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Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 1, 21–38

Persistent URL: `http://dml.cz/dmlcz/128508`

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

ON INFINITE PARTITION REPRESENTATIONS AND THEIR FINITE QUOTIENTS

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(Received September 1, 1992)

0. INTRODUCTION

The problem of representing finite lattices as intervals in subgroup lattices of finite groups has remained unsolved since its formulation in [3]. Its equivalent version—the problem of representing finite lattices as congruence lattices of finite algebras—was formulated even much earlier, see [1]. To prove that a given finite lattice L is not isomorphic to an interval in the subgroup lattice of any finite group $\mathbf{H}$ would require to exclude, in particular, top intervals in subgroup lattices of finite simple groups. It seems that this is beyond the reach of current understanding of the subgroup structure of finite simple groups despite the classification of finite simple groups and the subsequent description of their maximal subgroups in a series of papers that have been published since mid-eighties.

On the other hand, to represent a given lattice L as an interval in the subgroup lattice of a finite group $\mathbf{H}$ does not require any use of the classification of finite simple groups. It is sufficient to describe the group and the interval. However, things are not so easy. Any construction of the group $\mathbf{H}$ would have to take into account the O'Nan-Scott theorem classifying finite primitive groups of permutations, and no methods of constructions of finite groups based on the theorem have been developed so far.

One direction in which we could search for such constructions can be extracted from [6], where it is proved that every algebraic lattice L is isomorphic to an interval in the subgroup lattice of an infinite group $\mathbf{H}$. The emphasis in [6] is on the combinatorial aspects of the construction. From the group-theoretical point of view

This paper was written while the author was an Alexander von Humboldt-Fellow at Fachbereich Mathematik, Technische Hochschule Darmstadt. Excellent conditions provided by Arbeitsgruppe 1 are gratefully acknowledged.

the construction of $\mathbf{H}$ goes as follows. First we represent the lattice $\Pi(C(L))$, the lattice of all partitions of the set $C(L)$ of compact elements of L, as an interval $[G_0, G]$ in the subgroup lattice of a group $\mathbf{G}$. There is a canonical representation of L as a meet-subsemilattice of $\Pi(C(L))$ (recall that L is isomorphic to the lattice of non-empty ideals of the partially ordered set $C(L)$ with the order inherited from L). To every non-empty ideal I in $C(L)$ we assign the partition of $C(L)$ with just one non-trivial block I. Then we take an infinite group $\mathbf{F}$ (the free product of many copies of the two-element group), and consider the imprimitive action of the wreath product of $\mathbf{F}$ and the action of $\mathbf{G}$ on the set A of the left cosets of G_0. On the set $F \times A$ we define a relation $\sim$. Then we consider the subgroup $\mathbf{H}$ of the wreath product of $\mathbf{F}$ and $\mathbf{G}$ containing all permutations preserving the relation $\sim$. The relation has the property that $\mathbf{H}$ contains as a proper subgroup a semidirect product of $\mathbf{F}$ and $\mathbf{G}$. The interval in the subgroup lattice of $\mathbf{H}$ between the stabilizer of a block of the equivalence closure of $\sim$ and the group $\mathbf{H}$ itself is isomorphic to L.

Which parts of the construction could be made finite, if we start with a finite lattice L? The paper [5] contains representations of finite boolean lattices, finite partition lattices, and finite quasi-ordering lattices as intervals in subgroup lattices of finite groups. Any of these classes has the property that every finite lattice can be represented as a meet-subsemilattice of a lattice from the class. The major problem is to replace the infinite free product $\mathbf{F}$ by a finite group $\mathbf{W}$.

In this paper we investigate a special case of the problem of replacing the free product $\mathbf{F}$ by a finite group $\mathbf{W}$. A complete solution of the case would result in a new (and much more efficient) proof of the finite partition lattice representation theorem [4]. We use the concept of a twisting structure that appeared first in [6]. A twisting structure, roughly speaking, is a set of partial permutations of order 2 defined on subsets of a set A. We restrict ourselves to permutations of order 2 for technical reasons to avoid the necessity of writing exponents to various powers of the given permutation or its inverse. The mathematical content of the paper would not change by relaxing this restriction. We formulate two fairly general sufficient conditions the group $\mathbf{W}$ has to satisfy and present a construction of a finite group satisfying one of the two conditions.

The paper is organized as follows. After some preliminaries in the first section we introduce the concept of twists and twisting structures in the second section and prove the existence of the twisting structures we need. The main result of the third and fourth sections is a construction of an infinite partition representation of finite lattices based on twisting structures. The third section deals with injectivity and join-preservation. The fourth section introduces some fairly general conditions under which the representation preserves meets. In the last section we look for the possibility of finding a finite representation based on twisting structures.

Finally, let us mention that the idea of twists and twisting structures can be formalized also in the language of HNN-extensions of the combinatorial theory of groups. This approach enabled the author to find in [8] a short and direct proof that every lattice can be embedded in an infinite partition lattice.

1. Meet-preserving mappings into partition lattices

Let L and M be complete lattices. A mapping $\varphi\colon L \to M$ is a complete meet-morphism if it preserves arbitrary meets. In particular, φ maps the greatest element of L to the greatest element of M.

A complete meet-morphism φ from a lattice L into a partition lattice $\Pi(A)$ induces a metric-like structure on A: for $a, b \in A$ we define an element $d_\varphi(a,b)$ as the least element $x \in L$ such that $(a,b) \in \varphi(x)$. The element $d_\varphi(a,b)$ is well-defined since φ preserves arbitrary meets; in fact

$$d_\varphi(a,b) = \bigwedge\{x \in L\colon (a,b) \in \varphi(x)\}. \tag{1.1}$$

We shall refer to $d_\varphi(a,b)$ as the *distance* of a and b with respect to φ. It follows immediately that for any $x \in L$ we have

$$(a,b) \in \varphi(x) \quad \text{if and only if} \quad d_\varphi(a,b) \leqslant x. \tag{1.2}$$

There is also a triangle inequality:

$$d_\varphi(a,c) \leqslant d_\varphi(a,b) \vee_L d_\varphi(b,c) \quad \text{for every } a,b,c \in A. \tag{1.3}$$

To prove this observe that $(a,b) \in \varphi(d_\varphi(a,b))$ and $(b,c) \in \varphi(d_\varphi(b,c))$. It follows that the pair (a,c) is contained in $\varphi(d_\varphi(a,b)) \vee \varphi(d_\varphi(b,c)) \leqslant \varphi(d_\varphi(a,b) \vee_L d_\varphi(b,c))$, since φ is order-preserving. Now (1.3) follows from (1.2).

A set $B \subseteq A$ is called *balanced* with respect to φ if for every $a \in A$ the set of distances $\{d_\varphi(a,b)\colon b \in B\}$ contains a least element. Every one-element subset of A and the set A itself are balanced with respect to every $\varphi\colon L \to \Pi(A)$. Before presenting more interesting examples of balanced sets we recall some elementary facts about symmetric groups.

