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ON NONCONVEX VALUED VOLTERRA INTEGRAL INCLUSIONS IN BANACH SPACES

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1. INTRODUCTION

In a recent paper [22], we examined Volterra integral inclusions of the form

$$x(t) \in p(t) + \int_0^t U(t,s)F\big(s,x(s)\big)\,\mathrm{d}s\,, \quad t \in T = [0,b] \tag{1}$$

in a separble Banach space X. In inclusion (1), $p(\cdot) \in C(T,X)$, $U(t,s) \in \mathcal{L}(X)$ for all $0 \leqslant s \leqslant t \leqslant b$ with $\|U(t,s)\|_{\mathcal{L}} \leqslant M$ and $F\colon T \times X \to 2^X \setminus \{\emptyset\}$ is a closed valued perturbation. Our assumption on the kernel $U(t,s)$ was general enough to allow interpretting $U(t,s)$ as an evolution operator generated by a family of unbounded, densely defined operations $\{A(t)\colon t \in T\}$. If this is the case, then (1) describes the mild soutions of the semilinear evolution inclusion $\dot{x}(t) \in A(t)x(t) + F\big(t,x(t)\big)$, $x(0) = x_0$, with $p(t) = U(t,0)x_0$. Such inclusions were studied by Papageorgiou [17] under the hypothesis that $U(t,s)$ is compact for all $t - s > 0$. In [22], the kernel $U(t,s)$ was not assumed to be compact for $t - s > 0$, and instead it was assumed that the orientor field $F(t,x)$ satisfied a compactness type hypothesis involving the Hausdorff (ball) measure of noncompactness.

In this paper we continue alogn the lines of [22]. In addition to (1), we also consider the following Volterra integral inclusion:

$$x(t) \in p(t) + \int_0^t U(t,s)\operatorname{ext} F\big(s,x(s)\big)\,\mathrm{d}s\,, \quad t \in T = [0,b] \tag{2}$$

where $\operatorname{ext} F\big(s,x(s)\big)$ denotes the set of extremal points of $F\big(s,x(s)\big)$. Problems of this from arise in the study of control systems, in particular in the derivation of "bang-bang principles".

We should note that the theory developed in [22] can no longer be applied on (2), because the multifunction $(t, x) \to \operatorname{ext} F(t, x)$ is not in general closed valued and we can not say anything about its continuity properties. So our results here extend the existence theorems obtained in [22], and also we prove a new, very general density (relaxation) result relating the solution sets of (1) and (2) above. Hence the work of this paper in addition of extending, also complements [22], by presenting a relaxation theorem, a result that is missing from the study conducted in [22]. Finally, we should mention that our work here also extends the single valued one by Szufla [25] and the multivalued ones by Ragimkhanov [23] and Bulgakov-Lyapin [5] (who studied Volterra integral inclusions in $\mathbb{R}^n$) and by Chuong [6] and Papageorgiou [20] (who studied Volterra itegral inclusions in Banach spaces, but under much more restrictive hypotheses on the data).

2. Preliminaries

In this section we establish our notation and recall some basic definitions and results about measurable and continuous multifunctions that we will need in the sequel.

Let (Ω, Σ) be a measurable spece and X a separable Banach space. Throughout this paper, we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{ nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{ nonempty, (weakly-) compact, (convex)}\}.$$

For any $A \in 2^X \setminus \{\emptyset\}$, we set $|A| = \sup\{\|x\|\colon x \in A\}$ (the "norm" of A), $\sigma(x^*, A) = \sup[(x^*, a)\colon a \in A]$, $x^* \in X^*$ (the "support function" of A) and for every $z \in X$, $d(z, A) = \inf[\|z - a\|\colon a \in A]$ (the "distance function" from A).

A multifunction (set-valued fucntion), is said to be measurable, if for all $U \subseteq X$ nonempty open, $F^-(U) = \{\omega \in \Omega\colon F(\omega) \cap U \neq \emptyset\} \in \Sigma$. If in addition, we assume that $F(\cdot)$ is $P_f(X)$-valued, then the above definition is equivalent to any one of the following two statements:

(i) for every $z \in X$, $\omega \to d(z, F(\omega))$ is measurable,

(ii) there exist $f_n\colon \Omega \to X$ $n \geqslant 1$ measurable functions s.t. $F(\omega) = \overline{\{f_n(\omega)\}}_{n \geqslant 1}$ for all $\omega \in \Omega$.

These equivalent statements all imply

(iii) $\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\}$, with $B(X)$ being the Borel σ-field of X (graph measurability).

If there is a complete, σ-finite measure $\mu(\cdot)$ defined on Σ, then graph measurability is in fact equivalent to measurability for $P_f(X)$-valued multifunctions. For more details on the measurability of multifunctions, we refer to the survey paper of Wagner [28].

Now let (Ω, Σ, μ) be a finite measure space. Given $F\colon \Omega \to P_f(X)$ a measurable multifunction, we denote by S_F^1 the set of all selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^1(X)$; i.e. $S_F^1 = \{f \in L^1(S)\colon f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. Clearly this set is closed, maybe empty and using Aumann's selection theorem (see Wagner [28], theorem 5.10), we can check that S_F^1 is nonempty if and only if $\omega \mapsto \inf[\|x\|\colon x \in F(\omega)] \in L_+^1$. This is the case if $\omega \to |F(\omega)| \in L_+^1$ and such a multifunction is called "integrably bounded". A detailed study of S_F^1 can be found in [21]. Using S_F^1 we can define a set-valued integral for $F(\cdot)$, by setting $\int_\Omega F(\omega)\,\mathrm{d}\mu(\omega) = \{\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega)\colon f \in S_F^1\}$. The properties of this integral were studied by Kandilakis-Papageorgiou [11].

