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Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 1, 149–169

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

FORCING SEQUENCES OF POSITIVE INTEGERS

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(Received March 1, 1993)

1. Introduction

Let c be a nonzero complex number. It is easy to prove that if c^n is in the right half plane for every positive integer n, then c is a positive number. Hershkowitz and Schneider extended this result in [5], by generalizing the domain which contains all the powers, and by considering only partial sequence of the set of all positive integers as exponents. More specifically, let T be a domain in the complex plane, and let $0 \leqslant \alpha < \pi$. A sequence $(r_1, r_2, \ldots)$ of positive integers is said to be a (T, α)-*forcing sequence* if for every $c \in T$, if c^{r_i}, $i = 1, 2, \ldots$, are all in a wedge of width 2α symmetrically located around the nonnegative real axis, then c is a positive number. The paper [5] contains a few sufficient conditions and a few necessary conditions for certain sequences to be forcing, however, in general, the problem of characterizing forcing sequences is open.

In this paper we generalize some theorems of [5], and further investigate several problems concerning forcing sequences.

Most of our notation and definitions are given in Section 2.

Let $0 \leqslant \alpha \leqslant \pi$, let n be a positive integer, and let $R = (r_1, \ldots, r_n)$ be a sequence of positive integers. Clearly, the set of all nonzero complex numbers with argument in absolute value less than or equal to α / r_n is contained in the set of all complex numbers c all of whose powers c^{r_i}, $i = 1, 2, \ldots, n$, have argument in absolute value less than or equal to α. As is observed in [5], the study of the equality of these two sets is important to finding necessary conditions (under some additional conditions) and sufficient conditions for a sequence to be a forcing sequence. In Section 3 we prove new necessary and sufficient conditions for that equality to hold. While the results in [5] are of geometric flavor, our conditions are algebraic. Our result provides an alternative proof to results in [5] as well as some new assertions.

In Section 4 we discuss forcing sequences. We prove the infinite version of the main result of Section 3 to obtain an algebraic characterization of forcing sequences, in the form of existence of solution to a system of diophantine inequalities. We then use this result to characterize all geometric forcing sequences.

In Section 5 we define the concept of Kronecker sequences, and we investigate the relations between such sequences and forcing sequences. The main theorem of this chapter characterizes the arithmetic forcing sequences, using Kronecker's Theorem.

In Section 6 we define a minimal forcing sequence as a forcing sequence with no forcing subsequence. We show that minimal forcing sequences exist, and discuss the question whether every forcing sequence contains a minimal forcing sequence. This question is, in general, an open problem. As a possible approach for a further study we introduce an algorithm that prunes a forcing sequence without losing the forcing property. Under certain conditions, our algorithm constructs a minimal forcing subsequence for a given sequence.

2. Notation and definitions

This section contains almost all the notation and definitions used in this paper. We follow the notation used in [5].

2.1 Notation. The cardinality of a set S is denoted by $|S|$. The set of all complex numbers is denoted by $\mathbb{C}$. The set of all real numbers is denoted by $\mathbb{R}$. The set of all positive real numbers is denoted by $\mathbb{R}_+$. The set of all rationals is denoted by $\mathbb{Q}$. For a positive integer n, the set $\{1,\dots,n\}$ is denoted by $\langle n\rangle$.

2.2 Convention. The argument $\arg(c)$ of a nonzero complex number c will be assumed to be in the interval $[0,2\pi)$, unless is stated otherwise explicitly.

2.3 Definition. Let $0\leqslant\alpha\leqslant\beta\leqslant 2\pi$. We define the closed wedge (excluding 0) $W[\alpha,\beta]$ to be the set $\{c\in\mathbb{C}\colon c\neq 0,\ \alpha\leqslant\arg(c)\leqslant\beta\}$. For $-\pi\leqslant\alpha<0\leqslant\beta\leqslant\pi$ we define $W[\alpha,\beta]$ to be the set $\{c\in\mathbb{C}\colon c\neq 0,\ 2\pi-\alpha\leqslant\arg(c) \text{ or } \arg(c)\leqslant\beta\}$. The open wedge $W(\alpha,\beta)$ is defined to be the interior of $W[\alpha,\beta]$ in the Euclidean topology.

2.4 Notation. For $0\leqslant\alpha\leqslant\pi$ we denote $W[\alpha]=W[-\alpha,\alpha]$ and $W(\alpha)=W(-\alpha,\alpha)$.

2.5 Convention. When we say "sequence of positive integers" in this paper, we always mean "strictly increasing sequence of positive integers".

2.6 Notation. Let $0 \leqslant \alpha \leqslant \pi$ and let $R = (r_1, \dots, r_n)$ be a sequence of positive integers. We denote

$$\begin{aligned} S[R, \alpha, n] &= \{c \in \mathbb{C} \colon c^{r_k} \in W[\alpha],\ k \in \langle n \rangle\}, \\ S(R, \alpha, n) &= \{c \in \mathbb{C} \colon c^{r_k} \in W(\alpha),\ k \in \langle n \rangle\}. \end{aligned}$$

For an infinite sequence R we denote

$$\begin{aligned} S[R, \alpha, \infty] &= \{c \in \mathbb{C} \colon c^{r_k} \in W[\alpha],\ k = 1, 2, \dots\}, \\ S(R, \alpha, \infty) &= \{c \in \mathbb{C} \colon c^{r_k} \in W(\alpha),\ k = 1, 2, \dots\}. \end{aligned}$$

2.7 Notation. Let $0 \leqslant \alpha \leqslant \pi$ and let n be a positive integer. We denote

$$\begin{aligned} Q_n[\alpha] &= \{c \in \mathbb{C} \colon c^n \in W[\alpha]\}, \\ Q_n(\alpha) &= \{c \in \mathbb{C} \colon c^n \in W(\alpha)\}. \end{aligned}$$

Let $0 \leqslant \alpha \leqslant \pi$ and let n be a positive integer. As is observed in [5], we have

$$\tag{2.8} \begin{cases} Q_n[\alpha] = \bigcup\limits_{k=0}^{n-1} W\Big[\dfrac{2\pi k - \alpha}{n}, \dfrac{2\pi k + \alpha}{n}\Big], \\ Q_n(\alpha) = \bigcup\limits_{k=0}^{n-1} W\Big(\dfrac{2\pi k - \alpha}{n}, \dfrac{2\pi k + \alpha}{n}\Big). \end{cases}$$

It is also observed in [5] that for a sequence $R = (r_1, \dots, r_n)$ of positive integers we have

$$\tag{2.9} \begin{cases} S[R, \alpha, n] = \bigcap\limits_{i=1}^{n} Q_{r_i}[\alpha], \\ S(R, \alpha, n) = \bigcap\limits_{i=1}^{n} Q_{r_i}(\alpha). \end{cases}$$

2.10 Notation. For a positive integer n and a nonnegative integer k, $k \leqslant n - 1$, we denote by $W_n^k[\alpha]$ and by $W_n^k(\alpha)$ the wedges $W\big[\frac{2\pi k - \alpha}{n}, \frac{2\pi k + \alpha}{n}\big]$ and $W\big(\frac{2\pi k - \alpha}{n}, \frac{2\pi k + \alpha}{n}\big)$ respectively.

2.11 Definition. Let $0 \leqslant \alpha \leqslant \pi$, let $T \subseteq \mathbb{C}$, and let R be a (finite or infinite) sequence of positive integers. The sequence R is said to be a *(T, α)-forcing sequence* if $T \cap S[R, \alpha, |R|] \subseteq \mathbb{R}_+$. The sequence R is said to be a *(T, α)-semiforcing sequence* if $T \cap S(R, \alpha, |R|) \subseteq \mathbb{R}_+$. Each of the parameters T and α in our definition is optional, where the defaults are $\mathbb{C}$ and $\frac{1}{2}\pi$ respectively.

Let $0 \leqslant \alpha \leqslant \pi$, let n be a positive integer, and let $R = (r_1, \dots, r_n)$ be a sequence of positive integers. It follows from (2.8) and (2.9) that

$$W[\alpha/r_n] \subseteq S[R, \alpha, n]. \tag{3.1}$$

As is observed in [5], the study of the equality case in (3.1)

$$W[\alpha/r_n] = S[R, \alpha, n] \tag{3.2}$$

is important to finding necessary conditions and sufficient conditions for a sequence to be a forcing sequence. In this section we prove new necessary and sufficient conditions for (3.2) to hold. While the results in [5] are of geometric flavor, our conditions are algebraic. Our result provides an alternative proof to results in [5] as well as some new assertions.