By $\mathrm{Sym}(A)$ we denote the symmetric group on a set A. If π is a partition of A, then S_π denotes the set of permutations on A preserving all blocks of π, i.e.

$$S_\pi = \{s \in \mathrm{Sym}(A)\colon (a, s(a)) \in \pi \quad \text{for all } a \in A\}. \tag{1.4}$$

The subgroups S_π are usually called the *parabolic subgroups* of $\mathrm{Sym}(A)$. The mapping assigning to each $\pi \in \Pi(A)$ the parabolic subgroup S_π is a complete lattice embedding of $\Pi(A)$ into the subgroup lattice of $\mathrm{Sym}(A)$. Further, the subgroup lattice of any group G is embedded into the partition lattice $\Pi(G)$ in the obvious way: to a subgroup H of G we assign the partition of G into the left cosets of H. The composition of these two mappings (in the case $G = \mathrm{Sym}(A)$) gives a complete lattice embedding ε of $\Pi(A)$ into $\Pi(\mathrm{Sym}(A))$. The embedding can also be described directly:

$$(1.5) \qquad (p,q) \in \varepsilon(\pi) \quad \text{if and only if} \quad p^{-1}q \in S_\pi.$$

Next we describe the distance of two permutations $p, q \in \mathrm{Sym}(A)$ with respect to ε. If $r \in \mathrm{Sym}(A)$, then $\gamma(r)$ denotes the partition of A into the orbits of r, i.e. $(a,b) \in \gamma(r)$ if and only if a, b belong to the same cycle of r. Then we have

$$(1.6) \qquad d_\varepsilon(p,q) = \gamma(p^{-1}q).$$

Take a partition $\pi \in \Pi(A)$. By (1.5), $(p,q) \in \varepsilon(\pi)$ if and only if $p^{-1}q \in S_\pi$. But $p^{-1}q \in S_\pi$ if and only if $(a, p^{-1}q(a)) \in \pi$ for all $a \in A$. It follows that $\pi \geqslant \gamma(p^{-1}q)$. Hence $d_\varepsilon(p,q) \geqslant \gamma(p^{-1}q)$. Since obviously $p^{-1}q \in S_{\gamma(p^{-1}q)}$, we get $(p,q) \in \varepsilon(\gamma(p^{-1}q))$. This proves $d_\varepsilon(p,q) \leqslant \gamma(p^{-1}q)$.

Note also that the left regular representation of $\mathrm{Sym}(A)$ preserves every partition $\varepsilon(\pi)$:

$$(1.7) \qquad \text{if } (p,q) \in \varepsilon(\pi) \text{ and } r \in \mathrm{Sym}(A), \text{ then } (rp, rq) \in \varepsilon(\pi).$$

It follows that

$$(1.8) \qquad d_\varepsilon(p,q) = d_\varepsilon(rp, rq) \quad \text{for every } p, q, r \in \mathrm{Sym}(A).$$

This fact will be used in the proof of the following proposition describing a class of balanced sets.

Proposition 1.1. *Suppose that $\pi \in \Pi(A)$ has just one block containing more than one element. Then every left coset of S_π is balanced with respect to ε.*

P r o o f. Let us take a coset qS_π and $p \in \mathrm{Sym}(A)$. We have to prove that the set $\{d_\varepsilon(p, qs) \colon s \in S_\pi\}$ contains a least element. By (1.8), we may assume that p is the identity permutation i on A. Denote the non-trivial block of π by B. We may further assume that $d_\varepsilon(i,q)$ is a minimal element of the set of distances $\{d_\varepsilon(i, qs) \colon s \in S_\pi\}$. If $a, b \in B$ are different and belong to the same block of $d_\varepsilon(i,q) = \gamma(q)$, then take the element $qt_{ab} \in qS_\pi$, where t_{ab} denotes the transposition of a, b. The partition $\gamma(qt_{ab})$ is strictly smaller than $\gamma(q)$—the block of $\gamma(q)$ containing a, b is divided into two strictly smaller blocks in $\gamma(qt_{ab})$. This contradicts the minimality of $d_\varepsilon(i,q)$. Thus $d_\varepsilon(i,q) = \gamma(q)$ contains no two different elements of B in the same block. If $i \neq s \in S_\pi$, then $\gamma(qs)$ is strictly greater than $\gamma(q)$ since, for every $b \in B$, the blocks of $\gamma(q)$ containing b and bs are contained in the same block of $\gamma(qs)$. This proves $d_\varepsilon(i,q) \leqslant d_\varepsilon(i,qs)$ for every $s \in S_\pi$ and completes the proof that qS_π is balanced with respect to ε. □

Proposition 1.2. *Let $\psi\colon M \to \Pi(A)$ and $\varphi\colon L \to M$ be complete meet-morphisms. If $B \subseteq A$ is balanced with respect to ψ, then it is also balanced with respect to $\psi\varphi\colon L \to \Pi(A)$.*

Proof. First observe that for every $a, b \in A$ we have $(a,b) \in \psi\varphi(x)$ if and only if $d_\psi(a,b) \leqslant \varphi(x)$. It follows that $d_{\psi\varphi}(a,b)$ is the least element $x \in L$ such that $d_\psi(a,b) \leqslant \varphi(x)$.

Now choose any $a \in A$ and let $c \in B$ be such that $d_\psi(a,c)$ is the least element of the set $\{d_\psi(a,b)\colon b \in B\}$. If $y \in L$ satisfies $\varphi(y) \geqslant d_\psi(a,b)$, then also $\varphi(y) \geqslant d_\psi(a,c)$. Hence $y \geqslant d_{\psi\varphi}(a,c)$ for any such y. In particular, $d_{\psi\varphi}(a,b) \geqslant d_{\psi\varphi}(a,c)$. □

2. Twists and twisting structures

By a (symmetric) *twist* on a set A we mean a permutation defined on a subset of A. The *domain* $D(t)$ of t is the subset of A on which t is defined. If $a \in D(t)$, then the value of t in a will be written as at. This way of writing the value of a twist at a point will be kept throughout the whole paper.

Twists can be used to distinguish certain partitions of $\Pi(A)$. Let t be a twist on A and π a partition of A. We say that π is *preserved* by t if

$$\text{(2.1)} \qquad \text{whenever } (a,b) \in \pi \text{ and } a,b \in D(t), \text{ then } (at,bt) \in \pi.$$

A partition $\pi \in \Pi(A)$ is preserved by a set $\mathcal{T}$ of twists if it is preserved by every $t \in \mathcal{T}$. By $L_\mathcal{T}$ we denote the set of all partitions of A preserved by $\mathcal{T}$. The proof of the following proposition is straightforward and left to the reader.

Proposition 2.1. *The set $L_\mathcal{T}$ ordered by the relation of inclusion is a complete meet-subsemilattice of $\Pi(A)$.*

We say that a set $\mathcal{T}$ of twists on a set A is a *twisting structure* on A if all twists $t \in \mathcal{T}$ have order two.

Example 2.2. Let L be a finite lattice. We define a twisting structure $\mathcal{T}$ on the set L in the following way. All twists of $\mathcal{T}$ will be transpositions defined on certain three-element subsets of L. If $0 < x < y$, then we define a twist t on the subset $\{0, x, y\}$ of L, t transposes 0 and x. If $x \neq y$ are not comparable, then we take to $\mathcal{T}$ one of the following two transpositions defined on the subset $\{x, y, x \vee y\}$ of L: either the transposition of x and $x \vee y$ or the transposition of y and $x \vee y$. The former twists are called the *twists of the first type*, the latter ones are called the *twists of the second type.*

Next we state two simple properties of the twisting structure $\mathcal{T}$ on L.