It is a simple consequence of the Banach-Dieudonne theorem (see for example, Bourbaki [3]), that $A \in P_{kc}(X)$ if and only if $\sigma(\cdot, A)$ is sequentially continuous on $X_{w^*}^*$ (here $X_{w^*}^*$ denotes the Banach space X^* equipped with the weak* topology; recall that since by hypothesis X is separable, on bounded subsets of X^* the relative w^*-topology is metrizable). Using this fact, we see that $P_{kc}(X)$ can be embedded as a convex cone with vertex zero in separable Banach space Z (in particular, $Z = C(B_{w^*}^*)$, where $B_{w^*}^*$ denotes the unit ball of X^* equipped with the weak*-topology) and the embedding is additive, positively homogeneous and isometric. This is the well-known "Radström Embedding Theorem" (see for example, Klein-Thompson [13], theorem 17.2.1, p. 189). In particular, then if $F\colon \Omega \to P_{kc}(X)$ is an integrably bouded multifunction, then the set-valued integral $\int_\Omega F(\omega)\,\mathrm{d}\mu(\omega)$ is equal to the Bochner integral of $F(\cdot)$, when it is viewed as an element in $L^1(Z)$ (see Hiai-Umagaki [9], theorem 4.5 (i) and Papageorgiou [16], proposition 3.1). Therefore if $F\colon \Omega \to P_{kc}(X)$ is integrably bounded, then $\int_\Omega F(\omega)\,\mathrm{d}\mu(\omega) \in P_{kc}(X)$.

On $P_f(X)$ we can define a generalized metric, known in the literature as Hausdorff metric, by setting

$$h(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right]$$

for all A, $B \in P_f(X)$. It is well known that $\big(P_f(X), h\big)$ is a complete metric space, while $\big(P_k(X), h\big)$ is a Polish space (i.e. complete, separable, metric space). A multifunction $F\colon X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from X into $\big(P_f(X), h\big)$.

If V, W are Hausdorff topological spaces and $G\colon V \to 2^W \setminus \{\emptyset\}$, then $G(\cdot)$ is lower semicontinuous (l.s.c.), if for all $U \subseteq W$ open, $G^-(U) = \{v \in V\colon G(v) \cap U \neq \emptyset\}$ is open. If V, W are metric spaces, then this definition is equivalent to saying that if $v_n \to v$ in V, then $G(v) \subseteq \underline{\lim} G(v_n) = \{w \in W\colon \lim d_W(w, G(w)) = 0\} = \{w \in W\colon w = \lim w_n, w_n \in G(v_n), n \geqslant 1\}$ (here $d_W(\cdot,\cdot)$ denotes the metric on W).

Let $\mathcal{B}$ denote the collection of all bounded subsets of X. The Hausdorff (ball) measure of noncompactness $\beta\colon \mathcal{B} \to \mathbb{R}_+$ is defined by

$$\beta(B) = \inf\{r > 0\colon B \text{ can be convered by finitely many balls of radius } r\}.$$

Recall that $\beta(\cdot)$ is nonexpansive with respect to the Hausdorff pseudo-metric on $(2^X \setminus \{\emptyset\}) \cap \mathcal{B}$. For a comprehensive introduction to the subject of measures of noncompactness and their applications, we refer to Banas-Goebel [1].

3. Existence theorem

For the rest of this paper, $T = [0, b]$ and X is a separable Banach space. By $L^1(X)$ we will denote the Banach space of all equivalence classes of Bochner integrable functions $x\colon T \to X$, equipped with the usual norm $\|x\|_1 = \int_0^b \|x(t)\|\,\mathrm{d}t$. Also by $L^1_w(X)$, we will denote the space of all equivalence classes of Bochner integrable functions $x\colon T \to X$, equipped with the norm (weak norm) $\|x\|_w = \sup\limits_{0\leqslant t\leqslant b} \|\int_0^t x(\tau)\,\mathrm{d}\tau\|$ (or equivalently $\|x\|_w = \sup\limits_{0\leqslant s\leqslant t\leqslant b} \|\int_s^t x(\tau)\,\mathrm{d}\tau\|$).

In this section we address the problem of existence of solution for inclusions (2). By a solution of (2), we mean a function of $x(\cdot) \in C(T, X)$ such that $x(t) = p(t) + \int_0^t U(t,s)f(s)\,\mathrm{d}s$, $t \in T$, $f \in L^1(X)$, $f(s) \in \operatorname{ext} F(s, x(s))$ a.e. on T. We will need the following hypotheses on the data:

$H(F)$: $F\colon T \times X \to P_{fc}(X)$ is multifunction s.t.

(1) $t \to F(t,x)$ is measurable,

(2) $x \to F(t,x)$ is h-continuous,

(3) $\beta(F(t,B)) \leqslant k(t)\beta(B)$ a.e. for all $B \subseteq X$ nonempty, bounded (i.e. $B \in \mathcal{B}$) and with $k(\cdot) \in L^1_+(T)$,

(4) $|F(t,x)| \leqslant a(t) + c(t)\|x\|$ a.e., with $a(\cdot)$, $c(\cdot) \in L^1_+(T)$.

Remark. Note that hypothesis $H(F)$ (1) and (2) and theorem 3.3 of Papageorgiou [19], imply that $(t,x) \to F(t,x)$ is measurable on $T \times X$. Also from hypothesis $H(F)$ (3), we have that for all $t \in T \setminus N$ (N is a Lebesgue null subset of T) and all $x \in X$, $F(t,x) \in P_{kc}(X)$. Hence by modifying, if necessary, the orientor field F on the Lebesgue null set $N \subseteq T$, we may assume without any loss of generality that $F(t,x) \in P_{kc}(X)$ for all $(t,x) \in T \times X$. So in what follows we will assume that for all $(t,x) \in T \times X$, $F(t,x) \in P_{kc}(X)$ and so $\operatorname{ext} F(t,x) \neq \emptyset$. Then from theorem 9.3

of Himmelberg [10], we have that $(t,x) \to \operatorname{ext} F(t,x)$ is graph measurable (recall that X^* equipped with the w^*-topology (and hence with any topology compatible with duality (X^*,X)) is separable; see Wilansky [29], p. 144). So in particular, if $x\colon T \to X$ is measurable, then $t \to \operatorname{ext} F\bigl(t,x(t)\bigr)$ is graph measurable.

$H(U)$: $U\colon \Delta = \{0 \leqslant s \leqslant t \leqslant b\} \to \mathcal{L}(X)$ is a map s.t.