3.3 Lemma. *Let $0 \leqslant \alpha \leqslant \pi$, and let k_1, k_2, r_1, r_2 be numbers, $r_1, r_2 > 0$. We have*

$$\left[\frac{2\pi k_1 - \alpha}{r_1}, \frac{2\pi k_1 + \alpha}{r_1}\right] \cap \left[\frac{2\pi k_2 - \alpha}{r_2}, \frac{2\pi k_2 + \alpha}{r_2}\right] \neq \emptyset \tag{3.4}$$

if and only if

$$|k_1 r_2 - k_2 r_1| \leqslant \frac{\alpha(r_1 + r_2)}{2\pi}. \tag{3.5}$$

Proof. (3.4) holds if and only if $\frac{2\pi k_2 - \alpha}{r_2} \leqslant \frac{2\pi k_1 + \alpha}{r_1}$ and $\frac{2\pi k_1 - \alpha}{r_1} \leqslant \frac{2\pi k_2 + \alpha}{r_2}$, which is equivalent to $2\pi(k_2 r_1 - k_1 r_2) \leqslant \alpha(r_1 + r_2)$ and $2\pi(k_1 r_2 - k_2 r_1) \leqslant \alpha(r_1 + r_2)$, which together form (3.5). □

3.6 Lemma. *Let $0 \leqslant \alpha \leqslant \pi$, and let k_1, k_2, r_1, r_2 be nonnegative integers satisfying $k_i \leqslant r_i - 1$, $i = 1, 2$. Then the following are equivalent.*

(i) $W_{r_1}^{k_1}[\alpha] \cap W_{r_2}^{k_2}[\alpha] \neq \emptyset$.

(ii) *Either* (3.5) *holds, or*

$$k_1 = 0 \quad \textit{and} \quad r_1(r_2 - k_2) \leqslant \frac{\alpha(r_1 + r_2)}{2\pi}, \tag{3.7}$$

or

$$k_2 = 0 \quad \textit{and} \quad r_2(r_1 - k_1) \leqslant \frac{\alpha(r_1 + r_2)}{2\pi}. \tag{3.8}$$

Proof. Since $0 \leqslant k_i \leqslant r_i - 1$, $i = 1, 2$, by Notation 2.10, (i) can be written as

$$W\Big[\frac{2\pi k_1 - \alpha}{r_1}, \frac{2\pi k_1 + \alpha}{r_1}\Big] \cap W\Big[\frac{2\pi k_2 - \alpha}{r_2}, \frac{2\pi k_2 + \alpha}{r_2}\Big] \neq \emptyset. \tag{3.9}$$

We distinguish between three cases:

(i) Both k_1 and k_2 are positive. Here, since $k_i \leqslant r_i - 1$, $i = 1, 2$, (3.9) is equivalent to (3.4), which, by Lemma 3.3, is equivalent to (3.5).

(ii) $k_1 = 0$. In this case, (3.9) holds if and only if $2\pi - \frac{\alpha}{r_1} \leqslant \frac{2\pi k_2 + \alpha}{r_2}$ or $\frac{2\pi k_2 - \alpha}{r_2} \leqslant \frac{\alpha}{r_1}$, which is equivalent to $r_1(r_2 - k_2) \leqslant \alpha(r_1 + r_2)/2\pi$ or $k_2 r_1 \leqslant \alpha(r_1 + r_2)/2\pi$. Thus, in this case, (3.9) is equivalent to (3.5) or (3.7).

(iii) $k_2 = 0$. Similarly to the previous case, in this case (3.9) is equivalent to (3.5) or (3.8). □

We can now state the main result of this section.

3.10 Theorem. *Let $0 \leqslant \alpha < \pi$, let n be a positive integer, and let $R = (r_1, \ldots, r_n)$ be a sequence of positive integers. The following are equivalent.*

(i) $W[\alpha/r_n] \neq S[R, \alpha, n]$.

(ii) *There exist nonnegative integers $k_1, \ldots, k_n$ such that*

$$\begin{cases} k_i \leqslant r_i - 1, & i \in \langle n-1 \rangle, \\ 0 < k_n \leqslant r_n - 1, \\ |k_i r_j - k_j r_i| \leqslant \alpha(r_i + r_j)/2\pi, & i, j \in \langle n \rangle,\ i \neq j. \end{cases}$$

(iii) *There exist integers $k_1, \ldots, k_n$ such that*

$$\begin{cases} k_n \not\equiv 0 \pmod{r_n}, \\ |k_i r_j - k_j r_i| \leqslant \alpha(r_i + r_j)/2\pi, & i, j \in \langle n \rangle,\ i \neq j. \end{cases} \tag{3.11}$$

Proof. (i)$\Longrightarrow$(ii). Assume that (i) holds. In view of (3.1), let $x \in S[R, \alpha, n] \setminus W[\alpha/r_n]$. Since both $S[R, \alpha, n]$ and $W[\alpha/r_n]$ are symmetric with respect to the real axis, and since nonnegative numbers belong to both $S[R, \alpha, n]$ and $W[\alpha/r_n]$, we may assume, without loss of generality, that

$$0 < \arg(x) \leqslant \pi. \tag{3.12}$$

By (2.8) and (2.9) we have

$$x \in \bigcap_{i=1}^{n} Q_{r_i}[\alpha] = \bigcap_{i=1}^{n} \Big(\bigcup_{k=0}^{r_i - 1} W_{r_i}^k[\alpha] \Big),$$

and hence there exist nonnegative integers $k_1, \ldots, k_n$, $k_i \leqslant r_i - 1$, $i \in \langle n \rangle$, such that

$$x \in W_{r_i}^{k_i}[\alpha], \qquad i \in \langle n \rangle. \tag{3.13}$$

Also, $x \notin W[\alpha/r_n] = W_{r_n}^0[\alpha]$, and thus $k_n > 0$. It follows from (3.13) by Lemma 3.6 that for all $i, j \in \langle n \rangle$, $i \neq j$, we have either

$$|k_i r_j - k_j r_i| \leqslant \frac{\alpha(r_i + r_j)}{2\pi}, \tag{3.14}$$

or

$$k_i = 0 \quad \text{and} \quad r_i(r_j - k_j) \leqslant \frac{\alpha(r_i + r_j)}{2\pi}, \tag{3.15}$$

or

$$k_j = 0 \quad \text{and} \quad r_j(r_i - k_i) \leqslant \frac{\alpha(r_i + r_j)}{2\pi}.$$

Assume that (3.14) does not hold for some i and j, and without loss of generality assume that (3.15) holds. Since (3.14) does not hold, we have

$$k_j r_i > \frac{\alpha(r_i + r_j)}{2\pi}. \tag{3.16}$$

It follows from (3.15) and (3.16) that $k_j r_i > r_i(r_j - k_j)$, implying that $k_j > \frac{1}{2} r_j$. Since both k_j and r_j are integers, it now follows that $k_j \geqslant \frac{1}{2}(r_j + 1)$. Since $x \in W_{r_j}^{k_j}[\alpha]$, it now follows by Notation 2.10 that

$$\arg(x) \geqslant \frac{2\pi k_j - \alpha}{r_j} \geqslant \frac{2\pi \frac{r_j + 1}{2} - \alpha}{r_j} = \pi + \frac{\pi - \alpha}{r_j} > \pi,$$

in contradiction to (3.12). Therefore, our assumption that (3.14) does not hold for some i and j is false, and so (ii) holds.

(ii)$\Longrightarrow$(iii) is trivial.