Proposition 2.3. *If* $|L| = n + 1$, *then* $|\mathcal{T}| = \binom{n}{2}$.

For every $x \in L$ we define a partition $\varphi(x)$ of L by

(2.2) $$\varphi(x) \text{ has just one non-trivial block } \{y\colon y \leqslant x\}.$$

The mapping $\varphi\colon L \to \Pi(L)$ is obviously injective and preserves arbitrary meets. In fact, we have the following proposition.

Proposition 2.4. *A partition* $\pi \in \Pi(L)$ *is preserved by the twisting structure* $\mathcal{T}$ *of Example 2.2 if and only if it is of the form* $\varphi(x)$ *for some* $x \in L$. *Hence* φ *is an isomorphism between* L *and* $L_{\mathcal{T}}$.

Proof. It is easy to observe that all $\varphi(x)$, $x \in L$, are preserved by every twist $t \in \mathcal{T}$.

On the other hand, if $\pi \in \Pi(L)$ is preserved by $\mathcal{T}$ and $(y, z) \in \pi$ for some non-comparable $y, z \in L$, then we apply the twist t defined on the subset $\{y, z, y \vee z\}$ to the pair (y, z). Then either $(yt, zt) = (y \vee z, z)$ or $(yt, zt) = (y, y \vee z)$. In both cases we get that $y \vee z$ belongs to the block of π containing y, z. Hence every non-trivial block of π contains a largest element x. From the fact that every block of π is also closed under the twists of the first type we obtain that every non-trivial block of π coincides with the interval of L between 0 and the largest element x of the block. Hence there is just one non-trivial block and $\pi = \varphi(x)$. □

One disadvantage of the twisting structure $\mathcal{T}$ described in Example 2.2 is that the domains of twists of $\mathcal{T}$ are not balanced with respect to the injective meet-morphism φ defined by (2.2). That is why we modify the twisting structure $\mathcal{T}$ in the following way.

Example 2.5. We set $A = \mathrm{Sym}(L)$. Let $t \in \mathcal{T}$ be a twist on L. Thus t is defined on a 3-element subset $\{x, y, z\}$ of L and transposes, say, x and y. By t_{xy} we denote the transposition of x and y defined on the whole set L, and by π we denote the partition of L with just one non-trivial block $\{x, y, z\}$. For every coset pS_π of S_π we define a twist u on A such that

$$D(u) = pS_\pi \quad \text{and} \quad su = st_{xy}. \tag{2.3}$$

The twist u depends on the twist t and the coset pS_π. We will call it the *lift* of t to the coset pS_π, or briefly, a lift of t. The twist t will be referred to as the *base* of u. The set of lifts of all $t \in \mathcal{T}$ will be denoted by $\mathcal{U}$. We will call the twisting structure $\mathcal{U}$ the *symmetric lift* of $\mathcal{T}$.

Proposition 2.6. *If* $|L| = n+1$, *then* $|A| = (n+1)!$ *and* $|\mathcal{U}| = \frac{1}{6}\binom{n}{2}(n+1)!$.

Proof. Both formulas follow immediately from Proposition 2.3 and from the definitions of A and $\mathcal{U}$.

Recall from the previous section that ε denotes the complete lattice embedding of $\Pi(L)$ into $\Pi(\mathrm{Sym}(L))$ defined by (1.5). □

Proposition 2.7. *If* $\sigma \in \Pi(L)$, *then the least element of* $\Pi(\mathrm{Sym}(L))$ *containing* $\varepsilon(\sigma)$ *and preserved by* $\mathcal{U}$ *is* $\varepsilon(\varrho)$, *where* ϱ *is the least element of* $L_{\mathcal{T}}$ *containing* σ.

Proof. Let us denote by β the least partition of $A = \mathrm{Sym}(L)$ containing $\varepsilon(\sigma)$ and preserved by $\mathcal{U}$. We shall prove that $\varepsilon(\varrho) \subseteq \beta$. Take a twist $t \in \mathcal{T}$. The domain of t is a set $\{x, y, z\} \subseteq L$ and let t transpose x and y. By π we denote the partition of L with exactly one non-trivial block $\{x, y, z\}$, and by π_{xy}, π_{xz}, π_{yz}, we denote the partition of L with the only non-trivial block $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, respectively.

Suppose σ is not preserved by t. We can restrict ourselves to the case $(x, z) \in \sigma$. Then $(xt, zt) = (y, z)$. Take $p, q \in A$, $q \in pS_\pi$. Let u be the lift of t to the coset pS_π. First of all we prove that

$$\text{if } (p, q) \in \varepsilon(\pi_{yz}), \text{ then } (pu, qu) \in \varepsilon(\pi_{xz}). \tag{2.4}$$

Indeed, suppose $(p, q) \in \varepsilon(\pi_{yz})$, p, q different. By (1.5), $q = pt_{yz}$. By the definition of u we obtain $pu = pt_{xy}$ and $qu = qt_{xy} = pt_{yz}t_{xy}$. Then $(pu)^{-1}(qu) = t_{xy}t_{yz}t_{xy} = t_{xz}$. This implies $(pu, qu) \in \varepsilon(\pi_{xz})$, by another application of (1.5). If $p = q$, then (2.4) is trivial.

Since $(x, z) \in \sigma$, we have $\pi_{xz} \subseteq \sigma$. Take an arbitrary pair $(p, q) \in \varepsilon(\pi_{yz})$. Then $q \in pS_\pi$. By (2.4), $(pu, qu) \in \varepsilon(\pi_{xz}) \subseteq \varepsilon(\sigma)$. Since $\beta \supseteq \varepsilon(\sigma)$ is preserved by $\mathcal{U}$, we obtain $(p, q) = (puu, quu) \in \beta$. Hence $\varepsilon(\pi_{yz}) \subseteq \beta$. By repeating the whole process with different t's we get that also $\varepsilon(\varrho) \subseteq \beta$.

It remains to prove that if ϱ is preserved by $\mathcal{T}$, then $\varepsilon(\varrho)$ is preserved by $\mathcal{U}$. Let $p, q \in A$, $(p,q) \in \varepsilon(\varrho)$, and $p, q \in D(u)$ for a twist $u \in \mathcal{U}$. Take the base t of u. The twist t is defined on a set $\{x, y, z\} \subseteq L$ and transposes, say, x and y. Recall that by (1.6), $d_\varepsilon(p,q) = \gamma(p^{-1}q)$. Hence $\gamma(p^{-1}q) \leqslant \varrho$. Since p, q belong to the same coset of S_π, we get also $\gamma(p^{-1}q) \leqslant \pi$. If $p = q$, then trivially $(pu, qu) \in \varepsilon(\varrho)$. So assume that $p \neq q$. Then there are just two possibilities for $\pi \cap \varrho$, either $\pi \cap \varrho = \pi_{xy}$ or $\pi \cap \varrho = \pi$. In the former case $q = pt_{xy}$, and $pu = q$, $qu = p$. Hence $(pu, qu) \in \varepsilon(\varrho)$. And in the latter case we have $\pi \subseteq \varrho$, hence $(pu, qu) \in \varepsilon(\pi) \subseteq \varepsilon(\varrho)$. □

Proposition 2.8. *For every finite lattice L there exist a finite set A, a twisting structure $\mathcal{U}$ on A, and a complete lattice embedding $\psi\colon L \to L_{\mathcal{U}}$. Moreover, the domain of every twist $u \in \mathcal{U}$ is balanced with respect to ψ.*

Proof. Let ψ be the composition of the mappings $\varphi\colon L \to \Pi(L)$ defined by (2.2) and $\varepsilon\colon \Pi(L) \to \Pi(\mathrm{Sym}(L))$. The mapping ψ preserves arbitrary meets, the least and the greatest elements, because both φ and ε preserve them. It is injective as it is the composition of two injective mappings. To prove that it also preserves joins, take $x, y \in L$ and consider the least element of $L_{\mathcal{U}}$ containing both $\varepsilon\varphi(x)$ and $\varepsilon\varphi(y)$. By Proposition 2.7, it is equal to $\varepsilon\varphi(x \vee y)$.

