting ourselves to the smooth curves with values in ε.

One sees easily that $T_0 f = T_0 g\colon T_X E_1 \to T_X E_2$ is an affine bundle morphism over the base map $\varphi\colon E_{1x} \to E_{2x}$ with the derived linear morphism $T\varphi\colon T(E_{1x}) \to T(E_{2x})$. Indeed, let x^i be some local coordinates on M, y^p or z^a be some additional coordinates on E_1 or E_2 and

$$x^i = f^i(t), \qquad z^a = f^a(y^p, t) \tag{2}$$

be the coordinate expression of $f(t)$. Write

$$Y^p = \mathrm{d}y^p, \quad Z^a = \mathrm{d}z^a, \quad \varphi^a(y) = f^a(y, 0), \quad \Phi^a(y) = \frac{\partial f^a(y^p, 0)}{\partial t}.$$

Then the coordinate form of (1) is

$$Z^a = \frac{\partial \varphi^a(y)}{\partial y^p} Y^p + \Phi^a(y). \tag{3}$$

Hence the tangent vector to (2) is locally characterized by two systems of numbers and two systems of functions

$$x^i = f^i(0), \quad X^i = \frac{\partial f^i(0)}{\partial t}, \quad \varphi^a(y^p), \quad \Phi^a(y^p). \tag{4}$$

The following lemma gives a global assertion of such a type.

Lemma 1. *Let $F\colon T_X E_1 \to T_X E_2$ be an affine bundle morphism over $\varphi\colon E_{1x} \to E_{2x}$ with the derived linear morphism $T\varphi\colon T(E_{1x}) \to T(E_{2x})$. Then there exists a smooth curve $f\colon \mathbb{R} \to \mathscr{F}(E_1, E_2)$ such that $F = \tilde{A}$ for the tangent vector $A = \frac{\partial}{\partial t}\big|_0 f(t)$.*

Proof. Consider some local trivializations $U \times S_1$ and $U \times S_2$ of E_1 and E_2 over a neighborhood $U \subset M$ of x. Then $\mathscr{F}(U \times S_1, U \times S_2) = U \times C^\infty(S_1, S_2)$. The restriction of F to $Y^p = 0$ represents a map $\bar{F}\colon S_1 \to TS_2$ along φ. By Proposition 5 from [16] there exists a smooth curve $\gamma\colon \mathbb{R} \to C^\infty(S_1, S_2)$ such that $\bar{F}(y) = \frac{\partial \tilde{\gamma}(y,0)}{\partial t}$, where $\tilde{\gamma}\colon \mathbb{R} \times S_1 \to S_2$ is defined by $\tilde{\gamma}(y,t) = \gamma(t)(y)$. If $\delta\colon \mathbb{R} \to U$ is any curve with $\frac{\partial}{\partial t}\big|_0 \delta = X$, then the curve $(\delta, \gamma)\colon \mathbb{R} \to U \times C^\infty(S_1, S_2)$ has the required property. □

Now we show that each fiber of $T\mathscr{F}(E_1, E_2) \to \mathscr{F}(E_1, E_2)$ is a vector space. Consider $\tilde{A}_1 : T_{X_1}E_1 \to T_{X_1}E_2$ and $\tilde{A}_2\colon T_{X_2}E_1 \to T_{X_2}E_2$ over the same φ. Given $Y \in (T_{X_1+X_2}E_1)_y$, $y \in E_{1x}$, we take any $W \in (T_{X_1}E_1)_y$, so that $Y - W \in (T_{X_2}E_1)_y$, and we define

$$\widetilde{A_1 + A_2}(Y) = \tilde{A}_1(W) + \tilde{A}_2(Y - W).$$

If we select another $\overline{W} \in (T_{X_1}E_1)_y$, then $W - \overline{W}$ is a vertical vector. Hence

$$\tilde{A}_1(\overline{W}) = \tilde{A}_1(W) + T\varphi(\overline{W} - W), \qquad \tilde{A}_2(Y - \overline{W}) = \tilde{A}_2(Y - W) + T\varphi(W - \overline{W}),$$

so that our definition is correct. Further, for $0 \neq k \in \mathbb{R}$ we define

$$\widetilde{kA}\colon T_{kX}E_1 \to T_{kX}E_2 \quad \text{by} \quad \widetilde{kA}(Y) = k\tilde{A}\Big(\frac{1}{k}Y\Big)$$

while for $k = 0$ we prescribe $\widetilde{0A}$ to be $T\varphi\colon T_0E_{1x} \to T_0E_{2x}$. In coordinates, if $A_1 = (x^i, X^i_1, \varphi^a, \Phi^a_1)$ and $A_2 = (x^i, X^i_2, \varphi^a, \Phi^a_2)$, then

$$A_1 + A_2 = (x^i, X^i_1 + X^i_2, \varphi^a, \Phi^a_1 + \Phi^a_2), kA_1 = (x^j, kX^i_1, \varphi^a, k\Phi^a_1). \tag{5}$$

This proves that each $\pi^{-1}(\varphi)$ is a vector space.

In general, consider another pair $E_3 \to N, E_4 \to N$ of fiber bundles over the same base and subset $\varepsilon \subset \mathscr{F}(E_1, E_2)$.

Definition 3. A map $f\colon \varepsilon \to \mathscr{F}(E_3, E_4)$ is called smooth, if $f \circ g\colon Q \to \mathscr{F}(E_3, E_4)$ is smooth for every smooth map $g\colon Q \to \varepsilon$.

Definition 4. A vector field on $\mathscr{F}(E_1, E_2)$ is a smooth map $A\colon \mathscr{F}(E_1, E_2) \to T\mathscr{F}(E_1, E_2)$ satisfying $\pi \circ A = \mathrm{id}$. We say that A is projectable, if there exists a classical smooth vector field $A^0\colon M \to TM$ such that $A^0 \circ \pi = Tp \circ A$.

Write $V\mathscr{F}(E_1, E_2)$ for the kernel of $Tp\colon T\mathscr{F}(E_1, E_2) \to TM$, which will be called the vertical tangent bundle of $\mathscr{F}(E_1, E_2)$. Then we have an exact sequence

$$0 \to V\mathscr{F}(E_1, E_2) \to T\mathscr{F}(E_1, E_2) \to \mathscr{F}(E_1, E_2) \underset{M}{\times} TM \to 0 \tag{6}$$

Consider a linear splitting $\Gamma\colon \mathscr{F}(E_1,E_2)\underset{M}{\times}TM \to T\mathscr{F}(E_1,E_2)$, i.e. $\pi\circ\Gamma = pr_1, Tp\circ\Gamma = pr_2$ and $\Gamma(\varphi,-)\colon T_xM \to T_\varphi\mathscr{F}(E_1,E_2)$ is a linear map for each $\varphi\in\mathscr{F}(E_1,E_2), x=\pi(\varphi)$. Then for every vector field $X\colon M\to TM$ we have defined its Γ-lift $\Gamma X\colon \mathscr{F}(E_1,E_2)\to T\mathscr{F}(E_1,E_2)$. We say that Γ is smooth, if ΓX is smooth for every classical smooth vector field $X\colon M\to TM$.