(1) $U(\cdot,s)$ is strongly continuous on $[s,b]$, $U(t,\cdot)$ is strongly continuous on $[0,t]$ for all $t \in T$, $U(t,t) = I$,

(2) $\int_0^t \|U(t',s) - U(t,s)\|\psi(s)\,\mathrm{d}s = \eta(t',t) \to 0$ as $t'-t \to 0^+$ with t' or t fixed and with $\psi(s) = a(s) + c(s)$ (see $H(F)$ (4)).

Remark. If $\|U(t',s) - U(t,s)\|_{\mathcal{L}} \leqslant \frac{c(t'-t)}{t-s}$, then we can easily check that hypothesis $H(U)$ (2) is satisfied, if for example, a, $c \in L^2_+(T)$ (hence $\psi \in L^2_+(T)$). In turn, this estimate is valid, if $U(t,s)$ is the evolution operator (fundamental solution) generated by $\{A(t)\colon t \in T\}$ a family of linear, generally unbounded operators s.t. (i) $\overline{D\bigl(A(t)\bigr)} = X$ and $D\bigl(A(t)\bigr)$ is independent of $t \in T$, (ii) for each $t \in [0,b]$, the resolvent $R\bigl(\lambda, A(t)\bigr)$ exists for all $\operatorname{Re}\lambda \leqslant 0$ and $\bigl\|R\bigl(\lambda,A(t)\bigr)\bigr\|_{\mathcal{L}} \leqslant \frac{c}{|\lambda|+1}$ ($\operatorname{Re}\lambda \leqslant 0$) and (iii) $\bigl\|\bigl(A(t') - A(t)\bigr)A^{-1}(0)\bigr\| \leqslant c|r'-t|^{\alpha}$ $\alpha \in (0,1)$. For details, we refer to the books of Friedman [7], Ladas-Lakshmikantham [14] and Tanabe [26]. In particular, this is the case if $X \to H \to X^*$ is an evolution triple of separable Hilbert spaces and $A(t)\colon X \to X^*$ is a linear, continuous, strongly monotone operator (see Tanabe [26], chapter 5, section 4) or if $U(t,s) = K(t-s)$ with $K(\cdot)$ being an analytic semigroup (autonomous case; see [7], [14], [26]). Therefore our formulation incorporates large classes of semilinear evolution equations.

$H(p)$: $p(\cdot) \in C(T,X)$.

Theorem 3.1. *If hypotheses $H(F)$, $H(U)$ and $H(p)$ hold, then problem* (2) *admits a solution.*

Proof. First we will obtain an a priori bound for the solutions of (1) (hence for thoose of (2) too). So let $x(\cdot) \in C(T,X)$ be such a solution. We have:

$$x(t) = p(t) + \int_0^t U(t,s)f(s)\,\mathrm{d}s\,, \quad t \in T$$

with $f \in L^1(X)$ and $f(t) \in F\bigl(t,x(t)\bigr)$ a.e. Then we have

$$\begin{aligned}\|x(t)\| &\leqslant \|p\|_\infty + \int_0^t \|U(t,s)\|_{\mathcal{L}} \cdot \|f(s)\|\,\mathrm{d}s\\ &\leqslant \|p\|_\infty + \int_0^t M\bigl(a(s) + c(s)\|x(s)\|\bigr)\,\mathrm{d}s\,.\end{aligned}$$

Invoking Gronwall's inequality, we deduce that there exists $M_1 > 0$ s.t.

$$\|x(t)\| \leqslant M_1$$

for all $t \in T$ and all solutions $x(\cdot)$ of (1). Let $\varphi(t) = a(t) + M_1 c(t)$, $\varphi(\cdot) \in L^1_+(T)$. Then we may assume without any loss of generality that $|F(t,x)| \leqslant \varphi(t)$ a.e. (otherwise we replace $F(\cdot, x(\cdot))$, by $F(\cdot, p_{M_1}(x(\cdot)))$, with $p_{M_1}(\cdot)$ being the M_1-radial retraction).

Next we claim that we can find $G\colon T \to P_{kc}(X)$ an h-continuous multifunction s.t. for all $t \in T$, we have

$$G(t) = p(t) + \int_0^t U(t,s)\, \overline{\mathrm{conv}}\, F\big(s, G(s)\big)\, \mathrm{d}s\,, \quad t \in T.$$

To this end, consider the following Caratheodory type approximations:

$$G_n(t) = \begin{cases} p(t), & 0 \leqslant t \leqslant \frac{b}{n}, \\ p(t) + \displaystyle\int_0^{t-\frac{b}{n}} U(t - \tfrac{b}{n}, s)\, \overline{\mathrm{conv}}\, F\big(s, G_n(s)\big)\, \mathrm{d}s\,, & \frac{b}{n} \leqslant t \leqslant b. \end{cases}$$

Also for every $n \geqslant 1$, let

$$H_n(t) = p(t) + \int_0^t U(t,s)\, \overline{\mathrm{conv}}\, F\big(s, G_n(s)\big)\, \mathrm{d}s\,, \quad t \in T.$$

Recall tha since $F(t,\cdot)$ is h-continuous and $P_{kc}(X)$-valued, it maps compact sets into compact sets (see Klein-Thompson [13]). So by virtue of the Radström embedding theorem (see section 2), we have that for all $n \geqslant 1$ ans all $t \in T$, $G_n(t)$, $H_n(t) \in P_{kc}(X)$.