The second assertion follows from Propositions 1.1 and 1.2. □

To close this section we describe a natural action of the symmetric group $\mathrm{Sym}(L)$ on the twisting structure $\mathcal{U}$. Let $t \in \mathcal{T}$ be a twist on L, $D(t) = \{x, y, z\}$. Recall that π denotes the partition of L with just one non-trivial block $\{x, y, z\}$. Take the lift u of t to a coset rS_π and an arbitrary permutation $q \in \mathrm{Sym}(L)$. Then qrS_π is another coset of S_π. If we denote the lift of t to the coset qrS_π by u^q, then we get the following identity:

$$(qs)u^q = q(su), \quad s \in D(u). \tag{2.5}$$

In this way the group $\mathrm{Sym}(L)$ acts on $\mathcal{U}$. The set of lifts of a fixed $t \in \mathcal{T}$ is an orbit of the action, and every orbit of the action can be described in this way.

3. Group envelopes of twisting structures

Our initial data in this section will be: a set A, a twisting structure $\mathcal{U}$ on A, a complete sublattice L of $L_{\mathcal{U}}$, a transitive group G of permutations on A such that every $\pi \in L$ is preserved by every $g \in G$, and an action of G on $\mathcal{U}$ satisfying the condition

$$D(u^g) = g(D(u)) \quad \text{and} \quad (g(a))u^g = g(au) \quad \text{for every } a \in D(u). \tag{3.1}$$

All these structures are supposed to be finite. Recall that the set $L_{\mathcal{U}}$ of all partitions of A that are preserved by $\mathcal{U}$ is a complete meet-subsemilattice of $\Pi(A)$, by Proposition 2.1. By $d(a,b)$ we denote the distance of a and b with respect to the identical mapping of L into $\Pi(A)$.

From now on we will identify $\mathcal{U}$ with a set of generators of order 2 in a group $\mathbf{W}$. Obvious examples of $\mathbf{W}$ are the free product $\mathbf{F}$ of $|\mathcal{U}|$-many copies of $\mathbf{Z}_2$, one copy for every twist $u \in \mathcal{U}$, or the direct product of $|\mathcal{U}|$-many copies of $\mathbf{Z}_2$. The action of G on $\mathcal{U}$ can be extended to an action on the free product $\mathbf{F}$ in the obvious way: if $w = u_1 u_2 \cdots u_k$ is an element of $\mathbf{F}$ and $g \in G$, then

$$w^g = u_1^g u_2^g \cdots u_k^g. \tag{3.2}$$

An arbitrary group $\mathbf{W}$ with $|\mathcal{U}|$-many generators of order 2 is a quotient of $\mathbf{F}$. The action of $g \in G$ on the set $\mathcal{U}$ of generators of $\mathbf{W}$ can be extended to an automorphism of $\mathbf{W}$ in at most one way. If such an extension exists, then it is given by (3.2). We will always assume that there is such an extension for every $g \in G$. In other words, we are interested only in those groups $\mathbf{W}$ for which the action of G on $\mathcal{U}$ extends to an action of G on the whole group $\mathbf{W}$. In this case we say that $\mathbf{W}$ is a *group envelope* of $\mathcal{U}$. The free product $\mathbf{F}$ is an example of a group envelope of $\mathcal{U}$. All other group envelopes of $\mathcal{U}$ can be constructed as quotients of $\mathbf{F}$ by suitable normal subgroups invariant under the action (3.2) of G on $\mathbf{F}$.

The semidirect product of $\mathbf{W}$ and G (defined by the action (3.2) of G on $\mathbf{W}$) has a natural permutation representation on the set $W \times A$. An element (w,g) acts on $(x,a) \in W \times A$ in the following way:

$$(x,a)^{(w,g)} = (wx^g, g(a)). \tag{3.3}$$

We will refer to this action of $\mathbf{W} \times \mathbf{G}$ as the *natural action* of $\mathbf{W} \times \mathbf{G}$ on $W \times A$. Note that the action is transitive.

The following definition of a relation $\sim$ is of an utmost importance. This is the relation $\sim$ mentioned in the introduction. The relation is defined on the set $W \times A$ by

$$(x,a) \sim (xu, au) \quad \text{for every } u \in \mathcal{U},\ a \in D(u) \text{ and } x \in W. \tag{3.4}$$

Proposition 3.1. *The natural action of the* $\mathbf{W} \times \mathbf{G}$ *on* $W \times A$ *preserves relation* $\sim$.

Proof. It is sufficient to prove that $\sim$ is preserved by the actions of elements (w,i) and (o,g), where $w \in \mathbf{W}$, $g \in G$, and i, o are the identity elements of the groups G and $\mathbf{W}$, respectively.

If $(x,a) \sim (xu, au)$, then $a \in D(u)$. We have $(x,a)^{(w,i)} = (wx, a)$ and similarly $(xu, au)^{(w,i)} = (wxu, au)$. From the definition of $\sim$ we get $(wx, a) \sim (wxu, au)$.

Further, we have $(x,a)^{(o,g)} = (x^g, g(a))$ and $(xu, au)^{(o,g)} = (x^g u^g, g(au)) = (x^g u^g, g(a) u^g)$. As $g(a) \in D(u^g)$, we conclude $(x^g, g(a)) \sim (x^g u^g, g(a) u^g)$. This completes the proof. $\square$

We regard the set $W \times A$ as a collection of copies of A indexed by the group $\mathbf{W}$. The set $A_w = \{(w,a)\colon a \in A\}$ is the *w-copy* of A. If $\pi \in \Pi(A)$, then the *w-copy* of π is the equivalence relation $\pi_w = \{((w,a),(w,b))\colon (a,b) \in \pi\}$ on A_w.

As the next step of our construction we define a mapping $\Phi\colon L \to \Pi(W \times A)$ as follows:

(3.5) $$\Phi(\pi) \text{ is the equivalence closure of } \bigcup_{w \in W} \pi_w \cup \sim .$$

Since every $\pi \in L$ is a congruence of the unary algebra (A, G), the natural action of $\mathbf{W} \times \mathbf{G}$ on $W \times A$ preserves the union $\bigcup\limits_{w} \pi_w$ of the w-copies of π. Because it also preserves $\sim$, we get

Proposition 3.2. *The natural action of* $\mathbf{W} \times \mathbf{G}$ *on* $W \times A$ *preserves* $\Phi(\pi)$ *for every* $\pi \in L$.

Next we prove the following proposition.

