Czechoslovak Mathematical Journal

Dănuţ Marcu

A note on graph colouring

Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 2, 231–233

Persistent URL: http://dml.cz/dmlcz/128528

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

A NOTE ON GRAPH COLOURING

DĂNUŢ MARCU, Bucharest

(Received March 1, 1993)

Graphs considered here are undirected, finite and simple (without loops or multiple edges), and [1] is followed for terminology. Let $G = (V, E)$ be a graph, with V the set of vertices and E the set of edges. We denote by $\gamma(G)$ the chromatic number of G. A *k-colouring* of G ($k \geqslant \gamma(G)$) is a partition of V into k classes such that no two vertices belonging to the same class are adjacent. For a graph G with $1 < \gamma(G) \leqslant 3$ we denote by $K(3, G)$ the number of 3-colourings of G.

Theorem 1. *If G is a connected graph with $n \geqslant 3$ vertices and $\gamma(G) = 2$, then $K(3, G) \leqslant 2^{n-2} - 1$, and the only connected graphs for which this upper bound is attained are trees with n vertices.*

Proof. We prove this result by induction on n. Obviously, for $n = 3$ the result is true. Let now G be a connected graph with $n+1$ ($n+1 \geqslant 3$) vertices and $\gamma(G) = 2$. According to [1], G contains at least two vertices which are not cut-vertices. Let v be such a vertex and $G - v$ the connected graph obtained from G by removing v. According to [2], a connected graph with the chromatic number equal to two has a unique 2-colouring. Thus, by induction hypothesis, we have

$$K(3, G) \leqslant 1 + 2K(3, G - v) \leqslant 1 + 2(2^{n-2} - 1) = 2^{n-1} - 1$$

since, on the one hand, in view of the fact that $G - v$ has a unique 2-colouring, there exists a unique 3-colouring of G which has a class consisting only of v, and, on the other hand, v may be added to a 3-colouring of $G - v$ in at most two different ways (the graph being connected, v is joined by an edge with at least one vertex of $G - v$ which belongs to a class of the 3-colouring of $G - v$). We observe that the equality holds only if v is joined with a single vertex of $G - v$ which is a tree. Thus, G is also a tree. □

According to [3], the chromatic polynomial of a cycle C_n with n vertices is given by

$$(1) \qquad P(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1).$$

On the other hand, also by [3], if G is a graph with n vertices and $\gamma(G) = k$, then the number of k-colourings of G is equal to

$$(2) \qquad P(G;k)/k!.$$

Since an odd cycle has the chromatic number equal to three, then, by (1) and (2), we have

Theorem 2. *If C_{2k+1} is a cycle with $2k+1$ vertices, then $K(3, C_{2k+1}) = \frac{1}{3}(4^k - 1)$.*

Theorem 3. *If G is a connected graph with $n \geqslant 3$ vertices and $\gamma(G) = 3$, then*

$$K(3,G) \leqslant \begin{cases} \frac{1}{3}(2^{n-1}-1) & \text{if } n \text{ is odd,} \\ \frac{2}{3}(2^{n-2}-1) & \text{if } n \text{ is even.} \end{cases}$$

If n is odd, the single connected graph for which this upper bound is attained is the cycle C_n, and if n is even, the single connected graph for which this upper bound is attained consists of the cycle C_{n-1} and another vertex which is joined by an edge with a vertex of C_{n-1}.

Proof. We denote

$$g(n) = \begin{cases} \frac{1}{3}(2^{n-1}-1) & \text{if } n \text{ is odd,} \\ \frac{2}{3}(2^{n-2}-1) & \text{if } n \text{ is even.} \end{cases}$$

We prove this result by induction on n. For $n = 3$, the single connected graph G with three vertices and $\gamma(G) = 3$ is a 3-clique and this graph is uniquely 3-colourable, the result being true. So, suppose that the result is true for n and let G be a connected graph with $n+1$ ($n+1 \geqslant 3$) vertices and $\gamma(G) = 3$. As in the proof of Theorem 1, let v be a vertex of G such that $G - v$ is connected. If $\gamma(G-v) = 3$, then

$$K(3,G) \leqslant 2K(3,G-v) \leqslant 2g(n) \leqslant g(n+1),$$

since $2g(n) = g(n+1)$ if n is odd, $2g(n)+1 = g(n+1)$ if n is even and, G being connected, v is joined by an edge with at least one vertex which belongs to a class of every 3-colouring of $G - v$, that is, v may be placed into at most two classes of a

3-colouring of $G - v$. By induction hypothesis, the equality of $K(3, G)$ and $g(n+1)$ holds only if n is odd and v is joined by an edge with a single vertex of the odd cycle C_n.

If $\gamma(G - v) = 2$, then there exists a 3-colouring of G of the form $\{v\}$, I_1, I_2, where I_1 and I_2 are independent sets. If G is the complementary graph of a perfect graph, then, by definition, it contains a triangle, that is, a cycle (v, u_1, v_1, v) such that $u_1 \in I_1$ and $v_1 \in I_2$.

Otherwise, by Berge's Theorem [1], it contains an odd cycle $(v, u_1, v_1, u_2, v_2, \ldots, u_k, v_k, v)$ such that $u_i \in I_1$ and $v_i \in I_2$ for $1 \leqslant i \leqslant k$ and $k \geqslant 2$. Thus, in any case, G contains an odd cycle $C_{2k+1} = (v, u_1, v_1, u_2, v_2, \ldots, u_k, v_k, v)$ with $u_i \in I_1$, $v_i \in I_2$, $1 \leqslant i \leqslant k$ and $k \geqslant 1$. Then we can remove the edges which are the diagonals of this odd cycle, and the number of 3-colourings of the graph increases. Moreover, the subgraph induced by an odd cycle has the chromatic number equal to three and, therefore, by this operation, the chromatic number of the graph does not decrease and the graph remains connected. Since G is connected, there exists at least one vertex w of G which does not belong to C_{2k+1} and which is joined by an edge with at least one vertex of the cycle. If there are more edges joining w with vertices of the cycle, we may preserve a single edge and the resulting graph is also connected, with the chromatic number equal to three, and the number of its 3-colourings increases. If we find a vertex z which is joined by an edge with at least one vertex from the set $\{v, w, u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$, we repeat this construction, etc. Thus,

$$K(3, G) \leqslant 2^{n-2k} \cdot K(3, C_{2k+1}) \leqslant g(n+1),$$

since $2g(n) = g(n+1)$ for n odd, $2g(n) < g(n+1)$ for n even, as the reader can easily verify, and every new vertex may be added to at most two classes of the 3-colouring already obtained (since the graph is connected). Now, by using the induction hypothesis and the definition of $g(n)$, it follows that the graph for which this upper bound is attained is unique and has the form indicated in the theorem. □

References

[1] *C. Berge*: Graphes et Hypergraphes. Dunod, Paris, 1970.
[2] *F. Harary, S. T. Hedetniemi and R. Robinson*: Uniquely colorable graphs. J. Combin. Theory *6* (1969), 264–270.
[3] *R. C. Read*: An introduction to chromatic polynomials. J. Combin. Theory *4* (1968), 52–71.

Author's address: Str. Pasului 3, Sect. 2, 70241 Bucharest, Romania.