(iii)$\Longrightarrow$(i). Assume that (iii) holds. By Lemma 3.3 we have

$$\left[\frac{2\pi k_i - \alpha}{r_i}, \frac{2\pi k_i + \alpha}{r_i}\right] \cap \left[\frac{2\pi k_j - \alpha}{r_j}, \frac{2\pi k_j + \alpha}{r_j}\right] \neq \emptyset, \qquad i, j \in \langle n \rangle,\ i \neq j.$$

By Helly's Theorem [3], the intersection $\bigcap_{i=1}^{n} \left[\frac{2\pi k_i - \alpha}{r_i}, \frac{2\pi k_i + \alpha}{r_i}\right]$ is non-empty, and so there exists $c \in \bigcap_{i=1}^{n} W_{r_i}^{k_i \pmod{r_i}}[\alpha] \subseteq \bigcap_{i=1}^{n} Q_{r_i}[\alpha] = S[R, \alpha, n]$. Since $W_{r_n}^s[\alpha] \cap W_{r_n}^t[\alpha] = \emptyset$ whenever $0 \leqslant s$, $t < r_n$, $s \neq t$, and since $k_n \not\equiv 0 \pmod{r_n}$, it now follows that $c \notin W_{r_n}^0[\alpha] = W[\alpha/r_n]$, and so (i) holds. □

In the open wedge case we obtain the following theorem. The proof is essentially the same and thus omitted.

3.17 Theorem. *Let $0 \leqslant \alpha < \pi$, let n be a positive integer, and let $R = (r_1, \ldots, r_n)$ be a sequence of positive integers. The following are equivalent.*

(i) $W(\alpha/r_n) \neq S(R, \alpha, n)$.

(ii) *There exist nonnegative integers $k_1, \ldots, k_n$ such that*

$$\begin{cases} k_i \leqslant r_i - 1, \quad i \in \langle n-1 \rangle, \\ 0 < k_n \leqslant r_n - 1, \\ |k_i r_j - k_j r_i| < \alpha(r_i + r_j)/2\pi, \quad i, j \in \langle n \rangle,\ i \neq j. \end{cases}$$

(iii) *There exist integers $k_1, \ldots, k_n$ such that*

$$\begin{cases} k_n \not\equiv 0 \pmod{r_n}, \\ |k_i r_j - k_j r_i| < \alpha(r_i + r_j)/2\pi, \quad i, j \in \langle n \rangle,\ i \neq j. \end{cases} \tag{3.18}$$

Theorems 3.10 and 3.17 provide an algebraic characterization, that is the existence of solution to a system (3.11) of diophantine inequalities, to the geometric property $W[\alpha/r_n] \neq S[R, \alpha, n]$. The following theorem provides a sufficient conditions for the existence of a solution to inequalities of the type (3.11).

3.19 Theorem. *Let $0 \leqslant \alpha < \pi$, let n be a positive integer, and let $R = (r_1, \ldots, r_n)$ be a sequence of positive integers. If*

$$\prod_{i=1}^{n-1} \frac{\alpha}{2\pi}(r_i + r_n) \geqslant r_n^{n-2}, \tag{3.20}$$

then there exist integers $k_1, \ldots, k_n$ such that

$$\begin{cases} 0 \leqslant k_i \leqslant r_i, \quad i \in \langle n-1 \rangle, \\ 0 < k_n < r_n, \\ |k_i r_n - k_n r_i| \leqslant \alpha(r_i + r_n)/2\pi, \quad i \in \langle n-1 \rangle. \end{cases}$$

Proof. Define the homogeneous linear forms $\xi_i = r_n x_i - r_i x_n$, $i \in \langle n-1 \rangle$, and $\xi_n = x_n$. Observe that the determinant of the coefficients matrix A is the positive integer r_n^{n-1}. Define the positive numbers $\lambda_i = \alpha(r_i + r_n)/2\pi + \varepsilon$, $i \in \langle n-1 \rangle$, where $\varepsilon > 0$, and choose ε sufficiently small such that

$$[\lambda_i] = [\alpha(r_i + r_n)/2\pi], \qquad i \in \langle n-1 \rangle, \tag{3.21}$$

where $[a]$ is the largest integer that is less than or equal to a. We now define $\lambda_n = r_n - \delta$, where $\delta > 0$. If (3.20) holds then it follows that for δ sufficiently small we have $\prod_{i=1}^{n} \lambda_i \geqslant r_n^{n-1} = \det(A)$. By Minkowski's Theorem [2, Theorem 448, p. 395] there exist integers $k_1, \ldots, k_n$, not all zero, for which

$$\text{(3.22)} \qquad \begin{cases} |k_n| \leqslant \lambda_n < r_n, \\ |k_i r_n - k_n r_i| \leqslant \lambda_i, \quad i \in \langle n-1 \rangle. \end{cases}$$

Since $k_i r_n - k_n r_i$ is integer, it follows from (3.21) and (3.22) that

$$\text{(3.23)} \qquad \begin{cases} |k_n| < r_n, \\ |k_i r_n - k_n r_i| \leqslant \alpha(r_i + r_n)/2\pi, \quad i \in \langle n-1 \rangle. \end{cases}$$

Note that if $k_1, \ldots, k_n$ solve (3.23) then $-k_1, \ldots, -k_n$ solve (3.23). Therefore, without loss of generality we may assume that $k_n \geqslant 0$. If $k_n = 0$ then it follows from (3.23) that

$$|k_i| r_n \leqslant \frac{\alpha(r_i + r_n)}{2\pi} < \tfrac{1}{2}(r_i + r_n) < \tfrac{1}{2}(r_n + r_n) = r_n, \qquad i \in \langle n-1 \rangle,$$

implying that $k_i = 0$, $i \in \langle n-1 \rangle$. But this contradicts the fact that not all the k_i's are zero. Therefore, we have

$$\text{(3.24)} \qquad 0 < k_n < r_n.$$

Assume now that for some $i \in \langle n-1 \rangle$ we have $k_i < 0$. Then, in view of $k_n > 0$, we have

$$k_i r_n - k_n r_i < k_i r_n \leqslant -r_n = -\tfrac{1}{2}(r_n + r_n) < -\frac{\alpha(r_i + r_n)}{2\pi},$$

in contradiction to (3.23). Therefore, we have $k_i \geqslant 0$, $i \in \langle n-1 \rangle$. Now, let $k_i' = r_i - k_i$, $i \in \langle n \rangle$. Observe that $k_i' r_n - k_n' r_i = k_n r_i - k_i r_n$, and hence $k_1', \ldots, k_n'$ solve the inequalities

$$\text{(3.25)} \qquad |k_i' r_n - k_n' r_i| \leqslant \frac{\alpha(r_i + r_n)}{2\pi}, \qquad i \in \langle n-1 \rangle.$$

By (3.24) we have $0 < k_n' < r_n$. Assume that for some $i \in \langle n-1 \rangle$ we have $k_i' < 0$. As before, we have

$$k_i' r_n - k_n' r_i < k_i' r_n < -r_n = -\tfrac{1}{2}(r_n + r_n) < -\frac{\alpha(r_i + r_n)}{2\pi},$$

in contradiction to (3.25). Therefore, we have $k_i' \geqslant 0$, $i \in \langle n-1 \rangle$, and so $0 \leqslant k_i \leqslant r_i$, $i \in \langle n-1 \rangle$. Since we have (3.23) and (3.24), our proof is now completed. □

3.26 Remark. (i) Since $r_i < r_n$, $i \in \langle n-1 \rangle$, we have $\prod_{i=1}^{n-1} \frac{r_i + r_n}{2} < r_n^{n-1}$. Thus, if (3.20) holds then $r_n > \left(\frac{\pi}{\alpha}\right)^{n-1}$.

(ii) Since $\prod_{i=1}^{n-1} \frac{\alpha}{2\pi}(r_i + r_n) > \left(\frac{\alpha}{2\pi} r_n\right)^{n-1}$, it follows that if $\left(\frac{\alpha}{2\pi}\right)^{n-1} r_n \geqslant 1$, or, equivalently, if $r_n \geqslant \left(\frac{2\pi}{\alpha}\right)^{n-1}$, then (3.20) holds.

(iii) Note that by Theorem 3.19, condition (3.2) provides a sufficient condition for (3.18) to hold for $i \in \langle n \rangle$ and $j = n$.

Similarly to Theorem 3.19, we obtain the following sufficient condition for the existence of a solution to inequalities of the type (3.18).