Proposition 3.3. *The mapping* Φ *is order- and join-preserving.*

Proof. The fact that Φ is order-preserving follows immediately from (3.5). Let us denote the join operation in L by $\vee_L$. If $\pi, \varrho \in L$, then $\Phi(\pi \vee_L \varrho) \supseteq \Phi(\pi) \vee \Phi(\varrho)$. To prove the opposite inclusion observe that the equivalence relation $\Phi(\pi) \vee \Phi(\varrho)$ intersects every w-copy of A in the same relation σ. More exactly, if we define

(3.6) $$(a,b) \in \sigma \quad \text{if and only if} \quad ((o,a),(o,b)) \in \Phi(\pi) \vee \Phi(\varrho),$$

then, for any $w \in W$, $((w,a),(w,b)) \in \Phi(\pi) \vee \Phi(\varrho)$ if and only if $(a,b) \in \sigma$. This is a direct consequence of the previous proposition. Obviously, both $\pi \subseteq \sigma$ and $\varrho \subseteq \sigma$. This implies $\pi \vee \varrho \subseteq \sigma$ ($\vee$ denotes the join in $\Pi(A)$).

Next we prove that σ is preserved by $\mathcal{U}$. Let $u \in \mathcal{U}$, $a, b \in D(u)$, and $(a,b) \in \sigma$. Then $((o,a),(o,b)) \in \Phi(\pi) \vee \Phi(\varrho)$, $(o,a) \sim (u, au)$, and $(o,b) \sim (u, bu)$. Hence $((u,au),(u,bu)) \in \Phi(\pi) \vee \Phi(\varrho)$, therefore $(au, bu) \in \sigma$. Since $\pi \vee_L \varrho$ is the least partition of A preserved by $\mathcal{U}$ and containing $\pi \vee \varrho$, by Proposition 2.1, it follows that $\pi \vee_L \varrho \leqslant \sigma$. Thus $\Phi(\pi \vee_L \varrho) = \{((w,a),(w,b))\colon (a,b) \in \pi \vee_L \varrho,\ w \in W\} \subseteq \{((w,a),(w,b))\colon (a,b) \in \sigma,\ w \in W\} \subseteq \Phi(\pi) \vee \Phi(\varrho)$. $\square$

It is more complicated to find conditions on $\mathbf{W}$ under which the mapping Φ is injective and meet-preserving. It is especially true for the latter of the two conditions. In the remaining part of this section we find a condition to ensure that Φ is injective.

By a *graph* we mean a vertex set V together with an edge set E and two mappings $\alpha, \omega\colon E \to V$. The vertex $\alpha(e)$ is called the *initial* vertex of e and $\omega(e)$ is the *terminal* vertex of e. It is possible that $\alpha(e) = \omega(e)$. In this case we say that e is a *loop*. The graph is *symmetric* if we are also given a mapping ${}^{-1}\colon E \to E$ assigning to every edge e an *inverse* edge e^{-1} such that $\alpha(e^{-1}) = \omega(e)$ and $\omega(e^{-1}) = \alpha(e)$. If e is a loop, then we assume that $e^{-1} = e$.

We define the *graph* $\mathcal{D}$ of the twisting structure $\mathcal{U}$ on A in the following way. It is an oriented symmetric graph with the edges valued by elements of $\mathcal{U}$. The vertex set of $\mathcal{D}$ is A. Every pair $u \in \mathcal{U}$ and $a \in D(u)$ defines an edge e such that $\alpha(e) = a$ and $\omega(e) = au$. The value of e is $\mu(e) = u$. Loosely speaking, the graph $\mathcal{D}$ is the union of the graphs of all twists $u \in \mathcal{U}$. We say that the twisting structure $\mathcal{U}$ is *connected* if its graph is connected. Since u has order two, we have $\alpha(e^{-1}) = au$ and $\omega(e^{-1}) = a$.

Let $\pi \in L$. A sequence $p = e_1 e_2 \cdots e_k$ of edges of $\mathcal{D}$ is said to be a *π-path* if $(\omega(e_i), \alpha(e_{i+1})) \in \pi$ for every $i = 1, \ldots, k-1$. We say that p is a π-path from a to b if, moreover, $(a, \alpha(e_1)) \in \pi$ and $(\omega(e_k), b) \in \pi$. From the definition we get

$$\text{(3.7)} \qquad \text{if } \pi \leqslant \varrho, \text{ then every } \pi\text{-path is also a } \varrho\text{-path.}$$

Moreover, each path in $\mathcal{D}$ is a π-path for every $\pi \in L$. So if $\mathcal{U}$ is connected and $a, b \in A$, then there is a π-path from a to b for every $\pi \in L$.

A π-path $p = e_1 e_2 \cdots e_k$ is a *π-cycle* if also $(\omega(e_k), \alpha(e_1)) \in \pi$. We also say that p is a π-cycle at a if $(a, \alpha(e_1)) \in \pi$. If $p = e_1 e_2 \cdots e_k$ is a π-path, then $e_k^{-1} \cdots e_1^{-1}$ is also a π-path. We will call it the *inverse* of p and denote it by p^{-1}.

Suppose now that a group envelope $\mathbf{W}$ of $\mathcal{U}$ is given. For every π-path $p = e_1 e_2 \cdots e_k$ we define its *value* as the element $\mu(p) = \mu(e_1)\mu(e_2)\cdots\mu(e_k)$ of $\mathbf{W}$. The value depends on the envelope $\mathbf{W}$ of $\mathcal{U}$ and, if necessary, we say that $\mu(p)$ is the $\mathbf{W}$-*value* of p to make clear which envelope we have in mind. Obviously, $\mu(p^{-1}) = \mu(p)^{-1}$. The π-path p is *reduced* if the word $\mu(e_1)\mu(e_2)\cdots\mu(e_k)$ is reduced, i.e. if $\mu(e_i) \neq \mu(e_{i+1})$ for all $i = 1, \ldots, k-1$.

Proposition 3.4. *Let $\mathbf{W}$ be a group envelope of $\mathcal{U}$. Then $((x,a),(y,b)) \in \Phi(\pi)$ if and only if there is a π-path p from a to b such that $\mu(p) = x^{-1}y$.*

Proof. Suppose that $p = e_1 e_2 \cdots e_k$ is a π-path from a to b, $\mu(e_i) = u_i$ for $i = 1, \ldots, k$, and $\mu(p) = u_1 u_2 \cdots u_k = x^{-1}y$. Then $\alpha(e_i) = a_i$ and $\omega(e_i) = a_i u_i$ for some $a_i \in D(u_i)$. For every $i = 1, \ldots, k$ we denote the word $x u_1 \cdots u_i$ by w_i.

We have $(a, a_1) \in \pi$, hence $((x,a),(x,a_1)) \in \pi_x$. Moreover, $(x, a_1) \sim (xu_1, a_1u_1)$. Further, for every $i = 1 \dots k-1$, $(\omega(e_i), \alpha(e_{i+1})) = (a_iu_i, a_{i+1}) \in \pi$. Hence $((w_i, a_iu_i),(w_i, a_{i+1})) \in \pi_{w_i}$ for all $i = 1, \dots, k-1$. From the definition of $\sim$ we also obtain $(w_i, a_{i+1}) \sim (w_iu_{i+1}, a_{i+1}u_{i+1}) = (w_{i+1}, a_{i+1}u_{i+1})$. Finally, $(a_ku_k, b) = (\omega(e_k), b) \in \pi$, therefore $((w_k, a_ku_k),(w_k, b)) \in \pi_{w_k}$. But $w_k = xu_1 \cdots u_k = x\mu(p) = xx^{-1}y = y$. Thus any two subsequent elements in the sequence

$$\begin{aligned}&(x,a),\ (x,a_1),\ (xu_1, a_1u_1) = (w_1, a_1u_1),\ (w_1, a_2),\ (w_2, a_2x_2), \dots,\\ &(w_k, a_ku_k),\ (w_k, b) = (y, b)\end{aligned}$$

are equivalent in $\Phi(\pi)$. This completes the proof that $((x,a),(y,b)) \in \Phi(\pi)$.

