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ON THE JAKUBÍK PROBLEM ON RADICAL CLASSES OF LATTICE ORDERED GROUPS

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The Jakubík Problem on radical classes of lattice-ordered groups is an open question raised by J. Jakubík in 1977 in [1] as follows:

Let σ be a radical class with $A'(\sigma) \neq \emptyset$ ($A'(\sigma)$ denotes the class of all antiatoms over σ). Is there $\tau \in S$ with $\sigma < \tau < \varepsilon'(\sigma)$ such that $A'(\tau) = \emptyset$, where S is the lattice of all radical classes?

J. Jakubík himself solved the above problem in the case of σ being a principal radical class in [1]. In this paper, a complete solution for the general case is given (Theorem 4).

All of the notions and terminologies that concern radical classes in this paper are from [1], those related to lattice-ordered groups from [2].

The radical class of all lattice-ordered groups will be denoted by $\mathcal{G}$.

Let $G \in \mathcal{G}$. The least radical class containing G is called the principal radical class generated by G. We denote it by $T(G)$.

Let $\sigma \in S$. The symbol $\sigma(G)$ stands for the largest solid subgroup of G which belongs to σ.

Let $\sigma, \eta \in S$, and $\sigma < \eta$. The interval $[\sigma, \eta]$ is the class of all radical τ with $\sigma \leqslant \tau \leqslant \eta$. If $[\sigma, \eta]$ contains exactly two elements, then we call η an atom over σ; in this case, we also say that η covers σ (alternatively, σ is covered by η). If there are no atoms over σ contained in $[\sigma, \eta]$, then we call η an antiatom over σ. $A'(\sigma)$ denotes the class of all antiatoms over σ. Note that $\varepsilon'(\sigma)$, the supremum of $A'(\sigma)$, is also an element of $A'(\sigma)$.

In this paper, $\omega(\alpha)$ has the usual meaning, i.e. it is the least ordinal having cardinality α.

The following lemma is proved in [1].

Lemma 1. *Let $\sigma, \eta \in S$, $A'(\sigma) \neq \emptyset \neq A'(\eta)$, suppose $\sigma < \eta < \varepsilon'(\sigma)$. Then $\varepsilon'(\eta) \leqslant \varepsilon'(\sigma)$. If σ is a principal radical class, then $\varepsilon'(\eta) < \varepsilon'(\sigma)$.*

For a principal radical class, from the above lemma and Proposition 5.8 in [1] we infer that $A'(\varepsilon'(\eta)) = \emptyset$. Therefore, the answer to Jakubík Problem is affirmative in this case.

Let α be an infinite cardinal and let I be a dual ideal of $\omega(\alpha)$. For $G \in \mathcal{G}$, put $G(\alpha) = (\vec{\otimes} G_i) \vec{\otimes} G$, $G_i = Z$ $(i \in I)$, Z being the additive group of integers with the usual order. Note that $G(\alpha)$ is the lexicographic product of these G_i and G $(i \in I)$ with the ordering from left to right. Write $G_t^0 = \{g \in G(\alpha) \mid g(i) = 0$, for each $i \in I\}$. Obviously, G_t^0 is isomorphic to G.

Lemma 2. *Suppose $G_t^0 \subsetneq H \in C(G(\alpha))$. Then H has a solid subgroup isomorphic with $G(\alpha)$.*

This is Lemma 3.1 in [1].

From the proof of Proposition 3.3 in [1] we infer

Proposition 3. *For the above $G(\alpha)$, $T(G(\alpha))$ covers $T(G)$ and $T(G)(G(\alpha)) = G_t^0$.*

We can now construct a radical class which satisfies all conditions in the Jakubík question, and therefore gives an affirmative answer to the question.