3.27 Theorem. *Let $0 \leqslant \alpha \leqslant \pi$, let n be a positive integer, and let $R = (r_1, \ldots, r_n)$ be a sequence of positive integers. If*

$$\prod_{i=1}^{n-1} \frac{\alpha}{2\pi}(r_i + r_n) > r_n^{n-2},$$

then there exist integers $k_1, \ldots, k_n$ such that

$$\begin{cases} 0 \leqslant k_i \leqslant r_i, & i \in \langle n-1 \rangle, \\ 0 < k_n < r_n, & \\ |k_i r_n - k_n r_i| < \alpha(r_i + r_n)/2\pi, & i \in \langle n-1 \rangle. \end{cases}$$

In [5], the cases $S[R, \alpha, 1] = W[\alpha/r_1]$ and $S(R, \alpha, 1) = W(\alpha/r_1)$ are characterized. We conclude this section by using our results to obtain a simple characterization for the cases $S[R, \alpha, 2] = W[\alpha/r_2]$ and $S(R, \alpha, 2) = W(\alpha/r_2)$.

3.28 Proposition. *Let $0 \leqslant \alpha < \pi$, and let $R = (r_1, r_2)$ be a sequence of positive integers. We have $S[R, \alpha, 2] \neq W[\alpha/r_2]$ if and only if either $r_1 + r_2 \geqslant 2\pi/\alpha$ or r_1 and r_2 are not co-prime.*

Proof. If r_1 and r_2 are not co-prime then let d be their greatest common divisor. Choose $k_i = r_i/d \leqslant r_i - 1$, $i = 1, 2$. We have $0 = |k_1 r_2 - k_2 r_1| \leqslant \alpha(r_1 + r_2)/2\pi$, and since $0 < k_i \leqslant r_i - 1$, $i = 1, 2$, it follows by Theorem 3.10 that $S[R, \alpha, 2] \neq W[\alpha/r_2]$. If $r_1 + r_2 \geqslant 2\pi/\alpha$ then, by Theorem 3.19, there exist integers k_1 and k_2, $0 \leqslant k_1 \leqslant r_1$, $0 < k_2 < r_2$, such that $|k_1 r_2 - k_2 r_1| \leqslant \alpha(r_1 + r_2)/2\pi$, and by Theorem 3.10 we have $S[R, \alpha, 2] \neq W[\alpha/r_2]$. Conversely, if $S[R, \alpha, 2] \neq W[\alpha/r_2]$ then, by Theorem 3.10, there exist integers k_1 and k_2, $0 \leqslant k_1 \leqslant r_1 - 1$, $0 < k_2 \leqslant r_2 - 1$, such that $|k_1 r_2 - k_2 r_1| \leqslant \alpha(r_1 + r_2)/2\pi$. If r_1 and r_2 are co-prime then $r_1/r_2 \neq k_1/k_2$ and hence $|k_1 r_2 - k_2 r_1| \geqslant 1$. It now follows that $\alpha(r_1 + r_2)/2\pi \geqslant 1$, which is equivalent to $r_1 + r_2 \geqslant 2\pi/\alpha$. □

For open wedges we similarly obtain

3.29 Proposition. *Let $0 \leqslant \alpha < \pi$, and let $R = (r_1, r_2)$ be a sequence of positive integers. We have $S(R, \alpha, 2) \neq W(\alpha/r_2)$ if and only if either $r_1 + r_2 > 2\pi/\alpha$ or r_1 and r_2 are not co-prime.*

We remark that the results of this section provide an alternative proof to results in [5]. For details see [1].

4. Forcing sequences

The following theorem is the infinite version of Theorem 3.10. It provides an algebraic characterization of forcing sequences, in the form of existence of solution to a system of diophantine inequalities.

4.1 Theorem. *Let $0 \leqslant \alpha < \pi$, and let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers. The following are equivalent.*

(i) *R is not α-forcing.*

(ii) *There exists a sequence $(k_1, k_2, \ldots)$ of integers such that*

$$\text{(4.2)} \qquad \begin{cases} 0 \leqslant k_i \leqslant r_i - 1, \quad i = 1, 2, \ldots, \\ 0 < k_n \quad \text{for some positive integer } n, \\ |k_i r_j - k_j r_i| \leqslant \alpha(r_i + r_j)/2\pi, \quad i, j \in \langle n \rangle,\ i \neq j. \end{cases}$$

(iii) *There exists a sequence $(k_1, k_2, \ldots)$ of integers such that*

$$\begin{cases} k_n \not\equiv 0 \pmod{r_n} \quad \text{for some positive integer } n, \\ |k_i r_j - k_j r_i| \leqslant \alpha(r_i + r_j)/2\pi, \quad i, j \in \langle n \rangle,\ i \neq j. \end{cases}$$

P r o o f. (i)$\Longrightarrow$(ii). Assume that R is not α-forcing. Then there exists $x \in S[R, \alpha, \infty] \setminus \mathbb{R}_+$. Without loss of generality we may assume that $0 < \arg(x) \leqslant \pi$. Since $x \in S[R, \alpha, \infty]$, we have

$$x \in \bigcap_{i=1}^{\infty} Q_{r_i}[\alpha] = \bigcap_{i=1}^{\infty} \Big(\bigcup_{k=0}^{r_i - 1} W_{r_i}^{k}[\alpha] \Big).$$

Hence, there exist nonnegative integers $k_1, k_2, \ldots, k_i \leqslant r_i - 1$, $i = 1, 2, \ldots$, such that $x \in W_{r_i}^{k_i}[\alpha]$, $i = 1, 2, \ldots$. Since $\bigcap_{i=1}^{\infty} W_{r_i}^{0}[\alpha] = \mathbb{R}_+$, and since $x \notin \mathbb{R}_+$, it follows that

$0 < k_n$ for some positive integer n. The rest of the proof of this implication follows as the proof of the implication (i)$\Longrightarrow$(ii) in Theorem 3.10.

(ii)$\Longrightarrow$(iii) is trivial.

(iii)$\Longrightarrow$(i). Let (iii) hold. As in the proof of the corresponding implication in Theorem 3.10, we prove that $\bigcap_{i=1}^{\infty} W_{r_i}^{k_i \pmod{r_i}}[\alpha] \neq \emptyset$. Since $\bigcap_{i=1}^{\infty} W_{r_i}^{k_i \pmod{r_i}}[\alpha] \subseteq \bigcap_{i=1}^{\infty} Q_{r_i}[\alpha] = S[R,\alpha,\infty]$, it follows that $W_{r_n}^{k_n \pmod{r_n}}[\alpha] \cap S[R,\alpha,\infty] \neq 0$. Since $W_{r_n}^{s}[\alpha] \cap W_{r_n}^{t}[\alpha] = \emptyset$ whenever $0 \leqslant s,\ t < r_n$, $s \neq t$, and since $k_n \neq 0 \pmod{r_n}$, it now follows that $S[R,\alpha,\infty] \nsubseteq W_{r_n}^{0}[\alpha]$. Since $\mathbb{R}_+ \subseteq W_{r_n}^{0}[\alpha]$, we thus obtain that $S[R,\alpha,\infty] \nsubseteq \mathbb{R}_+$, and so R is not α-forcing. □

We now use Theorem 4.1 to characterize those geometric sequences that are α-forcing. Our result generalizes Corollary 3.15 in [5].

4.3 Theorem. *Let R be the sequence defined by*

$$\begin{cases} r_i = 1, \\ r_m = a_0 p^{m-2}, \quad m = 2,3,\ldots, \end{cases}$$

where a_0 and p are positive integers, $p > 1$, and let $0 \leqslant \alpha < \pi$. Then the following are equivalent.

(i) *R is α-forcing.*

(ii) *We have $\alpha < \min\left\{\frac{2\pi}{p+1}, \frac{2\pi}{a_0}\right\}$.*

Proof. (i)$\Longrightarrow$(ii). Assume that (ii) does not hold. Let us first assume that

$$\alpha \geqslant \frac{2\pi}{p+1}, \tag{4.4}$$

and let $c = \mathrm{e}^{\frac{2\pi \mathrm{i}}{a_0(p+1)}}$. Since $p^m = \pm 1 \pmod{p+1}$ for every positive integer m, it follows that for $m \geqslant 2$ we have $c^{r_m} = \mathrm{e}^{\pm\frac{2\pi \mathrm{i}}{p+1}}$, and by (4.4) we have $c^{r_m} \in W[\alpha]$, $m \geqslant 2$. Also, $c^{r_1} \in W\left[\frac{2\pi}{a_0(p+1)}\right] \subseteq W\left[\frac{2\pi}{p+1}\right] \subseteq W[\alpha]$. Thus, we have $c \in S[R,\alpha,\infty] \setminus \mathbb{R}_+$, and so R is not α-forcing. Assume now that (4.4) does not hold. Then

$$\alpha \geqslant \frac{2\pi}{a_0}. \tag{4.5}$$

Here we choose $c = \mathrm{e}^{\frac{2\pi \mathrm{i}}{a_0}}$. By (4.5), $c^{r_1} \in W[\alpha]$. For $m \geqslant 2$, since a_0 divides r_m we have $c^{r_m} = 1 \in W[\alpha]$. Thus again, we have $c \in S[R,\alpha,\infty] \setminus \mathbb{R}_+$, and so R is not α-forcing.