Definition 5. A connection (in tangent form) on $\mathscr{F}(E_1,E_2)$ is a smooth linear splitting $\Gamma\colon \mathscr{F}(E_1,E_2)\underset{M}{\times}TM\to T\mathscr{F}(E_1,E_2)$.

Remark 2. If E_1 is the trivial fibering $M\to M$, then $\mathscr{F}(E_1,E_2)=E_2$ and we obtain the standard connection on $E_2\to M$.

2. Jet prolongations of $\mathscr{F}(E_1,E_2)$

The simplest way how to define the r-th jet prolongation of $\mathscr{F}(E_1,E_2)$ is based on the concept of the fiber r-jet, [9], [10]. In general, given a fiber bundle $E\to M$ and a manifold N, two maps $f,g\colon E\to N$ are said to determine the same fiber r-jet $j^r_x f = j^r_x g$ at $x\in M$, if $j^r_y f = j^r_y g$ for all $y\in E_x$. Every smooth section s of $\mathscr{F}(E_1,E_2)$ determines the associated base-preserving morphism $\tilde{s}\colon E_1\to E_2$, $\tilde{s}(y)=s(p_1 y)(y)$.

Definition 6. Two sections $s_1,s_2\colon M\to\mathscr{F}(E_1,E_2)$ determine the same r-jet $j^r_x s_1 = j^r_x s_2$ at $x\in M$, if $j^r_x\tilde{s}_1 = j^r_x\tilde{s}_2$. The set $J^r\mathscr{F}(E_1,E_2)$ of all r-jets of the local sections of $\mathscr{F}(E_1,E_2)$ is called the r-jet prolongation of $\mathscr{F}(E_1,E_2)$.

However, it will be useful to discuss another approach as well. Since $\tilde{s}\colon E_1\to E_2$ is a base-preserving morphism, we can construct its r-th jet prolongation $J^r\tilde{s}\colon J^rE_1\to J^rE_2$. Write $J^r_x\tilde{s} = J^r\tilde{s}\big|J^r_xE_1$, $x\in M$. By direct evaluation, one easily verifies.

Proposition 1. *We have $j^r_x s_1 = j^r_x s_2$ iff $J^r_x\tilde{s}_1 = J^r_x\tilde{s}_2$.*

Let $z^a = f^a(x^i,y^p)$ be the coordinate expression of $\tilde{s}$. Then the additional coordinate expression of $J^1_x\tilde{s}$ is

$$z^a_i = \frac{\partial f^a}{\partial x^i} + \frac{\partial f^a}{\partial y^p}y^p_i \tag{7}$$

where y^p_i or z^a_i are the induced coordinates on J^1E_1 or J^1E_2. For $x=0$, the functions $\varphi^a(y^p) := f^a(0,y^p)$ are the coordinates of the target $s(0)$ of $j^1_0 s_1$ and $J^1_0\tilde{s}$ has the form

$$z^a_i = \frac{\partial\varphi^a(y)}{\partial y^p}y^p_i + \varphi^a_i(y),\quad \varphi^a_i(y) = \frac{\partial f^a(0,y)}{\partial x^i} \tag{8}$$

It is well-known that $J^1_x E_1$ or $J^1_x E_2$ is an affine bundle over E_{1x} or E_{2x}, whose derived vector bundle is $V_x E_1 \otimes T^*_x M$ or $V_x E_2 \otimes T^*_x M$, respectively. Obviously, (8) is an affine bundle morphism over φ with the derived linear morphism $T\varphi \otimes \mathrm{id}_{T^*_x M}$. Similarly to §1, we denote by $\widetilde{j^1_x s}$ the associated map $J^1_x \tilde{s}\colon J^1_x E_1 \to J^1_x E_2$. Analogously to Lemma 1, one can prove

Lemma 2. *Let $S\colon J^1_x E_1 \to J^1_x E_2$ be an affine bundle morphism over $\varphi\colon E_{1x} \to E_{2x}$ with the derived linear morphism $T\varphi \otimes \mathrm{id}_{T^*_x M}$. Then there exists a local section s of $\mathscr{F}(E_1, E_2)$ such that $s(x) = \varphi$ and $\widetilde{j^1_x s} = S$.*

By (8), every $X = \frac{\partial}{\partial t}\big|_0 f \in T_x M$ and every $S = j^1_x s$ define a vector

$$S(X) = \frac{\partial}{\partial t}\Big|_0 (s \circ f) \in T_{s(x)} \mathscr{F}(E_1, E_2) \tag{9}$$

such that $Tp(S(X)) = X$.

Definition 7. A connection in the jet form on $\mathscr{F}(E_1, E_2)$ is a smooth section $\Gamma\colon \mathscr{F}(E_1, E_2) \to J^1 \mathscr{F}(E_1, E_2)$ of the target jet projection.

Proposition 2. *The map* (9) *establishes a bijection between the jet form and the tangent form of connections on $\mathscr{F}(E_1, E_2)$.*

Proof. Using (8) we find directly that (9) defines a bijection between the linear splittings $T_x M \to T_\varphi \mathscr{F}(E_1, E_2)$ of Tp and the elements of $J^1 \mathscr{F}(E_1, E_2)_\varphi$. Assume the jet form of Γ is smooth and $f\colon Q \to \mathscr{F}(E_1, E_2)$ is a smooth map, so that $\Gamma \circ f\colon Q \to J^1 \mathscr{F}(E_1, E_2)$ is smooth. For every smooth vector field $X\colon M \to TM$, the map $(\Gamma \circ f)(X \circ p \circ f)$ is also smooth, so that the tangent form of Γ is smooth. Conversely, take a local basis $X_1, \dots, X_m$ of vector fields on TM. Then $(\Gamma X_1) \circ f, \dots, (\Gamma X_m) \circ f$ are smooth maps $Q \to T\mathscr{F}(E_1, E_2)$. By (8) we deduce that $\Gamma \circ f\colon Q \to J^1 \mathscr{F}(E_1, E_2)$ is smooth. □

To define the curvature of a connection of $\mathscr{F}(E_1, E_2)$ in §5, we shall use the second semiholonomic prolongation of $\mathscr{F}(E_1, E_2)$. We recall that $J^1(J^1 E_1 \to M) := \tilde{J}^2 E_1$ is the classical second nonholonomic prolongation of $E_1 \to M$. If x^i, y^p, y^p_i are the above local coordinates of $J^1 E_1$, then the induced coordinates on $\tilde{J}^2 E_1$ are $y^p_{0i} = \frac{\partial y^p}{\partial x^i}$ and $y^p_{ij} = \frac{\partial y^p_i}{\partial x^j}$. We have the target jet projection $\beta_1\colon \tilde{J}^2 E_1 \to J^1 E_1$ and the induced map $J^1\beta\colon \tilde{J}^2 E_1 \to J^1 E_1$ of the target jet projection $\beta\colon J^1 E_1 \to E_1$. An element $Y \in \tilde{J}^2 E_1$ is said to be semiholonomic if $\beta_1(Y) = J^1\beta(Y)$. In coordinates this is characterized by $y^p_i = y^p_{0i}$. All semiholonomic elements form a subbundle $\bar{J}^2 E_1 \subset \tilde{J}^2 E_1$, and the second holonomic prolongation $J^2 E$ is a subbundle of $\bar{J}^2 E$.