Now suppose that $((x,a),(y,b)) \in \Phi(\pi)$. Then there is a sequence q of elements of $W \times A$ starting at (x,a) and ending at (y,b) such that every pair of two subsequent elements of the sequence belongs to the relation $\bigcup_w \pi_w \cup \sim$. Recall further that every π_w is an equivalence relation on A_w, different copies A_w of A are disjoint, and pairs in the relation $\sim$ contain elements from different copies of A. So if three subsequent elements of q belong to the same copy A_w, then the second one can be omitted from q and the first one and the third one are still equivalent in π_w. Thus we can suppose that the sequence q has the form

$$\begin{aligned}&(x,a),\ (x,a_1) \sim (xu_1, a_1u_1),\ (xu_1, a_2) \sim (xu_1u_2, a_2u_2), \dots\\ &(xu_1u_2 \cdots u_{k-1}, a_k) \sim (xu_1 \cdots u_k, a_ku_k),\ (y,b)\end{aligned} \tag{3.8}$$

for suitable $a_i \in A$, $u_i \in \mathcal{U}$, $i = 1, \dots, k$, satisfying $(a, a_1) \in \pi$, $(a_iu_i, a_{i+1}) \in \pi$ for all $i = 1, \dots, k-1$, $(a_ku_k, b) \in \pi$, and $xu_1 \cdots u_k = y$. Every pair $(xu_1 \cdots u_{i-1}, a_i) \sim (xu_1 \cdots u_i, a_iu_i)$ defines an edge e_i of $\mathcal{D}$ such that $\alpha(e_i) = a_i$, and $\omega(e_i) = a_iu_i$. Since $(a_iu_i, a_{i+1}) \in \pi$ for $i = 1, \dots, k-1$, the sequence $r = e_1e_2 \cdots e_k$ is a π-path from a to b, and $\mu(r) = x^{-1}y$. □

We also need the following proposition.

Proposition 3.5. *If a π-path $p = e_1e_2 \cdots e_k$ from a to b is not reduced, then there exists a subsequence q of p such that q is a reduced π-path from a to b and $\mu(q) = \mu(p)$.*

Proof. Let $\mu(e_i) = u_i$, $\alpha(e_i) = a_i$, and $\omega(e_i) = a_iu_i$ for every $i = 1, \dots, k$. Moreover, $(a, a_1) \in \pi$, $(a_ku_k, b) \in \pi$, and $(a_iu_i, a_{i+1}) \in \pi$ for every $i = 1, \dots, k-1$. If p is not reduced, then $u_{i+1} = u_i$ for some i. From $(a_iu_i, a_{i+1}) \in \pi$ and $a_iu_i, a_{i+1} \in D(u_i)$ we get $(a_iu_i^2, a_{i+1}u_i) = (a_i, a_{i+1}u_{i+1}) \in \pi$. Thus we may omit from the sequence p the two edges e_i and e_{i+1}. It is easily verified that the sequence $p' =$

$e_1 \cdots e_{i-1} e_{i+2} \cdots e_k$ is a π-path and its value is $\mu(p') = u_1 u_2 \cdots u_{i-1} u_{i+2} \cdots u_k = \mu(p)$. So if p is not reduced, then we can find a reduced path q with the required properties by repeated application of the procedure just described. $\square$

From now on we will assume that the group envelope $\mathbf{W}$ of $\mathcal{U}$ satisfies the following condition for every $\pi \in L$:

$$\text{(3.9)} \qquad \text{if } \mu(p) = o \text{ for a } \pi\text{-path } p, \text{ then } p \text{ is a } \pi\text{-cycle.}$$

Proposition 3.6. *Suppose that* $\mathbf{W}$ *satisfies the condition* (3.9) *and* $\pi \in L$. *Then*

$$\text{(3.10)} \qquad ((w,a),(w,b)) \in \Phi(\pi) \quad \textit{if and only if} \quad (a,b) \in \pi.$$

Moreover, the mapping Φ *is injective.*

Proof. If $(a,b) \in \pi$, then $((w,a),(w,b)) \in \Phi(\pi)$ from the definition of Φ. To prove the converse implication, suppose $((w,a),(w,b)) \in \Phi(\pi)$. Then, by Proposition 3.4, there exists a π-path from a to b such that $\mu(p) = o$. From (3.9) we get that p is a π-cycle, hence $(a,b) \in \pi$.

The second assertion follows immediately from (3.10): if $\pi > \varrho$, $\pi, \varrho \in L$, then there is a pair $(a,b) \in \pi - \varrho$. Hence $((o,a),(o,b)) \in \Phi(\pi) - \Phi(\varrho)$. $\square$

4. A homotopy condition

In this section we start with the same data as in the previous one. Moreover, we assume that the twisting structure $\mathcal{U}$ is connected and that $\mathbf{W}$ is a group envelope of $\mathcal{U}$ satisfying the condition (3.9). In Section 3 we proved that the mapping Φ: $L \to \Pi(W \times A)$ defined by (3.5) was join-preserving (Proposition 3.3) and, under the condition (3.9), injective (Proposition 3.6). In this section we find a condition under which the mapping Φ preserves meets.

Let us take a point $a \in A$. If $p = e_1 e_2 \cdots e_k$ is a π-cycle at a, then its $\mathbf{W}$-value $\mu(p)$ is an element of $\mathbf{W}$. If q is another π-cycle at a, then pq is also a π-cycle at a and $\mu(pq) = \mu(p)\mu(q)$. As $\mu(p^{-1}) = \mu(p)^{-1}$ and o is the value of the empty π-cycle, the set

$$\text{(4.1)} \qquad C_\pi = \{\mu(p) \colon p \text{ is a } \pi\text{-cycle at } a\}$$

is a subgroup of $\mathbf{W}$. We will refer to it as the *π-cycle subgroup* of $\mathbf{W}$ at the point a, or simply, a *cycle subgroup* at a. Since every $(\pi \wedge \varrho)$-cycle at a is both a π-cycle and a ϱ-cycle at a, we have

$$\text{(4.2)} \qquad C_{\pi \wedge \varrho} \subseteq C_\pi \cap C_\varrho \quad \text{for every } \pi, \varrho \in L.$$

The main result of this section is the following theorem.