(ii)$\Longrightarrow$(i). Assume that (i) does not hold. By Theorem 4.1 there exists a sequence $(k_1, k_2, \ldots)$ of integers such that (4.2) holds. For $i \geqslant 2$ we have

$$|k_i r_{i+1} - k_{i+1} r_i| = a_0 p^{i-2} |k_i p - k_{i+1}| \leqslant \frac{\alpha}{2\pi} a_0 p^{i-2}(p+1) = \frac{\alpha}{2\pi}(r_i + r_{i+1}),$$

implying that $|k_i p - k_{i+1}| \leqslant \frac{\alpha}{2\pi}(p+1)$. By (ii) we now obtain $|k_i p - k_{i+1}| < 1$ and hence $k_i p = k_{i+1}$, or, equivalently,

$$k_i = k_2 p^{i-2}, \qquad i = 2, 3, \ldots . \tag{4.6}$$

Since $0 \leqslant k_1 \leqslant r_1 - 1 = 0$ we have $k_1 = 0$, and hence $k_n > 0$ implies $n \geqslant 2$. It now follows from (4.6) that $k_2 > 0$. By applying (4.2) to 1 and i, $i \geqslant 2$, we obtain $|k_i r_1 - k_1 r_i| = k_i = k_2 p^{i-2} \leqslant \alpha(a_0 p^{i-2} + 1)/2\pi$. Hence, $1 \leqslant k_i \leqslant \alpha(a_0 + p^{2-i})/2\pi$. We let i approach ∞ and get $1 \leqslant \alpha a_0/2\pi$, or, equivalently, $\alpha \geqslant 2\pi/a_0$, in contradiction to (ii). □

In the semiforcing case we obtain the following, using essentially the same proofs.

4.7 Theorem. *Let $0 \leqslant \alpha \leqslant \pi$, and let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers. The following are equivalent.*

(i) *R is not α-semiforcing.*

(ii) *There exists a sequence $(k_1, k_2, \ldots)$ of integers such that*

$$\begin{cases} 0 \leqslant k_i \leqslant r_i - 1, & i = 1, 2, \ldots, \\ 0 < k_n & \text{for some positive integer } n, \\ |k_i r_j - k_j r_i| < \alpha(r_i + r_j)/2\pi, & i, j \in \langle n \rangle,\ i \neq j. \end{cases}$$

(iii) *There exists a sequence $(k_1, k_2, \ldots)$ of integers such that*

$$\begin{cases} k_n \not\equiv 0 \pmod{r_n} & \text{for some positive integer } n, \\ |k_i r_j - k_j r_i| < \alpha(r_i + r_j)/2\pi, & i, j \in \langle n \rangle,\ i \neq j. \end{cases}$$

The algebraic characterization of geometric semiforcing sequences is

4.8 Theorem. *Let R be the sequence defined by*

$$\begin{cases} r_1 = 1, \\ r_m = a_0 p^{m-2}, & m = 2, 3, \ldots, \end{cases}$$

where a_0 and p are positive integers, $p > 1$, and let $0 \leqslant \alpha < \pi$. Then the following are equivalent.

(i) *R is α-semiforcing.*

(ii) *We have $\alpha \leqslant \min\left\{\frac{2\pi}{p+1}, \frac{2\pi}{a_0}\right\}$.*

In this section we define Kronecker sequences, and discuss their relations with forcing sequences. We apply these relations to characterize arithmetic sequences that are forcing.

5.1 Notation. Let x be a nonnegative number and let y be a positive number. We denote $x \pmod{y} = x - y[x/y]$, where $[x/y]$ is the largest integer that is less than or equal to x/y. Observe that $0 \leqslant x \pmod{y} < y$.

5.2 Definition. The sequence $R = (r_1, r_2, \ldots)$ is said to be a *Kronecker sequence* if for every irrational number θ the sequence $\big(r_i\theta \pmod{1}\big)_{i=1}^{\infty}$ is dense in the interval $[0, 1]$.

In 1884 Kronecker proved the following theorem, e.g. [2, Theorem 439, p. 376].

5.3 Theorem. *The natural numbers form a Kronecker sequence.*

The following proposition states a relation between Kronecker sequences and forcing sequences.

5.4 Proposition. *Let* $T = \big\{c \in \mathbb{C}\colon \frac{\arg(c)}{2\pi} \notin \mathbb{Q}\big\}$, *let* $R = (r_1, r_2, \ldots)$ *be a Kronecker sequence, and let* $0 \leqslant \alpha < \pi$. *Then* R *is* (T, α)*-forcing.*

Proof. Let $c \in T$ and let $\theta = \frac{\arg(c)}{2\pi}$. Since θ is irrational, it follows that for some positive integer i we have $|r_i\theta \pmod{1} - 0.5| < \frac{\pi-\alpha}{2\pi}$. The latter implies that $|r_i \arg(c) \pmod{2\pi} - \pi| < \pi - \alpha$. Thus, $\alpha < \arg(c^{r_i}) < 2\pi - \alpha$, and so $c^{r_i} \notin W[\alpha]$. Therefore, we have $c \notin S[R, \alpha, \infty]$, implying that $S[R, \alpha, \infty] \cap T = \emptyset \subseteq \mathbb{R}_+$, proving that R is (T, α)-forcing. □

It is natural to ask whether the converse holds, that is: Does there exist an α, $0 \leqslant \alpha < \pi$, such that every (T, α)-forcing sequence is a Kronecker sequence? The answer to this question is negative for $\alpha < 2\pi/3$. It follows from the following theorem, proven in [6].

5.5 Theorem. *If there exists* $\alpha \in \mathbb{R}$, $\alpha > 1$, *such that* $q_n = \frac{r_{n+1}}{r_n} \geqslant \alpha$, $n = 1, 2, \ldots$, *then there exists* $\xi \in \mathbb{R} \setminus \mathbb{Q}$ *such that*

$$\beta \leqslant r_n\xi \pmod{1} \leqslant 1 - \beta, \qquad n = 1, 2, \ldots \tag{5.6}$$

for some $\beta > 0$.

Condition (5.6) implies that the sequence $\big(r_i\xi \pmod{1}\big)_{i=1}^{\infty}$ is not dense in the interval $[0, 1]$, and hence we have

5.7 Corollary. *If there exists $\alpha \in \mathbb{R}$, $\alpha > 1$, such that $q_n = \frac{r_{n+1}}{r_n} \geqslant \alpha$, $n = 1, 2, \ldots$, then the sequence $R = (r_1, r_2, \ldots)$ is not a Kronecker sequence.*

5.8 Corollary. *For every α, $0 \leqslant \alpha < 2\pi/3$ there exists a (T, α)-forcing sequence which is not a Kronecker sequence.*

Proof. By Theorem 4.3, with $a_0 = 1$, $p = 2$, the sequence $R = (2^{n-1})_{n=1}^{\infty}$ is α-forcing, and so clearly it is (T, α)-forcing. Since $q_n = 2$, $n = 1, 2, \ldots$, it follows from Corollary 5.7 that R is not a Kronecker sequence. □

We remark that the question whether there exist an α, $2\pi/3 \leqslant \alpha < \pi$, such that every (T, α)-forcing sequence is a Kronecker sequence, is still open.

The rest of this section is devoted to characterizing arithmetic forcing sequences.