Let t, $t' \in T$, $t < t'$. We have:

$$\begin{aligned} & h\big(H_n(t'), H_n(t)\big) \\ & = h\Big(p(t') + \int_0^{t'} U(t',s)\, \overline{\mathrm{conv}}\, F\big(s, G_n(s)\big)\, \mathrm{d}s\,, p(t) + \int_0^t U(t,s)\, \overline{\mathrm{conv}}\, F\big(s, G_n(s)\big)\, \mathrm{d}s\Big) \\ & \leqslant \|p(t') - p(t)\| \\ & \quad + h\Big(\int_0^{t'} U(t',s)\, \overline{\mathrm{conv}}\, F\big(s, G_n(s)\big)\, \mathrm{d}s\,, \int_0^t U(t,s)\, \overline{\mathrm{conv}}\, F\big(s, G_n(s)\big)\, \mathrm{d}s\Big) \end{aligned}$$

$$\leqslant \|p(t') - p(t)\| + \left| \int_t^{t'} U(t',s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s \right|$$

$$+ h\bigg(\int_0^t U(t',s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s\,, \int_0^t U(t,s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s \bigg)$$

$$\leqslant \|p(t') - p(t)\| + M \int_t^{t'} \varphi(s)\,\mathrm{d}s$$

$$+ \int_0^t h\Big(U(t',s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big), U(t,s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s\Big)$$

$$\leqslant \|p(t') - p(t)\| + M \int_t^{t'} \varphi(s)\,\mathrm{d}s$$

$$+ \int_0^t \|U(t',s) - U(t,s)\|_{\mathcal{L}} \varphi(s)\,\mathrm{d}s \;=\theta(t',t)$$

(here $M = \sup_{(s,t)\in\Delta} \|U(t,s)\|_{\mathcal{L}}$; see hypothesis $H(U)$ (1)). Clearly $\theta(t',t) \to 0$ as $t' - t \to 0^+$ with t' or t fixed (see hypothesis $H(U)$ (2)).

Furthermore, we have: if $t \in [0, \frac{b}{n}]$,

$$\begin{aligned} h\big(G_n(t), H_n(t)\big) &= h\Big(p(t), p(t) + \int_0^t U(t,s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s\Big) \\ &\leqslant \left| \int_0^t U(t,s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s \right| \\ &\leqslant M \int_0^t \varphi(s)\,\mathrm{d}s \leqslant M \int_0^{\frac{b}{n}} \varphi(s)\,\mathrm{d}s \;= \gamma_1(n) \end{aligned}$$

if $t \in [\frac{b}{n}, b]$,

$$\begin{aligned} &h\big(G_n(t), H_n(t)\big) \\ &\quad = h\Big(p(t) + \int_0^{t-\frac{b}{n}} U(t - \frac{b}{n}, s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s\,, \\ &\qquad\qquad p(t) + \int_0^t U(t,s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s\Big) \\ &\quad < \left| \int_{t-\frac{b}{n}}^t U(t,s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\,\mathrm{d}s \right| \\ &\qquad + \int_0^{t-\frac{b}{n}} h\Big(U(t,s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big), U(t-\tfrac{b}{n},s)\,\overline{\mathrm{conv}}\,F\big(s,G_n(s)\big)\Big)\,\mathrm{d}s \end{aligned}$$

$$\leqslant M \int_{t-\frac{b}{n}}^{t} \varphi(s)\,\mathrm{d}s + \int_0^{t-\frac{b}{n}} \|U(t,s) - U(t-\frac{b}{n}, s)\|_{\mathcal{L}}\varphi(s)\,\mathrm{d}s$$

$$\leqslant M \int_{t-\frac{b}{n}}^{t} \varphi(s)\,\mathrm{d}s + \eta(t, t-\tfrac{b}{n}) = \gamma_2(n).$$

Hence for all $t \in T$, we have

$$h\big(G_n(t), H_n(t)\big) = \gamma(n) = \max[\gamma_1(n), \gamma_2(n)] \to 0 \quad \text{as } n \to \infty.$$

Next let $V_n(t) = \bigcup_{m \geqslant n} G_n(T)$ and $W_n(t) = \bigcup_{m \geqslant n} H_n(t)$. Since

$$\beta\big(G_m(t)\big) = \beta\big(H_m(t)\big) = 0$$

for $m \in \{1, 2, \ldots, n\}$, we have that for all $n \geqslant 1$

$$\beta\big(V_n(t)\big) = \beta\big(V_1(t)\big) \quad \text{and} \quad \beta\big(W_n(t)\big) = \beta\big(W_1(t)\big).$$

From the properties of $\beta(\cdot)$ (see Banas-Goebel [1], p. 21 and section 2) we have

$$\big|\beta\big(W_1(t)\big) - \beta\big(V_1(t)\big)\big| \leqslant \gamma(n) \quad \text{for all } n \geqslant 1.$$

Since $\gamma(n) \to 0$ as $n \to \infty$, we get

$$\beta\big(W_1(t)\big) = \beta\big(V_1(t)\big) \quad \text{for all } t \in T.$$

Using once again the Lipschitz continuity of $\beta(\cdot)$ with respect to the Hausdorff metric, we get for t, $t' \in T$, $t < t'$:

$$\big|\beta\big(W_1(t')\big) - \beta\big(W_1(t)\big)\big| \leqslant \theta(t', t)$$
$$\Longrightarrow \big|\beta\big(V_1(t')\big) - \beta\big(V_1(t)\big)\big| \leqslant \theta(t', t)$$

(i.e. both $t \to V_1(t)$ and $t \to W_1(t)$ are continuous functions on T). Then we have:

$$\beta\big(V_1(t)\big) = \beta\big(W_1(t)\big) \leqslant \beta\Big[\int_0^t U(t,s)\,\overline{\mathrm{conv}}\, F\big(s, V_1(s)\big)\,\mathrm{d}s\Big].$$

Let $u_k \colon T \to X$, $k \geqslant 1$, be measurable function s.t. $V_1(s) = \overline{\{u_k(s)\}_{k \geqslant 1}}$ for all $s \in T$ (their exitence follows from the measurability of $V_1(\cdot)$; see section 2). Then $F\big(s, V_1(s)\big) = \overline{\bigcup_{k \geqslant 1} F\big(s, u_k(s)\big)}$ and for each $k \geqslant 1$, $s \to F\big(s, u_k(s)\big)$ is measurable (cf.