Since we have interpreted $J^1 \mathscr{F}(E_1, E_2)$ as a subset of $\mathscr{F}(J^1 E_1, J^1 E_2)$, we have defined $j^1_x \sigma$ for a local smooth section σ of $J^1 \mathscr{F}(E_1, E_2) \to M$ by $j^1_x \tilde{\sigma}$. In this way we introduce the second nonholonomic prolongation $\tilde{J}^2 \mathscr{F}(E_1, E_2)$ of $\mathscr{F}(E_1, E_2)$. An element $j^1_x\sigma$ is said to be semiholonomic, if $\sigma(x) = j^1_x(\beta\circ\sigma)$, where $\beta\colon J^1 \mathscr{F}(E_1, E_2) \to \mathscr{F}(E_1, E_2)$ is the target jet projection. This defines $\bar{J}^2 \mathscr{F}(E_1, E_2) \subset \tilde{J}^2 \mathscr{F}(E_1, E_2)$. The inclusion $J^2 \mathscr{F}(E_1, E_2) \subset \bar{J}^2 \mathscr{F}(E_1, E_2)$ is given by $j^2_x s \mapsto j^1_x(j^1 s)$.

Analogously to the first order case, $j^1_x\sigma$ determines a map $\widetilde{j^1_x\sigma}\colon \tilde{J}^2_x E_1 \to \tilde{J}^2_x E_2$. In coordinates, if $\sigma = (f^a(x, y), f^a_i(x, y))$, then $\tilde{s}$ is of the form

$$z^a = f^a(x,y), \qquad z^a_i = \frac{\partial f^a(x,y)}{\partial y^p} y^p_i + f^a_i(x,y). \tag{10}$$

Hence

$$\varphi^a(y) = f^a(0,y), \quad \varphi^a_i = f^a_i(0,y), \varphi^a_{0i} = \frac{\partial f^a(0,y)}{\partial x^i}, \quad \varphi^a_{ij} = \frac{\partial f^a_i(0,y)}{\partial x^j} \tag{11}$$

are the coordinates of $j^1_0\sigma$. From (10) we obtain the coordinate expression of $\widetilde{j^1_x\sigma}$ in the form $z^a = \varphi^a(y)$ and

$$z^a_i = \frac{\partial \varphi^a}{\partial y^p} y^p_i + \varphi^a_i, \qquad z^a_{0i} = \frac{\partial \varphi^a}{\partial y^p} y^p_{0i} + \varphi^a_{0i}, \tag{12}$$

$$z^a_{ij} = \varphi^a_{ij} + \frac{\partial \varphi^a_i}{\partial y^p} y^p_{0j} + \frac{\partial \varphi^a_{0j}}{\partial y^p} y^p_i + \frac{\partial^2 \varphi^a}{\partial y^p \partial y^q} y^p_i y^q_{0j} + \frac{\partial \varphi^a}{\partial y^p} y^p_{ij}.$$

Using (12) we deduce directly the following assertion.

Proposition 3. *$j^1_x\sigma$ is semiholonomic or holonomic iff $\widetilde{j^1_x\sigma}$ maps $\bar{J}^2_x E_1$ into $\bar{J}^2_x E_2$ or $J^2_x E_1$ into $J^2_x E_2$, respectively.*

In coordinates, an element of $\bar{J}^2 \mathscr{F}(E_1, E_2)$ is characterized by $\varphi^a_i = \varphi^a_{0i}$ and the additional condition for a holonomic element is $\varphi^a_{ij} = \varphi^a_{ji}$.

We remark that the higher order nonholonomic and semiholonomic prolongations of $\mathscr{F}(E_1, E_2)$ can be defined in a quite similar way.

3. The finite order case

Since both vector fields from §1 and the connections from §2 are defined on a functional bundle, they represent a kind of differential operators. We are going to describe the simplest case of finite order operators.

Definition 8. A projectable vector field $A\colon \mathscr{F}(E_1, E_2) \to T\mathscr{F}(E_1, E_2)$ over $A^0\colon M \to TM$ is of order r, if the condition $j^r_y\varphi = j^r_y\psi$, $\varphi, \psi \in C^\infty(E_{1x}, E_{2x})$, $y \in E_{1x}$ implies that the restrictions of $\widetilde{A(\varphi)}$ and $\widetilde{A(\psi)}$ over y coincide, i.e.

$$\widetilde{A(\varphi)}\big|(T_{A^0(x)}E_1)_y = \widetilde{A(\psi)}\big|(T_{A^0(x)}E_1)_y. \tag{13}$$

Let $S(TE_1, TE_2)$ be the set of all affine morphism $(T_X E_1)_y \to (T_X E_2)_z$, $p_1 y = p_2 z = \pi_M X$, where $\pi_M \colon TM \to M$ is the bundle projection. This is a fibered manifold over $E_1 \underset{M}{\times} E_2 \underset{M}{\times} TM$. Write

$$\mathscr{F} J^r(E_1, E_2) = \bigcup_{x \in M} J^r(E_{1x}, E_{2x}).$$

This is a classical manifold as well.

A projectable r-th order vector field $A\colon \mathscr{F}(E_1, E_2) \to T\mathscr{F}(E_1, E_2)$ over A^0 defines the associated map $\mathscr{A}\colon \mathscr{F} J^r(E_1, E_2) \to S(TE_1, TE_2)$ by

$$\mathscr{A}(j^r_y \varphi) = A(\varphi) \big| (T_{A^0(x)} E_1)_y. \tag{14}$$

Proposition 4. *The associated map of a projectable r-th order vector field on $\mathscr{F}(E_1, E_2)$ is a classical C^∞-map.*

P r o o f. This follows from the fact that A is smooth in the sense of Definition 3 quite analogously to [6]. □

The local coordinates on $\mathscr{F} J^r(E_1, E_2)$ induced by x^i, y^p and z^a are z^a_α, $1 \leqslant |\alpha| \leqslant r$, where α is a multiindex, the range of which is the fiber dimension of E_1. Hence the coordinate form of $\mathscr{A}$ is $X^i(x^j)$ and

$$\Phi^a = \Phi^a(x^i, y^p, z^a_\alpha), \qquad 0 \leqslant |\alpha| \leqslant r. \tag{15}$$

The derived linear map of each element of $S(TE_1, TE_2)$ is identified with an element of $\mathscr{F} J^1(E_1, E_2)$. This defines a map $D\colon S(TE_1, TE_2) \to \mathscr{F} J^1(E_1, E_2)$ and the following diagram commutes:

$$\begin{array}{ccc}
\mathscr{F} J^1(E_1, E_2) & & \\
\uparrow{\scriptstyle \beta_r} & \overset{D}{\nwarrow} & \\
\mathscr{F} J^r(E_1, E_2) & \xrightarrow{\mathscr{A}} & S(TE_1, TE_2) \\
\downarrow & & \downarrow \\
E_1 \underset{M}{\times} E_2 & \xrightarrow{\mathrm{id} \underset{M}{\times} A^0} & E_1 \underset{M}{\times} E_2 \underset{M}{\times} TM
\end{array}$$

where β_r is the jet projection. Conversely, let $\mathscr{A}\colon \mathscr{F} J^r(E_1, E_2) \to S(TE_1, TE_2)$ be a smooth map with an underlying vector field $A^0\colon M \to TM$ such that (16) commutes. Then the rule

$$\widetilde{A(\varphi)} = \bigcup_{y \in E_{1x}} \mathscr{A}(j^r_y \varphi) \tag{17}$$

defines a projectable r-th order vector field A on $\mathscr{F}(E_1, E_2)$.