5.9 Proposition. *Let m be a positive integer, $m \geqslant 2$, and let T_m be the set $\{e^{2\pi i k/m} : 1 \leqslant k \leqslant m,\ (k, m) = 1\}$ of the primitive mth roots of the unity. Let R be the arithmetic sequence $\big(r_0 + (n-1)q\big)_{n=1}^{\infty}$, where $(r_0, q) = 1$, let $d = (q, m)$, and let $0 \leqslant \alpha < \pi$. Then R is (T_m, α)-forcing if and only if*

$$\alpha < \begin{cases} \left(1 - \frac{1}{m}\right)\pi, & d = 1,\ m \text{ is odd}, \\ \left(1 - \frac{d-2}{m}\right)\pi, & d > 1,\ m/d \text{ is odd}, \\ \left(1 - \frac{d-1}{m}\right)\pi, & d \text{ is odd},\ m \text{ is even}, \\ \left(1 - \frac{2}{m}\right)\pi, & d = 2,\ m/d \text{ is even}, \\ \left(1 - \frac{d-4}{m}\right)\pi, & d = 2 \pmod 4,\ d > 2,\ m/d \text{ is even}, \\ \left(1 - \frac{d-2}{m}\right)\pi, & d = 0 \pmod 4,\ m/d \text{ is even}. \end{cases} \tag{5.10}$$

Proof. By definition, R is (T_m, α)-forcing if and only if for every k, $1 \leqslant k \leqslant m$, satisfying $(k, m) = 1$ there exists a nonnegative integer n such that $(e^{2\pi i k/m})^{r_0 + nq} \notin W[\alpha]$. The latter holds if and only if

$$m\frac{\alpha}{2\pi} < \big[k(r_0 + nq)\big] \pmod m < m\Big(1 - \frac{\alpha}{2\pi}\Big). \tag{5.11}$$

Let $d = (q, m)$. Since knq is divisible by d, $\big[k(r_0 + nq)\big] \pmod m$ can get one of the m/d values:

$$(kr_0 + jd) \pmod m, \qquad j = 0, \ldots, \frac{m}{d} - 1. \tag{5.12}$$

We now show that all the values in (5.12) are indeed attained as n goes from 0 to $m/d - 1$. Let

$$0 \leqslant n_1, n_2 \leqslant \frac{m}{d} - 1, \tag{5.13}$$

and assume that $[k(r_0+n_1q)] \pmod m = [k(r_0+n_2q)] \pmod m$. Then $k(n_1-n_2)q$ is divisible by m, and hence $k(n_1-n_2)q/d$ is divisible by m/d. Observe that $(m/d, q/d) = 1$. Also, since m and k are co-prime it follows that $(m/d, k) = 1$. Therefore, $n_1 - n_2$ is divisible by m/d. In view of (5.13) this is possible if and only if $n_1 = n_2$. Thus, as n goes from 0 to $m/d - 1$, $[k(r_0+nq)] \pmod m$ attains m/d distinct values, which are necessarily the values in (5.12). Since R is (T_m, α)-forcing if and only if for every k, $1 \leqslant k \leqslant m$, satisfying $(k,m) = 1$ there exists a nonnegative integer n such that (5.11) holds, it now follows that R is (T_m, α)-forcing if and only if for every k, $1 \leqslant k \leqslant m$, satisfying $(k,m) = 1$, we have

$$(5.14)\quad m\frac{\alpha}{2\pi} < (kr_0 + jd) \pmod m < m\Big(1 - \frac{\alpha}{2\pi}\Big), \qquad \text{for some } j,\ 0 \leqslant j \leqslant \frac{m}{d} - 1.$$

Observe that

$$(kr_0 + jd) \pmod m = (kr_0) \pmod d + j'd$$

whenever $j' = \left[\frac{(kr_0+jd) \pmod m}{d}\right]$ or, equivalently,

$$j = \Big(j' - \Big[\frac{(kr_0) \pmod m}{d}\Big]\Big) \pmod{m/d}.$$

Therefore, (5.14) holds if and only if

$$(5.15)\quad m\frac{\alpha}{2\pi} < (kr_0) \pmod d + jd < m\Big(1 - \frac{\alpha}{2\pi}\Big), \qquad \text{for some } j,\ 0 \leqslant j \leqslant m/d - 1.$$

We now show that $(kr_0) \pmod d$ attains all the values l, $1 \leqslant l \leqslant d$, satisfying $(l,d) = 1$, as k goes through all the values between 1 and m that are co-prime with m. Let k be such a number. Since d divides m, it follows that $(d,k) = 1$. It is given that $(r_0, q) = 1$, and hence, since d divides q, we have $(r_0, d) = 1$. Thus $(kr_0, d) = 1$, and hence $(kr_0) \pmod d$ is co-prime with d. Now let l be any integer l, $1 \leqslant l \leqslant d$, such that $(l,d) = 1$. Since $(r_0, d) = 1$ there exists an integer k, $1 \leqslant k \leqslant d$, such that

$$(5.16)\qquad kr_0 \pmod d = l.$$

Since $(k,d) = 1$, it follows by Dirichlet's Theorem on primes in arithmetic progressions (e.g. [2, Theorem 15, p. 13]) that there exists a prime number of the form $k + jd$. We thus have

$$(5.17)\qquad \big((k+jd) \pmod m, m\big) = 1.$$

Let $k' = (k + jd) \pmod m$. Since d divides m, we have $k' \pmod d = k$. Thus, by (5.16) we have $k'r_0 \pmod d = l$. In view of (5.17), we have just proven that (kr_0)

(mod d) attains all the values l, $1 \leqslant l \leqslant d$, satisfying $(l,d)=1$, as k goes through all the values between 1 and m that are co-prime with m. Therefore, (5.15) holds for every k, $1 \leqslant k \leqslant m$, satisfying $(k,m)=1$, if and only if for every $1 \leqslant l \leqslant d$, such that $(l,d)=1$, there exists j, $0 \leqslant j \leqslant m/d-1$, such that $m\frac{\alpha}{2\pi} < l+jd < m\left(1-\frac{\alpha}{2\pi}\right)$. Note that $\frac{m}{2}$ is the midpoint of the interval $I=\left(m\frac{\alpha}{2\pi}, m\left(1-\frac{\alpha}{2\pi}\right)\right)$. Denote by L the length $m\left(1-\frac{\alpha}{\pi}\right)$ of I. We distinguish between the various cases:

(i) $d=1$, m is even. Since $d=1$, all we require is that I contain an integral point. The midpoint $\frac{m}{2}$ is such a point.

(ii) $d=1$, m is odd. Here I contains an integral point if and only if $L>1$, which is equivalent to $\alpha < \left(1-\frac{1}{m}\right)\pi$.

(iii) $d=2,3,4$, m/d is even. Since m/d is even, we have $m/2=0$ (mod d). The only positive integers between 1 and d that are co-prime with d are 1 and $d-1$. Therefore, our condition is satisfied if and only if I contains the points $\frac{m}{2}-1$ and $\frac{m}{2}+1$. This happens if $L>2$, which is equivalent to $\alpha < \left(1-\frac{2}{m}\right)\pi$.

(iv) d is odd, $d>5$, m/d is even. Since $\left(\frac{d\pm1}{2}, d\right)=1$, I must contain the points $\frac{m}{2}\pm\frac{d-1}{2}$. This would be enough, because in this case I would contain d consecutive integral points, and hence all the values mod d. So, our condition is satisfied if and only if $L>2\frac{d-1}{2}$, which is equivalent to $\alpha < \left(1-\frac{d-1}{m}\right)\pi$.

(v) $d=0$ (mod 4), $d>5$, m/d is even. Since 2 divides $\frac{d}{2}$, it follows that $\frac{d}{2}\pm1$ is not divisible by 2. Thus, $\left(\frac{d}{2}\pm1, d\right)$ is not divisible by 2, and it follows that $\left(\frac{d}{2}\pm1, d\right)=1$. Hence, I must contain the points $\frac{m}{2}\pm\left(\frac{d}{2}-1\right)$. This would be enough, because in this case I would contain $d-1$ consecutive integral points, and hence all the values mod d that are co-prime with d. So, our condition is satisfied if and only if $L>2\left(\frac{d}{2}-1\right)$, which is equivalent to $\alpha < \left(1-\frac{d-2}{m}\right)\pi$.