hypothesis $H(F)$). So $s \to F(s, V_1(s))$ is measurable (see propostion 2.3 (i) of Himmelberg [10]) $\Longrightarrow s \to \overline{\text{conv}}\, F(s, V_1(s))$ is measurable (see theorem 9.1 of Himmelberg [10]). Let $v_k : T \to X$, $k \geqslant 1$, be measurable maps s.t. $\overline{\text{conv}}\, F(s, V_1(s)) = \overline{\{v_k(s)\}_{k \geqslant 1}}$. Then $U(t,s)\,\overline{\text{conv}}\, F(s, V_1(s)) = U(t,s)\overline{\{v_k(s)\}_{k \geqslant 1}} = \overline{\{U(t,s)v_k(s)\}_{k \geqslant 1}}$. So applying lemma 2.2 of Kisielewicz [12] (see also Heinz [8], theorem 3.1 and Mönch [15], proposition 1.6), we get

$$\begin{aligned}
\beta(V_1(t)) &\leqslant \beta\Big[\int_0^t U(t,s)v_k(s)\,\mathrm{d}s : k \geqslant 1\Big] \\
&\leqslant M \int_0^t \beta\big(v_k(s) \colon k \geqslant 1\big)\,\mathrm{d}s \\
&\leqslant M \int_0^t \beta\Big(\overline{\text{conv}}\, F\big(s, V_1(s)\big)\Big)\,\mathrm{d}s = M \int_0^t \beta(F(s, V_1(s)))\,\mathrm{d}s \\
&\leqslant M \int_0^t k(s)\beta\big(V_1(s)\big)\,\mathrm{d}s\,.
\end{aligned}$$

Invoking Gronwall's inequality, we get that

$$\beta\big(V_1(t)\big) = 0 \quad \text{for all } t \in T.$$

Next note that for every $n \geqslant 1$ and every $t \in T$, we have:

$$G_n(t) \subseteq \hat{V}_1(t)$$

where $\hat{V}_1(t) = \overline{\text{conv}}\,\big[V_1(t) \cup \big(-V_1(t)\big)\big]$. From Mazur's theorem $\hat{V}_1(t) \in P_{kc}(X)$ and is symmetric. Let $\lambda_n(t)(\cdot) = \sigma\big(\cdot, G_n(t)\big)$ and $\mu(t)(\cdot) = \sigma\big(\cdot, \hat{V}_1(t)\big)$. Recall (see section 2) that for all $n \geqslant 1$ and all $t \in T$, $\lambda_n(t)(\cdot)$, $\mu(t)(\cdot) \in C(B^*_{w^*})$, with $B^*_{w^*}$ being the unit ball of X^* equipped with the relative w^*-topology (hence $B^*_{w^*}$ is compact metrizable). Note that for t, $t' \in T$, $t < t'$ we have

$$\begin{aligned}
\|\lambda_n(t') - \lambda_n(t)\|_{C(B^*_{w^*})} &= \sup_{\|x^*\| \leqslant 1} \big|\sigma\big(x^*, G_n(t')\big) - \sigma\big(x^*, G_n(t)\big)\big| \\
&= h\big(G_n(t'), G_n(t)\big).
\end{aligned}$$

Observe that if $t \leqslant \frac{b}{n} \leqslant t'$,

$$\begin{aligned}
h\big(G_n(t'), G_n(t)\big) &= h\Big(p(t') + \int_0^{t' - \frac{b}{n}} U(t' - \tfrac{b}{n}, s)\,\overline{\text{conv}}\,\big(F(s, G(s))\big)\,\mathrm{d}s\,, p(t)\Big) \\
&\leqslant \|p(t') - p(t)\| + M \int_0^{t'-t} \varphi(s)\,\mathrm{d}s
\end{aligned}$$

if $\frac{b}{n} \leqslant t \leqslant t'$,

$$h\big(G_n(t'), G_n(t)\big)$$

$$\leqslant \|p(t') - p(t)\| + M \int_{t-\frac{b}{n}}^{t'-\frac{b}{n}} \varphi(s)\,\mathrm{d}s + \int_0^{t-\frac{b}{n}} \|U(t' - \tfrac{b}{n}, s) - U(t - \tfrac{b}{n}, s)\|_{\mathcal{L}} \varphi(s)\,\mathrm{d}s$$

$$= \|p(t') - p(t)\| + M \int_{t-\frac{b}{n}}^{t'-\frac{b}{n}} \varphi(s)\,\mathrm{d}s + \eta(t' - \tfrac{b}{n}, t - \tfrac{b}{n})$$

and this by virtue of hypothesis $H(U)$ (2), implies that $\{t \to \lambda_n(t)(\cdot)\}_{n\geqslant 1}$ is equicontinuous in $C\big(T, C(B^*_{w^*})\big)$.

Also for every $t \in T$, and for every x^*, $z^* \in B^*$, we have

$$\begin{aligned} |\lambda_n(t)(x^*) - \lambda_n(t)(Z^*)| &= \max[\lambda_n(t)(x^* - z^*), \lambda_n(t)(z^* - x^*)] \\ &\leqslant \mu(t)(x^* - z^*) \\ \Longrightarrow \{\lambda_n(t)(\cdot)\}_{n\geqslant 1} &\quad \text{is equicontinuous on } C(B^*_{w^*}). \end{aligned}$$

Hence from the Arzèla-Ascoli theorem, we get that for all $t \in T$, $\overline{\{\lambda_n(t)(\cdot)\}_{n\geqslant 1}}^{C(B^*_{w^*})}$ is compact in $C(B^*_{w^*})$. A new application of the Arzèla-Ascoli theorem, this time in the space $C\big(T, C(B^*_{w^*})\big)$, tells us that $\{\lambda_n\}_{n\geqslant 1}$ is relatively in $C\big(T, C(B^*_{w^*})\big)$. So passing to a subsequence if necessary, we may assume that $\lambda_n \to \lambda$ in $C\big(T, C(B^*_{w^*})\big)$. Clearly for every $t \in T$, $\lambda(t)(\cdot) \in C(B^*_{w^*})$ is sublinear. Thus there exists $G(t) \in P_{kc}(X)$ s.t. $\sigma\big(x^*, B(t)\big) = \lambda(t)(x^*)$, $x^* \in X^*$. Since $t \to \lambda(t)(\cdot)$ belongs in $C\big(T, C(B^*_{w^*})\big)$, we get that $t \to G(t)$ is h-continuous from T into $P_{kc}(X)$ and in fact $G_n(t) \xrightarrow{h} G(t)$ for all $t \in T$, hence $H_n(t) \xrightarrow{h} G(t)$ for all $t \in T$.