Since $T\mathscr{F}(E_1, E_2)$ is a subset of $\mathscr{F}(TE_1, TE_2)$, we can define the second tangent bundle $T(T\mathscr{F}(E_1, E_2))$. This will be described in more detail in §6. Here we restrict ourselves to a general remark, which is related to our study of the order of connections.

Definition 9. A vector field $A\colon \mathscr{F}(E_1, E_2) \to \mathscr{F}(E_1, E_2)$ is called differentiable if the formula

$$TA\Big(\frac{\partial}{\partial t}\Big|_0 f\Big) = \frac{\partial}{\partial t}\Big|_0 A \circ f \tag{18}$$

defines a smooth map $TA\colon T\mathscr{F}(E_1, E_2) \to TT\mathscr{F}(E_1, E_2)$.

From (16) we easily deduce (see the coordinate formula in §6) the following assertion.

Proposition 5. *Every r-th order vector field on $\mathscr{F}(E_1, E_2)$ is differentiable.*

Definition 10. A connection $\Gamma\colon \mathscr{F}(E_1, E_2) \to J^1\mathscr{F}(E_1, E_2)$ is of order r if the condition $j^r_y\varphi = j^r_y\psi$, $\varphi, \psi \in C^\infty(E_{1x}, E_{2x})$, $y \in E_{1x}$, implies

$$\widetilde{\Gamma(\varphi)}\big|J^1_y E_1 = \widetilde{\Gamma(\Psi)}\big|J^1_y E_1. \tag{19}$$

Let $S(J^1E_1, J^1E_2)$ be the set of all affine maps $(J^1E_1)_y \to (J^1E_2)_z$ with the derived linear map of the form

$$B \otimes \mathrm{id}_{T^*_x M} \qquad B \in \mathscr{L} \in (V_yE_1, V_zE_2). \tag{20}$$

An r-th order connection $\Gamma\colon \mathscr{F}(E_1, E_2) \to J^1\mathscr{F}(E_1, E_2)$ defines the associated map $\mathscr{G}\colon \mathscr{F}J^r(E_1, E_2) \to S(J^1E_1, J^1E_2)$ by

$$\mathscr{G}(j^r_y\varphi) = \widetilde{\Gamma(\varphi)}\big|J^1_y E_1. \tag{21}$$

The coordinate form of $\mathscr{G}$ is

$$\Phi^a_i = \Phi^a_i(x^i, y^p, z^a_\alpha), \qquad 0 \leqslant |\alpha| \leqslant r. \tag{22}$$

Analogously to Proposition 4, one proves

Proposition 6. *The associated map of an r-th order connection $\mathscr{F}(E_1, E_2) \to J^1\mathscr{F}(E_1, E_2)$ is a classical C^∞-map.*

Let $D\colon S(J^1E_1, J^1E_2) \to \mathscr{F}J^1(E_1,E_2)$ be the map defined by (20). Then the following diagram commutes

$$
\begin{array}{ccc}
\mathscr{F}J^1(E_1,E_2) & & \\
\Big\uparrow{\scriptstyle \beta_r} & \nwarrow{\scriptstyle D} & \\
\mathscr{F}J^r(E_1,E_2) & \xrightarrow{\mathscr{G}} & S(J^1E_1, J^1E_2)
\end{array}
\tag{23}
$$

Conversely, let $\mathscr{G}\colon \mathscr{F}J^r(E_1,E_2) \to S(J^1E_1, J^1E_2)$ be a smooth morphism over the identity of $E_1 \times E_2$ such that (23) commutes. Then the rule

$$\widetilde{\Gamma(\varphi)} = \bigcup_{y \in E_{1x}} \mathscr{G}(j^r_y \varphi) \tag{24}$$

defines an r-th order connection on $\mathscr{F}(E_1,E_2)$.

Analogously to Definition 9, we introduce

Definition 11. A connection $\Gamma\colon \mathscr{F}(E_1,E_2) \to J^1\mathscr{F}(E_1,E_2)$ is called differentiable if the formula

$$J^1\Gamma(j^1_x s) = j^1_x(\Gamma \circ s) \tag{25}$$

defines a smooth map $J^1\mathscr{F}(E_1,E_2) \to \tilde{J}^2\mathscr{F}(E_1,E_2)$.

Proposition 7. *Every r-th order connection is differentiable.*

Proof. We deduce from (22) the coordinate form of $J^1\Gamma$ in some coordinates x^i, φ^a, ψ^a_i on $J^1\mathscr{F}(E_1,E_2)$ and $x^i, \varphi^a, \varphi^a_i, \varphi^a_{0i}, \varphi^a_{ij}$ on $\tilde{J}^2\mathscr{F}(E_1,E_2)$. Take a section σ

$$z^a = \Psi^a(x^i, y^p) \tag{26}$$

so that $\varphi^a = \psi^a(0,y)$ and $\psi^a_i = \frac{\partial\psi^a(0,y)}{\partial x^i}$. Then we obtain for $\Gamma \circ \sigma$

$$z^a_i = \frac{\partial\psi^a(x,y)}{\partial y^p}\, y^p_i + \Phi^a_i(x, y, \partial_\alpha\psi^a(x,y)). \tag{27}$$

Now (26) yields

$$z^a_{0i} = \frac{\partial\psi^a(0,y)}{\partial y^p}\, y^p_{0i} + \frac{\partial\psi^a(0,y)}{\partial x^i}, \quad \text{i.e.} \quad \varphi^a_{0i} = \psi^a_i \tag{28}$$

and (27) implies

$$\begin{aligned} z^a_{ij} = {} & \frac{\partial\psi^a_j}{\partial y^p} y^p_i + \frac{\partial^2\varphi^a}{\partial y^p \partial y^q} y^p_i y^q_{0j} + \frac{\partial\varphi^a}{\partial y^p} y^p_{ij} + \frac{\partial\varphi^a_i}{\partial y^p} y^p_{0j} \\ & + \frac{\partial\Phi^a_i}{\partial x^j} + \frac{\partial\Phi^a_i}{\partial z^b}\partial_j\psi^b + \ldots + \frac{\partial\Phi^a_i}{\partial z^b_\alpha}\partial_\alpha\partial_j\psi^b. \end{aligned} \tag{29}$$

In particular, (29) shows that $J^1\Gamma$ is well-defined and smooth. □

Following Virsik, [17], if Γ is differentiable and Δ is another connection $\mathscr{F}(E_1, E_2) \to J^1 \mathscr{F}(E_1, E_2)$, we define a section

$$\Gamma * \Delta = J^1\Gamma \circ \Delta\colon \mathscr{F}(E_1, E_2) \to \tilde{J}^2 \mathscr{F}(E_1, E_2)\,. \tag{30}$$

The order of such a section can be introduced similarly to Definition 10.