(vi) $d=2$ (mod 4), $d>5$, m/d is even. Let $e=\left(\frac{d}{2}+2, d\right)$ or $e=\left(\frac{d}{2}-2, d\right)$. It follows that e divides 4. Since $\frac{d}{2}$ is odd, it follows that e is odd, and so $\left(\frac{d}{2}\pm2, d\right)=1$. Hence, I must contain the points $\frac{m}{2}\pm\left(\frac{d}{2}-2\right)$. To see that this is enough, observe that the interval $\left(\frac{m}{2}-\left(\frac{d}{2}-1\right), \frac{m}{2}+\left(\frac{d}{2}-1\right)\right)$ contains $d-1$ consecutive integral points and hence all values mod d that are co-prime with d. Since $\frac{d}{2}-1$ is even, we have $\left(\frac{d}{2}-1, d\right)\geqslant 2$, and hence the interval $\left(\frac{m}{2}-\left(\frac{d}{2}-2\right), \frac{m}{2}+\left(\frac{d}{2}-2\right)\right)$ contains all the values mod d that are co-prime with d. We now have $L>2\left(\frac{d}{2}-2\right)$, which is equivalent to $\alpha < \left(1-\frac{d-4}{m}\right)\pi$.

(vii) $d>1$, m/d is odd. Here $\frac{m}{2}=\frac{d}{2}$ (mod d). Here, I must contain the points $\frac{m}{2}\pm\left(\frac{d}{2}-1\right)$, which are equal to ±1 (mod d). This would be enough, because in this case I would contain $d-1$ consecutive integral points, and hence all the values mod d that are co-prime with d. So, our condition is satisfied if and only if $L>2\left(\frac{d}{2}-1\right)$, which is equivalent to $\alpha < \left(1-\frac{d-2}{m}\right)\pi$.

The proof of Proposition 5.9 is now completed. □

5.18 Lemma. *Let r_0 and q be positive integers. The arithmetic sequence $R = \big(r_n = r_0 + (n-1)q\big)_{n=1}^{\infty}$ is a Kronecker sequence.*

Proof. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $q\theta \in \mathbb{R} \setminus \mathbb{Q}$. By Theorem 5.3, the sequence $\big(nq\theta \pmod 1\big)_{n=1}^{\infty}$ is dense in the interval $[0,1]$. Clearly, this implies that the sequence $\big((r_0\theta - \theta + nq\theta) \pmod 1\big)_{n=1}^{\infty} = \big([r_0 + (n-1)q]\theta \pmod 1\big)_{n=1}^{\infty}$ is dense in the interval $[0,1]$, proving our assertion. □

Using the above results, we can now prove the characterization of arithmetic forcing sequences.

5.19 Theorem. *Let r_0 and q be positive integers, let R be the arithmetic sequence $\big(r_n = r_0 + (n-1)q\big)_{n=1}^{\infty}$, and let $0 \leqslant \alpha < \pi$. Then the following are equivalent.*
(i) R is α-forcing.
(ii) $(r_0, q) = 1$, and

$$\alpha < \begin{cases} \frac{2\pi}{3}, & q = 1, \\ \frac{\pi}{2}, & q = 2, \\ \frac{2\pi}{q}, & q \geqslant 3. \end{cases} \tag{5.20}$$

Proof. (i)$\Longrightarrow$(ii). Let $d = (r_0, q)$, and let $c = e^{\frac{2\pi i}{d}}$. Since d divides r_n, we have $c^{r_n} = 1$, $n \geqslant 1$. Therefore, $c \in S[R, \alpha, \infty]$. Since R is α-forcing, $c \in \mathbb{R}_+$, implying that $d = 1$. If $q = 1$ then let $c = e^{\frac{2\pi i}{3}}$. Since $c^n \in W\big[\frac{2\pi}{3}\big]$, $n \geqslant 1$, and since R is α-forcing, it follows that we must have $\alpha < \frac{2\pi}{3}$. If $q = 2$ then, since $(r_0, q) = 1$, the integer r_0 is odd, and hence r_n is odd for all $n \geqslant 1$. Therefore, for $c = e^{\frac{\pi i}{2}}$ we have $c^n \in W\big[\frac{\pi}{2}\big]$, $n \geqslant 1$. Since R is α-forcing, it follows that we must have $\alpha < \frac{\pi}{2}$. Finally, assume that $q \geqslant 3$. Let $m = q$, and let $d = (q, m) = q$. Since by (i) R is also (T_m, α)-forcing, it follows by Proposition 5.9 that $\alpha < \big(1 - \frac{d-2}{m}\big)\pi = \frac{2\pi}{q}$.

(ii)$\Longrightarrow$(i). Let m be a positive integer, $m \geqslant 2$, and let $d = (q, m)$. We distinguish between several cases.

(i) $q = 1$. Here $d = 1$. By (5.20) we have

$$\alpha < \frac{2\pi}{3} \leqslant \begin{cases} \pi, & m \text{ is even}, \\ \frac{m-1}{m}\pi, & m \text{ is odd}, \end{cases}$$

and by Proposition 5.9 R is (T_m, α)-forcing.

(ii) $q = 2$, m is even. Here $d = 2$. By (5.20) we have

$$\alpha < \frac{\pi}{2} \leqslant \begin{cases} \pi, & m/2 \text{ is odd}, \\ \frac{m-2}{m}\pi, & m/2 \text{ is even}, \end{cases}$$

and by Proposition 5.9 R is (T_m, α)-forcing.

(iii) $q = 2$, m is odd. Here $d = 1$. By (5.20) we have $\alpha < \frac{\pi}{2} < \frac{2\pi}{3} \leqslant \frac{m-1}{m}\pi$, and by Proposition 5.9 R is (T_m, α)-forcing.

(iv) $q = 3$, m is divisible by 3. Here $d = 3$. By (5.20) we have

$$\alpha < \frac{2\pi}{3} \leqslant \begin{cases} \dfrac{m-1}{m}\pi, & m/3 \text{ is odd}, \\ \dfrac{m-2}{m}\pi, & m/3 \text{ is even}, \end{cases}$$

and by Proposition 5.9 R is (T_m, α)-forcing.

(v) $q = 3$, m is not divisible by 3. Here $d = 1$. By (5.20) we have

$$\alpha < \frac{2\pi}{3} \leqslant \begin{cases} \pi, & m \text{ is even}, \\ \dfrac{m-1}{m}\pi, & m \text{ is odd}, \end{cases}$$

and by Proposition 5.9 R is (T_m, α)-forcing.

(vi) $q \geqslant 4$, $m = q$. Here $d = q$. By (5.20) we have $\alpha < \frac{2\pi}{m} = \left(1 - \frac{d-2}{m}\right)\pi$, and by Proposition 5.9 R is (T_m, α)-forcing.

(vii) $q \geqslant 4$, $m \neq q$, $d = 1$. By (5.20) we have

$$\alpha < \frac{2\pi}{q} < \frac{2\pi}{3} \leqslant \begin{cases} \pi, & m \text{ is even}, \\ \dfrac{m-1}{m}\pi, & m \text{ is odd}, \end{cases}$$

and by Proposition 5.9 R is (T_m, α)-forcing.

(viii) $q \geqslant 4$, $m \neq q$, $d \neq 1, 2$. Since $m \neq q$, we have $d \leqslant m/2$. By (5.20) we now obtain $\alpha < \frac{2\pi}{q} \leqslant \frac{\pi}{2} < \left(1 - \frac{m/2-1}{m}\right)\pi \leqslant \left(1 - \frac{d-1}{m}\right)\pi < \left(1 - \frac{d-2}{m}\right)\pi < \left(1 - \frac{d-4}{m}\right)\pi$, and by Proposition 5.9 R is (T_m, α)-forcing.

(ix) $q \geqslant 4$, $m \neq q$, $d = 2$. By (5.20) we have

$$\alpha < \frac{2\pi}{q} \leqslant \begin{cases} \pi, & m/2 \text{ is odd}, \\ \dfrac{m-2}{m}\pi, & m/2 \text{ is even}, \end{cases}$$

and by Proposition 5.9 R is (T_m, α)-forcing.

We have completed proving that R is (T_m, α)-forcing for every m, $m \geqslant 2$. Hence, R is $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing, where $\mathbb{C}_{\mathbb{Q}} = \left\{c \in \mathbb{C} \colon \frac{\arg(c)}{2\pi} \in \mathbb{Q}\right\}$. By Lemma 5.18, R is a Kronecker sequence, and so by Proposition 5.4 R is also $(\mathbb{C} \setminus \mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing. Therefore, R is an α-forcing sequence. □

6.1 Definition. A (T,α)-forcing sequence R is said to be a *minimal* (T,α)-*forcing sequence* if every proper subsequence of R is not (T,α)-forcing.

Minimal forcing sequences do exist, as follows from the following proposition.