Theorem 4.1. *Let A be a finite set, $\mathcal{U}$ a twisting structure on A, L a complete sublattice of $L_{\mathcal{U}}$, G a transitive group of permutations on A such that every $\pi \in L$ is preserved by every $g \in G$, $\mathbf{W}$ a group envelope of $\mathcal{U}$, and $\pi, \varrho \in L$. Then $\Phi(\pi \wedge \varrho) = \Phi(\pi) \wedge \Phi(\varrho)$, where the mapping $\Phi\colon L \to \Pi(W \times A)$ is defined by* (3.5), *if and only if for every $a \in A$ the cycle subgroups of $\mathbf{W}$ at a satisfy the condition*

$$C_\pi \cap C_\varrho = C_{\pi \wedge \varrho}. \tag{4.3}$$

P r o o f. Suppose that (4.3) holds. Since Φ is order-preserving (Proposition 3.3), we have $\Phi(\pi \wedge \varrho) \subseteq \Phi(\pi) \wedge \Phi(\varrho)$ for any $\pi, \varrho \in L$. To prove the opposite inclusion, suppose $((x, a), (y, b)) \in \Phi(\pi) \wedge \Phi(\varrho)$. By Proposition 3.4, there exist a π-path p from a to b such that $\mu(p) = x^{-1}y$ and a ϱ-path q from a to b with the same value $\mu(q) = x^{-1}y$. Since $\mathcal{U}$ is connected, there exists a $(\pi \wedge \varrho)$-path r from b to a. Any $(\pi \wedge \varrho)$-path is both a π-path and a ϱ-path. Thus pr is a π-cycle at a with the value $\mu(pr) = x^{-1}y\mu(r)$ and qr is a ϱ-cycle at a with the same value. Hence $x^{-1}y\mu(r) \in C_\pi \cap C_\varrho$. By (4.3), there is a $(\pi \wedge \varrho)$-cycle s at a with the value $\mu(s) = x^{-1}y\mu(r)$. Then sr^{-1} is a $(\pi \wedge \varrho)$-path from a to b with the value $\mu(sr^{-1}) = x^{-1}y$. Hence $((x, a), (y, b)) \in \Phi(\pi \wedge \varrho)$, by Proposition 3.4.

To prove the converse implication suppose that $\Phi(\pi \wedge \varrho) = \Phi(\pi) \wedge \Phi(\varrho)$ and take $w \in C_\pi \cap C_\varrho$. Then there are a π-cycle p' at a and a ϱ-cycle q' at a such that $\mu(p') = \mu(q') = w$. By Proposition 3.4, $((o, a), (w, a)) \in \Phi(\pi) \wedge \Phi(\varrho)$. Hence $((o, a), (w, a)) \in \Phi(\pi \wedge \varrho)$. By another application of Proposition 3.4, there exists a $(\pi \wedge \varrho)$-cycle r' at a such that $\mu(r') = w$. Thus $w \in C_{\pi \wedge \varrho}$. The opposite inclusion $C_{\pi \wedge \varrho} \subseteq C_\pi \cap C_\varrho$ is stated in (4.2). □

By making more use of the connectedness of $\mathcal{U}$ we can prove that it is sufficient to verify the equality (4.3) for a single element $a \in A$.

Corollary 4.2. *Under the assumptions of Theorem* 4.1, $\Phi(\pi \wedge \varrho) = \Phi(\pi) \wedge \Phi(\varrho)$ *if and only if there exists an element $a \in A$ such that the cycle subgroups of $\mathbf{W}$ at a satisfy the condition* (4.3).

P r o o f. Let b be another element of A. Take an arbitrary ω-path r from a to b, and for every $\sigma \in L$ consider the mapping Δ that assigns to every σ-cycle p at a the σ-cycle $r^{-1}pr$ at b. The mapping is obviously injective. To see that it is one-to-one, take an arbitrary σ-cycle q at b. Then rqr^{-1} is a σ-cycle at a and $\Delta(rqr^{-1}) = q$. Moreover, if p is a σ-cycle at a such that $\mu(p) = o$, then $\mu(\Delta(p)) = o$, too. Denote by D_σ the σ-cycle subgroup at b. Then the mapping assigning to every $w \in C_\sigma$ the element $\mu(r)^{-1}w\mu(r) \in D_\sigma$ is an isomorphism between C_σ and D_σ. Hence from the assumption that the equality (4.3) holds for the π- and ϱ-cycle subgroups at a we get that it also holds for the corresponding subgroups at b. The rest follows from Theorem 4.1. □

If we assume that the domains of twists $u \in \mathcal{U}$ are balanced with respect to L, then there is a simple way to find an infinite envelope $\mathbf{W}$ of $\mathcal{U}$ satisfying the condition (4.3) for any two elements $\pi, \varrho \in L$. This result, although not directly necessary in what follows, is a good illustration of our approach. A similar construction was used in [2] to prove that every algebraic lattice is isomorphic to an interval in the subgroup lattice of an infinite group.

Lemma 4.3. *Let $\mathbf{W}$ be an envelope of $\mathcal{U}$ and let the domains of twists of $\mathcal{U}$ be balanced with respect to L. Moreover, let $p = e_1e_2\cdots e_k$ be a π-path from a to b and $q = f_1f_2\cdots f_k$ a ϱ-path from a to b. If $\mu(e_i) = \mu(f_i) = u_i$ for all $i = 1,\dots,k$, then there exists a $(\pi \wedge \varrho)$-path $r = g_1g_2\cdots g_k$ from a to b such that $\mu(g_i) = u_i$, $i = 1,\dots,k$.*

Proof. We proceed by induction on k. If $k = 0$, then $\mu(p) = o$. By (3.10), $(a,b) \in \pi$. Similarly, $(a,b) \in \varrho$. Thus $(a,b) \in \pi \wedge \varrho$ and the empty path is a $(\pi \wedge \varrho)$-path from a to b.

Suppose now that $k \geqslant 1$ and the assertion holds for every two π- and ϱ-paths of length $k-1$ between the same pair of elements of A. Since $\mu(e_k) = \mu(f_k) = u_k$, we have $\alpha(e_k) = a_k$, $\omega(e_k) = a_ku_k$, $\alpha(f_k) = b_k$, and $\omega(f_k) = b_ku_k$ for some $a_k, b_k \in D(u_k)$. Since $(a_ku_k, b) \in \pi$, we get that the distance $d(a_ku_k, b)$ is less than or equal to π. Similarly, $d(b_ku_k, b) \leqslant \varrho$. Since $D(u_k)$ is balanced with respect to L, there is an element $c_ku_k \in D(u_k)$ such that $d(c_ku_k, b) \leqslant d(a_ku_k, b) \wedge d(b_ku_k, b) \leqslant \pi \wedge \varrho$. Thus $(c_ku_k, b) \in \pi \wedge \varrho$. It proves that the one-element path g_k, where $\alpha(g_k) = c_k$ and $\omega(g_k) = c_ku_k$, is a $(\pi \wedge \varrho)$-path from c_k to b. Moreover, $(a_ku_k, c_ku_k) \in \pi$ and $(b_ku_k, c_ku_k) \in \varrho$. Since both π and ϱ are preserved by u_k, we get $(a_k, c_k) \in \pi$, $(b_k, c_k) \in \varrho$, and also $(a_{k-1}u_{k-1}, c_k) \in \pi$, $(b_{k-1}u_{k-1}, c_k) \in \varrho$. Thus $p' = e_1e_2\cdots e_{k-1}$ is a π-path from a to c_k and $q' = f_1f_2\cdots f_{k-1}$ is a ϱ-path from a to c_k. By the induction hypothesis, there is a $(\pi \wedge \varrho)$-path $r' = g_1g_2\cdots g_{k-1}$ from a to c_k such that $\mu(g_i) = u_i$ for $i = 1,\dots,k-1$. Then $r = g_1g_2\cdots g_{k-1}g_k$ is a $(\pi \wedge \varrho)$-path satisfying the conclusions of the lemma. □