Proposition 8. *If* Γ *and* Δ *are connections of orders* r *and* s*, respectively, then* $\Gamma * \Delta$ *has the order* $r + s$.

Proof. We substitute the associated map of Δ into (28) and (29). □

To obtain an explicit formula for the associated map of $\Gamma * \Delta$, we introduce the following concept. Having a smooth function $f\colon \mathscr{F}J^r(E_1, E_2) \to \mathbb{R}$, we define its formal differential Df by

$$Df\colon \mathscr{F}J^{r+1}(E_1, E_2) \to V^*E_1, Df(j^{r+1}_y\varphi) = d_y f(j^r\varphi). \tag{31}$$

Then every vertical vector field μ on V^*E_1 determines $\langle Df, \mu\rangle : \mathscr{F}J^{r+1}(E_1, E_2) \to \mathbb{R}$. For the coordinate vector fields $\frac{\partial}{\partial y^p}$ we obtain the formal derivatives

$$D_p f = \frac{\partial f}{\partial y^p} + \frac{\partial f}{\partial z^a} z^a_p + \ldots + \frac{\partial f}{\partial z^a_\alpha} z^a_{\alpha+p}. \tag{32}$$

By iteration, we introduce $D_\beta f\colon \mathscr{F}J^{r+|\beta|}(E_1, E_2) \to \mathbb{R}$. Let $\Psi^a_i(x^i, y^p, z^a_\beta)$, $0 \leqslant |\beta| \leqslant s$, be associated map of Δ. Then the coordinate form of the main term of (29) is

$$\varphi^a_{ij} = \frac{\partial \Phi^a_i}{\partial x^j} + \frac{\partial \Phi^a_i}{\partial z^b}\Psi^b_j + \frac{\partial \Phi^a_i}{\partial z^b_p} D_p \Psi^b_j + \ldots + \frac{\partial \Phi^a_i}{\partial z^b_\alpha} D_\alpha \Psi^b_j. \tag{33}$$

Remark 3. In both cases of connections in the jet form and of projectable vector fields we have a situation somewhat similar to the vertical prolongation operators on classical fibered manifolds studied by Kosmann-Schwarzbach, [11], and the second author, [8]. In [10] Slovák deduced that every vertical prolongation operator is differentiable in the sense of our Definitions 9 and 11. However, his proof is based on quite sophisticated procedures in mathematical analysis, so that we have the feeling that such a problem in our setting is beyond the scope of the present paper.

We describe some properties of connections on a classical fibered manifold $p\colon E \to M$ in a way which can be generalized to $\mathscr{F}(E_1, E_2)$. Given $A \in J^1_y E$ and $B \in T_x M, x = py$, we denote by $A(B) \in T_y E$ the A-lift of B. We show that every $A \in \tilde{J}^2_y E$ induces similarly a lifting $\lambda A\colon TT_x M \to TT_y E$. If $A = J^1_x \sigma$ and $B = \frac{\partial}{\partial t}\big|_0 f(t) \in TT_x M$, then we construct $\sigma(\pi(f(t)))(f(t))\colon \mathbb{R} \to TE$ and set

$$\lambda A(B) = \frac{\partial}{\partial t}\Big|_0 \sigma(\pi(f(t)))(f(t)) \tag{34}$$

where $\pi\colon TM \to M$ is the bundle projection. Given some local fiber coordinates x^i, y^p on E, we have the induced coordinates y^p_i, y^p_{0i}, y^p_{ij} on $\tilde{J}^2 E$, the induced coordinates X^i, Y^p on TE and the additional coordinates on TEE denoted by a dot. Then one finds easily the following coordinate form of (34):

$$Y^p = y^p_i X^i, \quad \dot{y}^p = y^p_{0i}\dot{x}^i, \quad \dot{Y}^p = y^p_{ij} X^i \dot{x}^j + y^p_i \dot{X}^i. \tag{35}$$

Let κ be the canonical involution of the second tangent bundle. If $A \in \bar{J}^2_y E$, then $\kappa_E \circ \lambda A \circ \kappa_M\colon TT_x M \to TT_y E$ is the lifting of another element $\kappa A \in \bar{J}^2_y E$, [15]. In coordinates, $y^p_{ji}(\kappa A) = y^p_{ij}(A)$. Hence A is holonomic iff $\kappa A = A$. Since $\bar{J}^2_y E \to J^1 E$ is an affine bundle with the derived vector bundle $VE \otimes (\otimes^2 T^* M)$, the points κA and A determine a vector $\Delta(A) := \overrightarrow{(\kappa A) A} \in V_y E \otimes \Lambda^2 T^*_x M$, which is called the deviation (or difference tensor) of A, [7], [12]. The coordinates of $\Delta(A)$ are $y^p_{ij} - y^p_{ji}$. If $X_1, X_2 \in T_x M$, then we have $\Delta(A)(X_1, X_2) \in V_y E$.

Let $\pi_1 = \pi_{TM} = TTM \to TM$ and $\pi_2 = T\pi_{TM} = TTM \to TM$ be the canonical projections. Consider $C, D \in TT_x M$ satisfying

$$\pi_1(C) = \pi_2(D) \quad \text{and} \quad \pi_1(D) = \pi_2(C). \tag{36}$$

Since κ exchanges the two projections, C and κD are in the same fiber of TTM with respect to π_1 and satisfy $\pi_2(C - \kappa D) = 0$. Hence $C - \kappa D$ is a tangent vector to a fiber of TM and such a vector can be identified with an element of $T_x M$, which will be denoted by $C \dot{-} D$ and called the strong difference of C and D. In coordinates, if

$$C \equiv (a^i, b^i, c^i), D \equiv (b^i, a^i, d^i) \quad \text{then} \quad C \dot{-} D \equiv (c^i - d^i). \tag{37}$$

In [8] it is deduced the the bracket $[X, Y]$ of two vector fields $X, Y\colon M \to TM$ can be expressed by

$$[X, Y] = TY \circ X \dot{-} TX \circ Y. \tag{38}$$

Lemma 3. *Let $C, D \in TT_xM$ satisfy the condition* (36) *for the strong difference and $A \in \bar{J}^2_y E$. Then $\lambda A(C)$, $\lambda A(D)$ also satisfy* (36) *and*

$$\Delta A(\pi_1 C, \pi_2 C) = (\lambda A(C) \dot{-} \lambda A(D)) - \beta_1(A)(C \dot{-} D)$$

where $\beta_1 \colon \bar{J}^2_1 E \to J^1 E$ is the jet projection.