6.2 Proposition. *Let q be a positive integer, $q \geqslant 2$. The sequence $(q^{n-1})_{n=1}^{\infty}$ is a minimal α-forcing sequence for all α, $\frac{2\pi}{q^2} \leqslant \alpha < \frac{2\pi}{q+1}$.*

Proof. Since $\alpha < \frac{2\pi}{q+1}$, it follows by Theorem 4.3 that R is α-forcing. To see that R is a minimal α-forcing sequence let $c = e^{\frac{2\pi i}{q^m}}$, where m is a positive integer. If $n < m$ then $m-n+1 \geqslant 2$, and since $\frac{2\pi}{q^2} \leqslant \alpha$ we have $c^{r_n} = e^{\frac{2\pi i}{q^{m-n+1}}} \in W[\alpha]$. If $n = m$ then, since $\alpha < \frac{2\pi}{q+1}$, we have $c^{r_n} = e^{\frac{2\pi i}{q}} \notin W[\alpha]$. If $n > m$ then $c^{r_n} = 1 \in W[\alpha]$. Thus, r_m is the only element r_n of R for which $c^{r_n} \notin W[\alpha]$, and so r_m must belong to every forcing subsequence of R. □

As is observed in Remark 3.2.iii in [5], there exists no α-forcing sequence for $\alpha \geqslant \frac{2\pi}{3}$. As a corollary to Proposition 6.2 we obtain

6.3 Corollary. *There exists a minimal α-forcing sequence whenever $0 < \alpha < \frac{2\pi}{3}$.*

Proof. Let q be the smallest integer that is greater than or equal to $\frac{2\pi}{\alpha} - 2$. Then

$$\frac{2\pi}{q+2} \leqslant \alpha < \frac{2\pi}{q+1}. \tag{6.4}$$

Since $\alpha < \frac{2\pi}{3}$, we have $q > 1$, and so $q \geqslant 2$, implying that $q^2 \geqslant q + 2$. It now follows from (6.4) that $\frac{2\pi}{q^2} \leqslant \alpha < \frac{2\pi}{q+1}$, and by Proposition 6.2 the sequence $(q^{n-1})_{n=1}^{\infty}$ is a minimal α-forcing sequence. □

Corollary 6.3 raises the following natural question.

6.5 Question. Does every α-forcing sequence have a minimal α-forcing subsequence?

Question 6.5 is still an open problem. As a possible approach for a further study we now introduce an algorithm that prunes a forcing sequence without losing the forcing property. We remark that our algorithm does not suggest a computational method for constructing minimal forcing sequences, but it implies a positive answer to Question 6.5 under certain conditions. We restrict our discussion to $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequences, where $\mathbb{C}_{\mathbb{Q}} = \left\{c \in \mathbb{C} \colon \frac{\arg(c)}{2\pi} \in \mathbb{Q}\right\}$.

6.6 Algorithm. Let $0 < \alpha < \pi$, and let $R = (r_n)_{n=1}^{\infty}$ be a $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence. Let $(c_m)_{m=1}^{\infty}$ be an ordering of the primitive roots of the unity that are not equal to one. We construct a sequence $\{R^k\}_{k=0}^{\infty}$ of $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequences, satisfying $R = R^0 \supseteq R^1 \supseteq \ldots$.

Initialization: We let R^0 be the sequence R.

Step k, $k = 1, 2, \ldots$:

Step k.1: Since R^{k-1} is $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing, the set $S = \{r \in R^{k-1} \colon c_1^r \notin W[\alpha]\}$ is non-empty. We let

$$r_1^k = \begin{cases} r,\ r \in S \setminus \{r_k\}, & \text{if } S \setminus \{r_k\} \neq \emptyset, \\ r_k, & \text{if } S \setminus \{r_k\} = \emptyset. \end{cases}$$

Step k.j, $j = 2, 3, \ldots$: Obviously, the set $T = \{c_l \colon c_l^{r_i^k} \in W[\alpha],\ i = 1, \ldots, j-1\}$ contains the number $e^{\frac{2\pi i}{m}}$ for m sufficiently large, and thus is non-empty. Let $t = \min\{l \colon c_l \in T\}$. Since R^{k-1} is $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing, the set $S = \{r \in R^{k-1} \colon c_l^r \notin W[\alpha]\}$ is non-empty. We let

$$r_j^k = \begin{cases} r,\ r \in S \setminus \{r_k\}, & \text{if } S \setminus \{r_k\} \neq \emptyset, \\ r_k, & \text{if } S \setminus \{r_k\} = \emptyset. \end{cases}$$

Conclusion of step k: We order R^k in an increasing order. By our definition, for every j there exists $i \in \langle j \rangle$ such that $c_j^{r_i^k} \notin W[\alpha]$, and hence R^k is $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing. Also, we have $R^k \subseteq R^{k-1}$.

In order to show that in certain cases Algorithm 6.6 produces a minimal forcing sequence, we define

6.7 Definition. An element r of a (T, α)-forcing sequence is called (R, T, α)-*minimal* if $R \setminus \{r\}$ is not (T, α)-forcing.

6.8 Proposition. *For every positive integer k, the elements of $R^k \cap \{r_1, \ldots, r_k\}$ are $(R^k, \mathbb{C}_{\mathbb{Q}}, \alpha)$-minimal.*

Proof. Assume that $r_i \in R^k$ for some $i \in \langle k \rangle$. Since $R^k \subseteq R^i$, we have $r_i \in R^i$. By the construction in Algorithm 6.6, $r_i \in R^i$ if and only if there exists a root c of the unity such that r_i is the only element of R^{i-1} for which $c^{r^i} \notin W[\alpha]$. Since $R^k \subseteq R^{i-1}$, it follows that r_i is the only element of R^k for which $c^{r^i} \notin W[\alpha]$. Thus, r_i is $(R^k, \mathbb{C}_{\mathbb{Q}}, \alpha)$-minimal. □

6.9 Corollary. *If $R^{\infty} = \bigcap_{k=0}^{\infty} R^k$ is a $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence then R^{∞} is a minimal $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence.*

Finally, we now show that for every $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence R, there exists a subset $\mathbb{C}'$ of $\mathbb{C}$ such that R^∞ is a minimal $(\mathbb{C}', \alpha)$-forcing sequence.

6.10 Notation. Let R be a $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence, and let $c \in \mathbb{C}_{\mathbb{Q}} \setminus \mathbb{R}_+$. Then the set $S = \{r_l \in R\colon c^{r_1} \notin W[\alpha]\}$ is non-empty. We denote

$$i_{\max}(c) = \begin{cases} \max\{l\colon r_l \in S\}, & \text{if } S \text{ is finite,} \\ \infty, & \text{if } S \text{ is infinite.} \end{cases}$$

6.11 Proposition. *Let R be a $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence. If for some $c \in \mathbb{C}_{\mathbb{Q}}$ we have $i_{\max}(c) < \infty$ then there exists $r \in R^\infty$ for which $c^r \notin W[\alpha]$.*

Proof. Let $i = i_{\max}(c)$. By Algorithm 6.6 and Proposition 6.8, for every $l \geqslant k$ we have $R^l \cap \{r_1, \dots, r_k\} = R^k \cap \{r_1, \dots, r_k\}$. Therefore, if there exists $r \in R^{i-1} \cap \{r_1, \dots, r_{i-1}\}$ for which $c^r \notin W[\alpha]$ then we have $r \in \bigcap_{k=0}^{\infty} R^k$ and we are done. Otherwise, since R^{i-1} is a $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence, there exists $r \in R^{i-1} \setminus \{r_1, \dots, r_{i-1}\}$ for which $c^r \notin W[\alpha]$. Since $i_{\max}(c) = i$, it follows by Notation 6.10 that r_i is the only element of R^{i-1} for which $c^{r_i} \notin W[\alpha]$. By Algorithm 6.6 we have $r_i \in R^i \cap \{r_1, \dots, r_i\}$, and hence $r_i \in \bigcap_{k=0}^{\infty} R^k$. □

Let $\mathbb{C}'$ be the set $\{c \in \mathbb{C}_{\mathbb{Q}} : i_{\max}(c) < \infty\}$. We now obtain

6.12 Corollary. *Let R be a $(\mathbb{C}_{\mathbb{Q}}, \alpha)$-forcing sequence. Then R^∞ is a minimal $(\mathbb{C}', \alpha)$-forcing sequence.*

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