Now for every $x^* \in X^*$, we have

$$\sigma\Big(x^*, F\big(t, G_n(t)\big)\Big) = \sigma\Big(x^*, \bigcup_{x\in G_n(t)} F(t,x)\Big) = \sup_{x\in G_n(t)} \sigma\big(x^*, F(t,x)\big).$$

Since $G_n(t) \in P_{kc}(X)$ and $F(t,\cdot)$ is h-continuous (hence $x \to \sigma\big(x^*, F(t,x)\big)$ is continuous), we can find $x_n \in G_n(t)$ s.t.

$$\sigma\Big(x^*, F\big(t, G_n(t)\big)\Big) = \sigma\big(x^*, F(t, x_n)\big).$$

Similarly for $G(t) \in P_{kc}(X)$. Therefore we have:

$$\begin{aligned} &h\Big(\overline{\operatorname{conv}}\, F\big(t, G_n(t)\big), \overline{\operatorname{conv}}\, F\big(t, G(t)\big)\Big) \\ &\qquad = \sup_{\|x^*\|\leqslant 1} \Big|\sigma\Big(x^*, F\big(t, G_n(t)\big)\Big) - \sigma\Big(x^*, F\big(t, G(t)\big)\Big)\Big|. \end{aligned}$$

But $\sigma(\cdot, F(t, G_n(t))) - \sigma(\cdot, F(t, G(t))) \in C(B^*_{w^*})$ and so we can find $x_n^* \in B^*$ such that

$$h\left(\overline{\mathrm{conv}}\, F(t, G_n(t)), \overline{\mathrm{conv}}\, F(t, G(t))\right) = \left|\sigma\left(x_n^*, F(t, G_n(t))\right) - \sigma\left(x_n^*, F(t, G(t))\right)\right|.$$

Also by what was said above, we can find $v_n \in G_n(t)$, $w_n \in G(t)$ s.t.

$$\sigma\left(x_n^*, F(t, G_n(t))\right) = \sigma(x_n^*, F(t, v_n)) \quad \text{and} \quad \sigma\left(x_n^*, F(t, G(t))\right) = \sigma(x_n^*, F(t, w_n)).$$

Recall that $G_n(t)$, $G(t) \subseteq V_1(t) \in P_k(X)$. So by passing to a subsequence if necessary, we may assume that

$$x_n^* \xrightarrow{w^*} x^* \text{ in } X^* \quad \text{and} \quad v_n \xrightarrow{s} v,\ w_n \xrightarrow{s} w \text{ in } X.$$

Then

$$\begin{aligned}
&\left|\sigma(x_n^*, F(t, v_n)) - \sigma(x^* F(t, v))\right| \\
&\quad \leqslant \left|\sigma(x_n^*, F(t, v_n)) - \sigma(x_n^*, F(t, v))\right| + \left|\sigma(x_n^*, F(t, v)) - \sigma(x^*, F(t, v))\right| \\
&\quad \leqslant h\big(F(t, v_n), F(t, v)\big) + \sigma\big(x_n^* - x^*, \hat{V}_1(t)\big) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

since $\sigma(\cdot, \hat{V}_1(t))$ is continuous on bounded subsets of X^* endowed with the relative w^*-topology. Similarly, we get $\sigma(x_n^*, F(t, w_n)) \to \sigma(x^*, F(t, w))$. Therefore finally we have

$$\begin{aligned}
&\overline{\mathrm{conv}}\, F(t, G_n(t)) \xrightarrow{h} \overline{\mathrm{conv}}\, F(t, G(t)) \quad \text{in } P_{kc}(X) \\
\Longrightarrow\ & U(t, s)\, \overline{\mathrm{conv}}\, F(s, G_n(s)) \xrightarrow{h} U(t, s)\, \overline{\mathrm{conv}}\, F(s, G(s)) \text{ in } P_{kc}(X)\ (t \geqslant s \geqslant 0).
\end{aligned}$$

So using theorem 3.5 of Papageoriou [18] (or even the Radström embedding theorem), we get

$$p(t) + \int_0^t U(t, s)\, \overline{\mathrm{conv}}\, F(s, G_n(s))\, ds \xrightarrow{h} p(t) + \int_0^t U(t, s)\, \overline{\mathrm{conv}}\, F(s, G_n(s))\, ds$$

in $P_{kc}(X)$

$$\Longrightarrow G(t) = p(t) + \int_0^t U(t, s)\, \overline{\mathrm{conv}}\, F(s, G_n(s))\, ds\,,\ t \in T$$

and $t \to G(t)$ is h-continuous from T into $P_{kc}(X)$.

Now let $\Gamma \subseteq L^1(X)$ be defined by

$$\Gamma = \left\{f \in L^1(X) \colon f(t) \in \overline{\mathrm{conv}}\, F(t, G(t)) \text{ a.e.}\right\}$$

and $E \subseteq C(T,X)$ by

$$E = \left\{ y \in C(T,X) \colon y(t) = p(t) + \int_0^t U(t,s) f(s)\,\mathrm{d}s\,,\ t \in T,\ f \in \Gamma \right\}.$$

A straightforward application of the Arzèla-Ascoli theorem, shows that $E \in P_{kc}(C(T,X))$. Let $R \colon E \to 2^{L^1(X)} \setminus \{\emptyset\}$ be defined by

$$R(x) = S^1_{F(\cdot,x(\cdot))}.$$

Apply theorem 1.1 of Tolstonogov [27], to get $r : E \to L^1_w(X)$ continuous s.t. $r(y) \in \operatorname{ext} R(y)$ for all $y \in E$. But from Benamara [2], we know that $\operatorname{ext} R(y) = \exp S^1_{F(\cdot,y(\cdot))} = S^1_{\operatorname{ext} F(\cdot,y(\cdot))}$. Let $\xi \colon E \to C(T,X)$ be defined by

$$\xi(x)(t) = p(t) + \int_0^t U(t,s) r(x)\,\mathrm{d}s\,.$$

Note that from the definitions of the sets Γ and E, we have

$$\begin{aligned} x(t) \in G(t) &\quad \text{for all } t \in T \\ &\Longrightarrow\ r(x)(\cdot) \in \Gamma \\ &\Longrightarrow\ \xi(x) \in E; \text{ i.e. } \xi \colon E \to E. \end{aligned}$$