Proof. By (35) and (37) we have $\lambda A(C) = (y^p_i a^i, y^p_i b^i, y^p_{ij} a^i b^j + y^p_i c^i)$, $\lambda A(D) = (y^p_i b^i, y^p_i a^i, y^p_{ij} b^i a^j + y^p_i d^i)$. This implies our claim. □

According to Remark 2, two connections $\Gamma, \Delta \colon E \to J^1 E$ determine $\Gamma * \Delta = J^1\Gamma \circ \Delta \colon E \to \tilde{J}^2 E$. For $\Gamma = \Delta$ the values of $\Gamma * \Gamma$ lie in $\bar{J}^2_y E$. In this case we obtain a construction closely related to an idea by Ehresmann, [2].

Definition 12. The map $\tilde{\Gamma} = J^1\Gamma \circ \Gamma \colon E \to \bar{J}^2 E$ is the Ehresmann prolongation of Γ. The composition

$$C\Gamma := -\Delta \circ \tilde{\Gamma} \colon E \to VE \otimes \Lambda^2 T^* M \tag{39}$$

is the curvature of Γ.

To deduce that $C\Gamma$ coincides with the standard curvature of Γ, we need a property of the lifting map

$$\lambda\tilde{\Gamma} \colon E \underset{M}{\times} TTM \to TTE.$$

Consider two vector fields $X, Y \colon M \to TM$, so that $TX \circ Y \colon M \to TTM$.

Lemma 4. *We have*

$$\lambda\tilde{\Gamma}(TX \circ Y) = (T\Gamma X) \circ \Gamma Y \colon E \to TTE.$$

Proof. We have $\tilde{\Gamma}(y) = j^1_x(\Gamma \circ s)$, $j^1_x s = \Gamma(y)$. If $Y(x) = \frac{\partial}{\partial t}\big|_0 f(t)$, then

$$TX(Y(x)) = \frac{\partial}{\partial t}\Big|_0 (X \circ f).$$

By (34),

$$\lambda\tilde{\Gamma}(TX(Y(x))) = \frac{\partial}{\partial t}\Big|_0 \Gamma(s(f(t)))(X(f(t))) = (T\Gamma X \circ \Gamma Y)(y).$$

□

Proposition 9. *For every vector fields $X, Y : M \to TM$, we have*

$$C\Gamma(X,Y) = [\Gamma X, \Gamma Y] - \Gamma([X,Y]).$$

Proof. Consider $TX \circ Y$, $TY \circ X : M \to TTM$. By Lemma 4 we obtain

$$\lambda\tilde{\Gamma}(TX \circ Y) = T\Gamma X \circ \Gamma Y \quad \text{and} \quad \lambda\tilde{\Gamma}(TY \circ X) = T\Gamma Y \circ \Gamma X.$$

Then Lemma 3 and (38) imply

$$\Delta \circ \tilde{\Gamma}(X,Y) = (\lambda\tilde{\Gamma}(TX \circ Y)\dot{-}\lambda\tilde{\Gamma}(TY \circ X)) - \Gamma(TX \circ Y \dot{-} TY \circ X) = \\ = -[\Gamma X, \Gamma Y] + \Gamma([X,Y]).$$

□

5. The curvature of a connection on $\mathscr{F}(E_1, E_2)$

The deviation of an element $j^1_x\sigma \in \tilde{J}^2\mathscr{F}(E_1,E_2)$ can be defined by means of the associated map $\widetilde{j^1_x\sigma} : \bar{J}^2_x E_1 \to \bar{J}^2_x E_2$. In the semiholonomic case we have $\varphi^a_i = \varphi^a_{0i}$. So if we take a holonomic 2-jet $Y \in J^2_x E_1$, then the right-hand side of the second line in (12) is symmetric except the first term. Hence the deviation $\Delta(\widetilde{j^1_x\sigma}(Y))$ is independent of y^p_i and y^p_{ij}. This defines a map $\Delta(j^1_x\sigma): E_{1x} \to V_x E_2 \odot \Lambda^2 T^*_x M$ over φ, i.e. an element of $\mathscr{F}(E_1, VE_2 \otimes \Lambda^2 T^*M)$.

Definition 13. $\Delta(j^1_x\sigma)$ is called the deviation of $j^1_x\sigma$. The coordinate form of $\Delta(j^1_x\sigma)$ is $\varphi^a_{ij} - \varphi^a_{ji}$.

Definition 14. For a differentiable connection $\Gamma : \mathscr{F}(E_1,E_2) \to J^1\mathscr{F}(E_1,E_2)$, the map $\tilde{\Gamma} := J^1\Gamma \circ \Gamma : \mathscr{F}(E_1,E_2) \to \tilde{J}^2\mathscr{F}(E_1,E_2)$ is the Ehresmann prolongation of Γ.

Definition 15. The composition

$$C\Gamma := -\Delta \circ \tilde{\Gamma} : \mathscr{F}(E_1,E_2) \to \mathscr{F}(E_1, VE_2 \otimes \Lambda^2 T^*M)$$

is the curvature of a differentiable connection $\Gamma : \mathscr{F}(E_1,E_2) \to J^1\mathscr{F}(E_1,E_2)$.

Clearly, $C\Gamma$ is a section of the canonical projection $\mathscr{F}(E_1, VE_2 \otimes \Lambda^2 T^*M) \to \mathscr{F}(E_1,E_2)$.

Let Γ be an r-th order connection with the associated map $\Phi^a_i(x^i, y^p, z^a_\alpha)$. Then we obtain the associated map of $C\Gamma$ by setting $\Psi^a_i = \Phi^a_i$ in (33) and by antisymmetrizing in i and j. This implies

Proposition 10. *The curvature of an r-th order connection has the order $2r$.*

As remarked in §3, the inclusion $T\mathscr{F}(E_1, E_2) \subset \mathscr{F}(TE_1 \to TM, TE_2 \to TM)$ defines the second tangent bundle $T(T\mathscr{F}(E_1, E_2)) = TT\mathscr{F}(E_1, E_2)$. We have a projection $TTp\colon TT\mathscr{F}(E_1, E_2) \to TTM$ and two projections π_T, $T\pi\colon TT\mathscr{F}(E_1, E_2) \to T\mathscr{F}(E_1, E_2)$. In the above coordinates, consider an element $F \in TT\mathscr{F}(E_1, E_2)$ tangent to a curve $x^j(t)$, $X^i(t)$, $f^a(y,t)$ and

$$Z^{\dot{a}} = \frac{\partial f^a(y,t)}{\partial y^p} Y^p + \Phi^a(y,t).$$

Then its associated map $\tilde{F}\colon TT_xE_1 \to TT_xE_2$, $X = TTp(F)$, is of the form

$$Z^a = \frac{\partial \varphi^a}{\partial y^p} Y^p + \Phi^a(y), \dot{z}^a = \frac{\partial \varphi^a}{\partial y^p} \dot{y}^p + f^a(y) \tag{40}$$

$$\dot{Z}^a = F^a(y) + \frac{\partial \Phi^a}{\partial y^p}\dot{y}^p + \frac{\partial f^a}{\partial y^p} Y^p + \frac{\partial^2 \varphi^a}{\partial y^p \partial y^q} Y^p \dot{y}^q + \frac{\partial \varphi^a}{\partial y^p}\dot{Y}^p.$$

So φ^a, Φ^a, f^a, F^a are the functional coordinates of F, which are completed by the coordinates x^i, X^i, $\dot{x}^i$, $\dot{X}^i$ of $X \in TTM$. The coordinate form of π_T or $T\pi$ is

$$\pi_T(x^i, X^i, \dot{x}^i, \dot{X}^i, \varphi^a, \Phi^a, f^a, F^a) = (x^i, X^i, \varphi^a, \Phi^a),$$

$$T\pi(x^i, X^i, \dot{x}^i, \dot{X}^i, \varphi^a, \Phi^a, f^a, F^a) = (x^i, \dot{x}^i, \varphi^a, f^a).$$

Consider the canonical involution κ_{E_1} or κ_{E_1} of the second tangent bundle.