Proposition 4.4. *Let the domains of the twisting structure $\mathcal{U}$ be balanced with respect to the identical mapping from L to $\Pi(A)$, and let $\mathbf{F}$ be the free product of $|\mathcal{U}|$-many copies of $\mathbf{Z}_2$. Then $\mathbf{F}$ is a group envelope of $\mathcal{U}$ and satisfies the condition* (4.3) *for arbitrary $a \in A$ and any two partitions $\pi, \varrho \in L$.*

Proof. The fact that $\mathbf{F}$ is a group envelope of $\mathcal{U}$ follows immediately from (3.2). It remains to prove that $C_\pi \cap C_\varrho \subseteq C_{\pi\wedge\varrho}$. So suppose that $w = u_1u_2\cdots u_k \in C_\pi \cap C_\varrho$ and that w is a reduced word. Then there is a π-cycle $p = e_1e_2\cdots e_k$ at a such that $\mu(p) = w$. By Proposition 3.5, p may be taken reduced. Similarly, there is a reduced ϱ-cycle $q = f_1f_2\cdots f_k$ at a with the value $\mu(q) = w$. As $\mathbf{F}$ is the free product, two reduced words over the alphabet $\mathcal{U}$ are equal in $\mathbf{F}$ only if they coincide. Thus $\mu(e_i) = \mu(f_i) = u_i$ for all $i = 1, \ldots, k$. By the previous lemma, there is a $(\pi\wedge\varrho)$-cycle r at a such that $\mu(r) = w$. Hence $C_\pi \cap C_\varrho \subseteq C_{\pi\wedge\varrho}$. □

The previous proof points out the problems with constructing a finite group envelope $\mathbf{W}$ of $\mathcal{U}$ that satisfies the condition (4.3). The proof was made possible by the absence of non-trivial equalities among the words over the alphabet $\mathcal{U}$ in the group $\mathbf{F}$. As every finite group envelope $\mathbf{W}$ of $\mathcal{U}$ is a quotient of $\mathbf{F}$ (as a group), to find a finite $\mathbf{W}$ satisfying (4.3) means to introduce equalities among the words over the alphabet $\mathcal{U}$ that do not change (4.3).

5. Finite envelopes satisfying the condition (3.9)

We start with the same data as in Section 4. We are going to construct a finite envelope $\mathbf{H}$ of $\mathcal{U}$ satisfying the condition (3.9).

Take an arbitrary partition $\pi \in L \subseteq L_\mathcal{U}$. We define a quotient $\mathcal{D}_\pi$ of the graph $\mathcal{D}$ of $\mathcal{U}$ as follows. It is again a symmetric graph with edges valued by $\mathcal{U}$, the vertices of $\mathcal{D}_\pi$ are the blocks of the partition π. The edge set of $\mathcal{D}_\pi$ is also the set of blocks of an equivalence relation on the set of edges of $\mathcal{D}$, two edges are identified if and only if they have the same value u and their initial vertices belong to the same block of π. Since π is preserved by $\mathcal{U}$, it follows that the terminal vertices of the two edges are also equivalent in π. To every block of equivalent edges there corresponds a single edge of $\mathcal{D}_\pi$ with the initial vertex $\pi[\alpha(e)]$, the terminal vertex $\pi[\omega(e)]$ and the value $\mu(e)$, where e is an arbitrary representative of the block. Obviously, the definition does not depend on the choice of the representative e. Note also that a sequence of edges $p = e_1e_2\cdots e_k$ of $\mathcal{D}$ is a π-path if and only if the projection of p is a path in $\mathcal{D}_\pi$. Moreover, the values of a π-path in $\mathcal{D}$ and its projection in $\mathcal{D}_\pi$ coincide.

The edges of $\mathcal{D}_\pi$ with a given value u form the graph of a partial permutation on the factor set A/π. We extend this partial permutation to a full permutation u_π defined on A/π as follows:

$$(5.1) \qquad (\pi[a])u_\pi = \begin{cases} \pi[bu], & \text{if there is } b \in D(u) \text{ such that } (a,b) \in \pi; \\ \pi[a], & \text{otherwise.} \end{cases}$$

Thus each permutation u_π has order at most two. Let us denote by $\mathbf{H}_\pi$ the subgroup of the symmetric group on A/π generated by all u_π, $u \in \mathcal{U}$.

Lemma 5.1. *Let $p = e_1 e_2 \cdots e_k$ be a π-path. Then it is a π-path from a to b if and only if* $(\pi[a])\mu(e_1)_\pi \mu(e_2)_\pi \cdots \mu(e_k)_\pi = \pi[b]$.

Proof. The path p is a π-path from a to b if and only if its projection in $\mathcal{D}_\pi$ is a path from $\pi[a]$ to $\pi[b]$. The claim follows from the fact that each edge e_i projects to an edge in $\mathcal{D}_\pi$ with the same value $\mu(e_i)$ and this edge is an edge of the graph of the permutation $\mu(e_i)_\pi$. □

We define a group $\mathbf{H}$ as the subgroup of the direct product $\prod_{\pi \in L} \mathbf{H}_\pi$ generated by the elements $(u_\pi)_{\pi \in L}$, $u \in \mathcal{U}$. We identify each twist $u \in \mathcal{U}$ with the element $(u_\pi)_{\pi \in L} \in \mathbf{H}$. Different twists of $\mathcal{U}$ are identified with different elements of order two of the group $\mathbf{H}$, because of the component of the product $\prod_{\pi \in L} \mathbf{H}_\pi$ corresponding to the least element of L (which coincides with the least partition of A).

To prove that $\mathbf{H}$ is a group envelope of $\mathcal{U}$ we have to show that the action of G on $\mathcal{U}$ can be extended to an action on the whole group $\mathbf{H}$. To this end it is more convenient to write the value of a permutation $g \in G$ at a point a as ag instead of $g(a)$. Then the second part of (3.1) assumes the form

$$au^g = (a)g^{-1}ug \quad \text{for every } a \in D(u^g). \tag{5.2}$$

By extending the twists u^g and u to full permutations using (5.1) we get $(\pi[a])(u^g)_\pi = (\pi[a])g^{-1}u_\pi g$, since g^{-1} preserves π. So if we set

$$(u_\pi)^g = (u^g)_\pi = g^{-1}u_\pi g, \tag{5.3}$$

then we get that conjugating by elements of G defines an action of G on permutations u_π, and the action extends to an action of G on the whole group $\mathbf{H}_\pi$. It follows that the component-wise conjugation by elements $g \in G$ defines an action of G on the product $\prod_{\pi \in L} \mathbf{H}_\pi$ that coincides with the action of G on $\mathcal{U}$. This proves the first part of the following proposition.

Proposition 5.2. *The group $\mathbf{H}$ is a group envelope of $\mathcal{U}$ and satisfies the condition* (3.9).

Proof. It remains to prove that $\mathbf{H}$ satisfies the condition (3.9). Let $p = e_1 e_2 \cdots e_k$ be a π-path at a point a such that $\mu(p) = o$. Then, in particular,

$$(\pi[a])\mu(e_1)_\pi \mu(e_2)_\pi \cdots \mu(e_k)_\pi = \pi[a].$$

By Lemma 5.1, we get that p is a π-cycle at a. □

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