We claim that $\xi(\cdot)$ is continuous. Indeed let $x_n \to x$ in E. Then since $r(x_n)$, $r(x) \in S^1_{\hat{V}_1(\cdot)}$ and $\hat{V}_1(\cdot)$ is $P_{kc}(X)$-valued and integrably bounded, we have that $r(x_n) \xrightarrow{w} r(x)$ in $L^1(X)$ (see Schechter [24]). So for every $t \in T$

$$\int_0^t U(t,s) r(x_n)(s)\,\mathrm{d}s \xrightarrow{w} \int_0^t U(t,s) r(x)(s)\,\mathrm{d}s\,.$$

Set $q_n(t) = \int_0^t U(t,s) r(x_n)(s)\,\mathrm{d}s$, $q(t) = \int_0^t U(t,s) r(x)(s)\,\mathrm{d}s$, $g_n, q \in C(T,X)$. Using hypothesis $H(U)$ (2), we can easily check that $\{q_n(\cdot)\}_{n \geqslant 1} \subseteq C(T,X)$ is equicontinuous and for all $t \in T$, $q_n(t) \in \int_0^t U(t,s)\hat{V}_1(s)\,\mathrm{d}s \in P_{kc}(X)$. So by the Arzèla-Ascoli theorem, $\{q_n\}_{n \geqslant 1}$ is relatively compact $C(T,X)$, hence $q_n \to q$ in $C(T,X)$. Therefore $\xi(x_n) = p + q_n \to \xi(x) = p + q$ in $C(T,X) \Longrightarrow \xi(\cdot)$ is indeed continuous. Apply Schauder's fixed point theorem to get $x \in E$ s.t. $\xi(x) = x$. Clearly $x \in C(T,X)$ is the desired solution of (2). □

4. A density result

Let S and S_e be the solution sets of (1) and (2) respectively. In this section we show that we can approximate, with arbitrary degree of accuracy, elements in S using those in S_e. We already konw (see [22]), that under hypotheses $H(F)$, $H(U)$ and $H(p)$, S is a nonempty, compact subset of $C(T,X)$ (in fact, hypothesis $H(F)$ can be weakened further for establishing that). Here we will need the following stronger hypothesis on the orientor field:

$H(F)_1$:$F\colon T\times X \to P_{fc}(X)$ is a multifunction s.t.

(1) $t \to F(t,x)$ is measurable,

(2) $h\big(F(t,y),F(t,x)\big) \leqslant \ell(t)\|x-y\|$ a.e. with $\ell(\cdot) \in L^1_+(T)$,

(3) $\beta\big(F(t,B)\big) \leqslant k(t)\beta(B)$ a.e. for all $B \subseteq X$ nonempty bounded and with $k(\cdot) \in L^1_+(T)$,

(4) $|F(t,x)| \leqslant a(t) + c(t)\|x\|$ a.e. with $a(\cdot) \in L^1_+(T)$.

Remark. Again, because of $H(F)_1$ (3) and by modifying, if necessary, the orientor field on a Lebesgue null subset of T, we can assume without any loss of generality that $F(t,x) \in P_{kc}(X)$ for all $(t,x) \in T\times X$.

Theorem 2. *If hypotheses $H(F)_1$, $H(U)$ and $H(p)$ hold, then $\bar{S}_e = S$ the closure taken in $C(T,X)$.*

Proof. Let $x \in S$. Then by definition, we have

$$x(t) = p(t) + \int_0^t U(t,s)f(s)\,\mathrm{d}s$$

for all $t \in T$ and with $f \in L^1(X)$, $f(x) \in F\big(s,x(s)\big)$ a.e. Let $E \subseteq C(T,X)$ be as in the proof of theorem 1. Given $y \in E$ and $\varepsilon > 0$, let $H\colon T \to 2^X \setminus \{\emptyset\}$ be defined by

$$H(t) = \Big\{u \in X\colon \|f(t)-u\| < \varepsilon + d\Big(f(t),F\big(t,y(t)\big)\Big), u \in F\big(t,y(t)\big)\Big\}.$$

Using hypothesis $H(F)_1$, we can easily check that $\operatorname{Gr} H \in \mathcal{L}(T)\times B(X)$, with $\mathcal{L}(T)$ being the Lebesgue σ-field of T (i.e. the Lebesgue completion of the Borel σ-field $B(T)$). Applying Aumann's selection theorem, we can get $u\colon T\to X$ measurable s.t. for all $t \in T$, $u(t) \in H(t)$. Let $\Phi\colon E \to 2^{L^1(X)}\setminus\{\emptyset\}$ be defined by

$$\Phi(y) = \Big\{u \in S^1_{F(\cdot,y(\cdot))}\colon \|f(t)-u(t)\| < \varepsilon + d\Big(f(t),F\big(t,y(t)\big)\Big) \text{ a.e.}\Big\}.$$

From proposition 4 of Bressan-Colombo [4], we know that $\Phi(\cdot)$ is l.s.c. and has decomposable values (i.e. if $A \subseteq T$ is measurable and u_1, $u_2 \in \Phi(y)$, then $\chi_A u_1 +$

$\chi_{A^c}u_2 \in \Phi(y)$). Hence $y \to \overline{\Phi(y)}$ is l.s.c. and has decomposable values. Apply theorem 3 of Bressan-Colombo [4], to get $u_\varepsilon : E \to L^1(X)$ a continuous function s.t. $u_\varepsilon(y) \in \overline{\Phi(y)}$ for all $y \in E$. Hence $\|f(t) - u_\varepsilon(y)(t)\| \leqslant \varepsilon + ds\big(f(t), F\big(t, y(t)\big)\big) \leqslant \varepsilon + \ell(t)\|x(t) - y(t)\|$ a.e. Also apply theorem 1.1 of Tolstonogov [27], to get $v_\varepsilon : E \to L^1_w(X)$ s.t. $\|u_\varepsilon(z) - v_\varepsilon(z)\|_w < \varepsilon$ for all $z \in E$.