Theorem 4. *Let $\sigma \in S$, $A'(\sigma) \neq \emptyset$. Then there exists a radical class η such that $\sigma < \eta < \varepsilon'(\sigma)$ with $A'(\eta) = \emptyset$.*

Proof. There is an ℓ-group $G \in \varepsilon'(\sigma) \setminus \sigma$. Put $\tau = \sigma \vee T(G)$. There are two cases as follows:

(i) σ is comparable with $T(G)$. In this case, $\sigma < T(G)$ and $\tau = \sigma \vee T(G) = T(G)$ is a principal radical class. By Lemma 1 and Proposition 5.2 in [1], we have $A'(\tau) \neq \emptyset$. Thus $\tau < \varepsilon'(\sigma)$. Therefore, $\sigma < \tau < \varepsilon'(\tau) < \varepsilon'(\sigma)$. Put $\eta = \varepsilon'(\tau)$. Proposition 5.8 in [1] says that $A'(\eta) = \emptyset$.

(ii) σ is not comparable with $T(G)$. Then τ is finitely $\vee$-decomposable. By Proposition 3.5 in [1], $A'(\tau) \neq \emptyset$. Put $\eta = \varepsilon'(\tau)$, we have $\sigma < \eta \leqslant \varepsilon'(\sigma)$ and $A'(\eta) = \emptyset$.

Let $Z = \tau \vee T(G(\alpha))$, then $[\tau, Z] = [\sigma \vee T(G), \sigma \vee T(G) \vee T(G(\alpha))]$.

From the projectivity of intervals $[\sigma \vee T(G), \sigma \vee T(G \vee T(G(\alpha)))]$, $[(\sigma \vee T(G)) \wedge T(G(\alpha)), T(G(\alpha))]$ we infer that if $\sigma \vee T(G) \wedge T(G(\alpha)) = T(G(\alpha))$, then $\tau = Z$, that is $\sigma \vee T(G) = \sigma \vee T(G(\alpha))$. Hence $G(\alpha) = (\sigma \vee T(G))(G(\alpha)) = \sigma(G(\alpha)) \vee T(G)(G(\alpha)) = \sigma(G(\alpha)) \vee G_t^0$, which implies $G(\alpha) = \sigma(G(\alpha))$. Thus $G(\alpha) \in \sigma$ and consequently $G \in \sigma$, a contradiction. So it must be the case that $\big(\sigma \vee T(G)\big) \wedge T\big(G(\alpha)\big) < T\big(G(\alpha)\big)$. Note that $T\big(G(\alpha)\big)$ covers $T(G)$. Then $\sigma \vee T(G) = T(G)$. Therefore, Z covers τ.

Now suppose that Z is not an antiatom over σ. Then there is $Y \in S$ such that Y covers σ with $Y < Z$. Obviously, $Y \vee \tau \leqslant Z$ and $Y \wedge \tau = \sigma$. Moreover, from the projectivity of intervals $[\tau, Y \vee \tau]$ and $[Y \wedge \tau, Y] = [\sigma, Y]$ we obtain that τ is covered by $Y \vee \tau$. So we have $Z = Y \vee \tau$. Thus, $G(\alpha) = Z\big(G(\alpha)\big) = (Y \vee \tau)\big(G(\alpha)\big) = \big(Y \vee \sigma \vee T(G)\big)\big(G(\alpha)\big) = \big(Y \vee T(G)\big)\big(G(\alpha)\big) = Y\big(G(\alpha)\big) \vee G^0_t$. Hence, $G(\alpha) = Y\big(G(\alpha)\big)$, i.e. $G(\alpha) \in Y$, which implies $Y \geqslant \sigma \vee T\big(G(\alpha)\big) = Z$, then $Y = Z$, which is not the case. Therefore, $Z \in A'(\sigma)$.

We have $Z \leqslant \varepsilon'(\sigma)$. If $\varepsilon'(\sigma) = \varepsilon'(\tau)$, then τ is covered by Z and $Z \leqslant \varepsilon'(\tau)$, a contradiction. Therefore, $\varepsilon'(\sigma) > \varepsilon'(\tau) = \eta$. This completes the proof. □

References

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