Proposition 11. *For every $F \in TT\mathscr{F}(E_1, E_2)$ over $X \in TTM$ there exists a unique element $\kappa F \in TT\mathscr{F}(E_1, E_2)$ such that its associated map $\widetilde{\kappa F}\colon TT_{\kappa_M X}E_1 \to TT_{\kappa_M X}E_2$ is $\widetilde{\kappa F} = \kappa_{E_2} \circ \tilde{F} \circ \kappa_{E_1}$.*

Proof. This follows from (40). □

Obviously, the coordinate form of κ is

$$\kappa(x, X, \dot{x}, \dot{X}, \varphi, \Phi, f, F) = (x, \dot{x}, X, \dot{X}, \varphi, f, \Phi, F). \tag{41}$$

Consider $C, \overline{C} \in TT\mathscr{F}(E_1, E_2)$ over $X, \overline{X} \in TTM$ satisfying

$$\pi_T(C) = T\pi(\overline{C}) \quad \text{and} \quad \pi_T(\overline{C}) = T\pi(C). \tag{42}$$

Then we define the strong difference $C \mathbin{\dot{-}} \overline{C} \in T\mathscr{F}(E_1, E_2)$, $Tp(C \mathbin{\dot{-}} \overline{C}) = X \mathbin{\dot{-}} \overline{X}$, as follows. For every $B \in (T_{X \dot{-} \overline{X}}E_1)_y$ we take any Y, $\overline{Y} \in (TTE_1)_y$ over X, $\overline{X}$

such that $Y \dot{-} \bar{Y} = B$. Then one easily verifies that $C(Y)$, $\bar{C}(\bar{Y})$ also satisfy (42), $C(Y) \dot{-} \bar{C}(\bar{Y})$ depends on C, $\bar{C}$ and B only and represents the associated map of an element $C \dot{-} \bar{C} \in T\mathscr{F}(E_1, E_2)$, whose coordinates are

$$(x^i, \dot{X}^i - \dot{\bar{X}}^i, \varphi^a, F^a - \bar{F}^a). \tag{43}$$

Let A, B be two differentiable vector fields on $\mathscr{F}(E_1, E_2)$. Then the maps $TA \circ B, TB \circ A\colon \mathscr{F}(E_1, E_2) \to TT\mathscr{F}(E_1, E_2)$ satisfy the condition (42) at every $\varphi \in \mathscr{F}(E_1, E_2)$.

Definition 16. The vector field

$$[A, B] := TB \circ A \dot{-} TA \circ B\colon \mathscr{F}(E_1, E_2) \to T\mathscr{F}(E_1, E_2)$$

is called the bracket of A and B.

By (38) we immediately deduce

Proposition 12. *If A and B are projectable over A^0 and B^0, then $[A, B]$ is projectable over $[A^0, B^0]$.*

Assume A is of order r and B is of order s with the associated maps $X^i(x)$, $A^a(x^i, y^p, z^a_\alpha)$, $|\alpha| \leqslant r$ and $Y^i(x)$, $B^a(x^i, y^p, z^a_\beta)$, $|\beta| \leqslant s$, respectively. Analogously to §3, the fourth component of the associated map of $TA \circ B$ is

$$\frac{\partial A^a}{\partial x^i} Y^i + \frac{\partial A^a}{\partial z^b} B^b + \frac{\partial A^a}{\partial z^b_p} D_p B^b + \ldots + \frac{\partial A^a}{\partial Z^b_\alpha} D_\alpha B^b, \qquad |\alpha| \leqslant r \tag{44}$$

while the fourth component of the associated map of $TB \circ A$ is

$$\frac{\partial B^a}{\partial x^i} X^i + \frac{\partial B^a}{\partial z^b} A^b + \frac{\partial B^a}{\partial z^b_p} D_p A^b + \ldots + \frac{\partial B^a}{\partial z^b_\beta} D_\beta A^b, \qquad |\beta| \leqslant s. \tag{45}$$

Hence we can summarize by

Proposition 13. *The bracket $[A, B]$ has the order $r + s$ and its associated map is $[A^0, B^0]$ and the difference* (45)–(44).

We are going to generalize Proposition 9 to connections on $\mathscr{F}(E_1, E_2)$. First of all we remark that every $A = j^1_x \sigma \in \tilde{J}^2 \mathscr{F}(E_1, E_2)_\varphi$ defines a lifting $\lambda A\colon TT_x M \to TT_\varphi \mathscr{F}(E_1, E_2)$ by

$$\lambda A\Big(\frac{\partial}{\partial t}\Big|_0 f\Big) = \frac{\partial}{\partial t}\Big|_0 \sigma(\pi_M(f(t))(f(t)).$$

In coordinates, if $A = (x^i, \varphi^a, \varphi^a_i, \varphi^a_{0i}, \varphi^a_{ij})$ and $B = \frac{\partial}{\partial t}\big|_0 f = (x^i, X^i, \dot{x}^i, \dot{X}^i)$, then one easily finds the following coordinate form of $\lambda A(B)$:

$$(x^i, \varphi^a, \varphi^a_i X^i, \varphi^a_{0i}\dot{x}^i, \varphi^a_{ij} X^i \dot{x}^j + \varphi^a_i \dot{X}^i). \tag{46}$$

This directly implies the following generalization of Lemma 3.

Lemma 5. *Let $C, D \in TT_x M$ satisfy the condition* (36) *for the strong difference and $A \in \bar{J}^2 \mathscr{F}(E_1, E_2)$. Then $\lambda A(C)$, $\lambda A(D)$ satisfy* (42) *and*

$$\Delta A(\pi_T C, T\pi C) = (\lambda A(C) \dot{-} \lambda A(D)) - \beta_1(A)(C \dot{-} D).$$

Now we need an assumption of technical character (which is fulfilled for every finite order connection).

Definition 17. *A differentiable connection $\Gamma\colon \mathscr{F}(E_1, E_2) \to J^1 \mathscr{F}(E_1, E_2)$ is called strongly differentiable, if ΓX is a differentiable vector field on $\mathscr{F}(E_1, E_2)$ for every smooth vector field $X\colon M \to TM$.*

Proposition 14. *For every strongly differentiable connection Γ on $\mathscr{F}(E_1, E_2)$ and for all vector fields X, Y on M we have*

$$C\Gamma(X, Y) = [\Gamma X, \Gamma Y] - \Gamma([X, Y]).$$

Proof. In the same way as in Lemma 4 we deduce $\lambda\tilde{\Gamma}(TX \circ Y) = (T\Gamma X) \circ \Gamma Y$. Then we apply Lemma 5. □

7. The absolute differentiation

Let $A, B \in J^1 \mathscr{F}(E_1, E_2)_\varphi$ be two 1-jets with the same target φ. To deduce that their difference is an element $A - B \in \mathscr{F}(E_1, VE_2 \otimes T^*M)$ over φ, we consider the associated maps $\tilde{A}, \tilde{B}\colon J^1_x E_1 \to J^1_x E_2$,

$$A \equiv z^a_i = \frac{\partial \varphi^a}{\partial y^p} y^p_i + \Phi^p_i(y), \qquad B \equiv z^a_i = \frac{\partial \varphi^a(x, y)}{\partial y^p} y^p_i + \Psi^p_i(y).$$

The element $A(Y) - B(Y)$ is independent of the choice of $Y \in J^1_x E_1$, which defines a map $E_{1x} \to VE_2 \otimes T^*M$ over φ. (In this sense $J^1 \mathscr{F}(E_1, E_2)$ is an affine bundle with the derived vector bundle $\mathscr{F}(E_1, VE_2 \otimes T^*M)$ analogously to the classical case.)

Let $s\colon M \to \mathscr{F}(E_1, E_2)$ be a section and $\Gamma\colon \mathscr{F}(E_1, E_2) \to J^1 \mathscr{F}(E_1, E_2)$ a connection.

Definition 18. The absolute differential

$$\nabla s \colon M \to \mathscr{F}(E_1, VE_2 \otimes T^*M)$$

is the above difference $\nabla s(x) = j^1_x s - \Gamma(s(x))$.

If $X : M \to TM$ is a vector field, we define the absolute derivative of s with respect to X by

$$\nabla_X s = \langle \nabla s, X \rangle : M \to \mathscr{F}(E_1, VE_2) \tag{47}$$

where $\langle \;,\; \rangle$ is the extension of the evaluation map $T \underset{M}{\times} T^* \to \mathbb{R}$. Having an r-th order connection with the associated map (22) and a section s of the form $z^a = \varphi^a(x, y)$, then the coordinate form of ∇s is

$$\frac{\partial \varphi^a(x,y)}{\partial x^i} - \Phi^a_i(x^i, y^p, \partial_\alpha \varphi^a(x,y)). \tag{48}$$

To obtain $\nabla_x s$, we contract (48) with the coordinate functions $X^i(x)$ of X.

Remark 4. In the case $E_1 = E_2 := E$ we have a distinguished section $I\colon M \to \mathscr{F}(E, E)$, $I(x) = \mathrm{id}_{E_x}$. Analogously to the case of a classical linear connection on TM, the absolute differential $\nabla I \colon M \to \mathscr{F}(E, VE \otimes T^*M)$ can be called the torsion of a connection Γ on $\mathscr{F}(E, E)$. By (48), the coordinate form of the torsion of an r-th order connection is $-\Phi^p_i(x^i, y^p, y^p, \delta^p_q, 0, \dots, 0)$.

It might be instructive to discuss a special case in more detail. Let $E \to M$ be a vector bundle. Consider the subspace $LE \subset \mathscr{F}(E, E)$ of all linear maps, which is a classical vector bundle over M. A connection Γ on LE in our sense is a classical general connection on LE. Hence our approach leads to the original idea of the torsion of a general connection Γ on LE. If w^p_q are the induced fiber coordinates on LE, the usual coordinate expression of Γ is $\mathrm{d}w^p_q = F^p_{qi}(x^j, w^r_s)\,\mathrm{d}x^i$. Then $-F^p_{qi}(x^j, \delta^r_s)$ is the coordinate form of the torsion of Γ. Of course, if we take for Γ the tensor product $\Delta \otimes \Delta^*$ of a linear connection Δ on E and of the dual connection Δ^* on E^*, [10], then the torsion of $\Delta \otimes \Delta^*$ vanishes, for I is invariant with respect to $\Delta \otimes \Delta^*$.

8. The vector bundle case

Assume $p\colon E_2 \to M$ is a vector bundle. Then each fiber of $\mathscr{F}(E_1, E_2)$ is a vector space, provided the linear operations on $C^\infty(E_{1x}, E_{2x})$ are defined by extending the linear operations on E_{2x}. In other words, $\mathscr{F}(E_1, E_2) \to M$ is a vector bundle over sets, cf. [4]. Such a vector bundle structure is further extended to $J^1\mathscr{F}(E_1, E_2)$ by

$$j^1_x s_1 + j^1_x s_2 = j^1_x(\tilde{s}_1 + \tilde{s}_2), \qquad j^1_x(ks) = j^1_x k\tilde{s}, \qquad k \in \mathbb{R}$$

with addition and multiplication by reals in E_2. Hence $J^1\mathscr{F}(E_1, E_2) \to M$ also is a vector bundle over sets.

Definition 19. A connection $\Gamma\colon \mathscr{F}(E_1, E_2) \to J^1\mathscr{F}(E_1, E_2)$ is called linear if Γ is a linear morphism over M.

In the case of an r-th order linear connection, its associated map (22) has the form

$$\Phi^a_{ib}(x, y)z^b + \Phi^{aq}_{ib}(x, y)z^b_q + \ldots + \Phi^{a\alpha}_{ib} z^b_\alpha. \tag{49}$$

If E_2 is a vector bundle, then $VE_2 = E_2 \underset{M}{\times} E_2$, which implies

$$\mathscr{F}(E_1, VE_2 \otimes \Lambda^2 T^*M) = \mathscr{F}(E_1, E_2) \underset{M}{\times} \mathscr{F}(E_1, E_2 \otimes \Lambda^2 T^*M).$$

In this case, analogously to the classical situation, the curvature will be interpreted as the second component of the map from Definition 15,

$$C\Gamma\colon \mathscr{F}(E_1, E_2) \to \mathscr{F}(E_1, E_2 \otimes \Lambda^2 T^*M),$$

while the first component is the identity.

Proposition 15. *For every differentiable linear connection Γ, the map $C\Gamma$: $\mathscr{F}(E_1, E_2) \to \mathscr{F}(E_1, E_2 \otimes \Lambda^2 T^*M)$ is a linear morphism over M.*

Proof. One easily verifies that in the linear case both $\tilde{\Gamma}$ and Δ in Definition 15 are linear morphisms over M. □

Quite similarly, if E_2 is a vector bundle, then the absolute derivative $\nabla_X s$ of a section s with respect to a vector field X on M is identified with the second component of (47), so that it is section of $\mathscr{F}(E_1, E_2)$ as well.

We finally remark that several other ideas from the classical theory of connections can be generalized to the case of $\mathscr{F}(E_1, E_2)$. The most interesting ones could be the vertical prolongation of Γ, the connections on $T\mathscr{F}(E_1, E_2) \subset \mathscr{F}(TE_1 \to TM, TE_2 \to TM)$ or a detailed study of the absolute differentiation in the linear case. Such a research can be based on some general ideas from the theory of classical connections collected in the book [10].

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