Now let $\varepsilon_n \downarrow 0$ and set $u_n = u_{\varepsilon_n}$, $v_n = v_{\varepsilon_n}$. Let $x_n \in E$ s.t. $\theta(x_n) = x_n$. Their existence is guaranteed by Schauder's fixed point theorem (see the proof of theorem 1). Clearly $x_n \in S_e$. Recall that $E \subset C(T, X)$ is compact. So by passing to a subsequence if necessary, we may assume that $x_n \to \hat{x}$ in $C(T, X)$. For every $x^* \in X^*$, we have:

$$
\begin{aligned}
\big|\big(x^*, x(t) - x_n(t)\big)\big| &\leqslant \left|\left(x^*, \int_0^t U(t,s)\Big(f(s) - v_n\big(x_n(s)\big)\Big)\, ds\right)\right| \\
&= \left|\left(x^*, \int_0^t U(t,s)\big(f(s) - u_n(x_n)(s)\big)\, ds\right)\right| \\
&\quad + \left|\left(x^*, \int_0^t U(t,s)\Big(u_n\big(x_n(s) - v_n(x_n)(s)\big)\Big)\, ds\right)\right| \\
&\leqslant M\|x^*\| \int_0^t \|f(s) - u_n(x_n)(s)\|\, ds \\
&\quad + \left|\int_0^t \Big(x^*, U(t,s)\big(u_n(x_n)(s) - v_n(x_n)(s)\big)\Big)\, ds\right| \\
&\leqslant M\|x^*\|\varepsilon_n b + M\|x^*\| \int_0^t \ell(s)\|x(s) - x_n(s)\|\, ds \\
&\quad + \left|\int_0^t \Big(x^*, U(t,s)\big(u_n(x_n)(s) - v_n(x_n)(s)\big)\Big)\, ds\right|.
\end{aligned}
$$

But by construction $u_n(x_n) - v_n(x_n) \xrightarrow{\|\cdot\|_w} 0$ and since $u_n(x_n) - v_n(x_n) \in S^1_{\hat{V}_1}$ with $\hat{V}_1(\cdot)$ P_{kc}-valued and integrably bounded $\Longrightarrow$ $u_n(x_n) - v_n(x_n) \xrightarrow{w} 0$ in $L^1(X)$. So $\int_0^t \Big(x^*, U(t,s)\big(u_n(x_n)(s) - v_n(x_n)(s)\big)\Big)\, ds \to 0$ as $n \to \infty$. Hence in the limit as $n \to \infty$, we get

$$
\begin{gathered}
\|x(t) - \hat{x}(t)\| \leqslant M\|x^*\| \int_0^t \ell(s)\|x(s) - \hat{x}(s)\|\, ds \\
\Longrightarrow\ x = \hat{x} \quad \text{(Gronwall's inequality)}.
\end{gathered}
$$

So $x = \lim x_n$ in $C(T, X)$ with $x_n \in S_e$. Therefore $S = \bar{S}_e^{C(T,X)}$. □

In this section, we use theorem 2 to obtain a bang-bang principle for controlled Volterra integral equations. So we consider the following two systems:

$$x(t) = p(t) + \int_0^t U(t,s)\big[f\big(s,x(s)\big) + B(s)u(s)\big]\,\mathrm{d}s\,, \quad t \in T, \tag{3}$$
$$u(t) \in V(t) \text{ a.e., } u(\cdot)\text{-measurable}$$

and

$$x(t) = p(t) + \int_0^t U(t,s)\big[f\big(s,x(s)\big) + B(s)u(s)\big]\,\mathrm{d}s\,, \quad t \in T, \tag{4}$$
$$u(t) \in \operatorname{ext} V(t) \text{ a.e., } u(\cdot)\text{-measurable.}$$

These systems may correspond to controlled semilinear evolution equations, in which case $p(t) = U(t,0)x_0$, with $x_0 \in X$ (initial state) and $U(t,s)$ is the evolution operator, generated by a family $\{A(t)\colon t \in T\}$ of generally unbounded, densely defined linear operators. We model the control space by a separable Banach space Y.

Let S and S_e be the trajectories of (3) and (4) respectively. Also $R(t) = \{x(t)\colon x \in S\}$ and $R_e(t) = \{x(t)\colon x \in S_e\}$ be the corresponding reachable sets at time $t \in T$. We will need the following hypotheses:

$H(f)$: $f\colon T \times X \to X$ is a map s.t.
- (1) $t \to f(t,x)$ is measurable,
- (2) $\|f(t,x) - f(t,y)\| \leqslant \ell(t)\|x-y\|$ a.e. with $\ell(\cdot) \in L^1_+(T)$,
- (3) $\|f(t,x)\| \leqslant a(t) + c(t)\|x\|$ a.e. with $a,\ c \in L^1_+(T)$.

$H(B)$: $B\colon T \to \mathcal{L}(Y,X)$ is measurable for the strong operator topology on $\mathcal{L}(Y,X)$ (i.e. for all $u \in Y$, $t \to B(t)u$ is measurable) and for every $t \in T$, $B(t)$ is a compact operator and $\|B(t)\|_{\mathcal{L}} \leqslant k$, $k > 0$.

$H(V)$: $V\colon T \to P_{wkc}(Y)$ is a measurable multifunction s.t. $|V(t)| \leqslant N$ for all $t \in T$.

Theorem 3. *If hypotheses $H(f)$, $H(U)$, $H(B)$, $H(V)$ and $H(p)$ hold, then $S = \bar{S}_e^{C(T,X)}$ and for all $t \in T$, $R(t) = \overline{R_e(t)}^{\|\cdot\|}$.*

Proof. Let $F\colon T \times X \to P_{kc}(X)$ be defined by

$$F(t,x) = f(t,x) + B(t)V(t).$$

Let $C \subseteq X$ be nonempty bounded. We have

$$\beta\big(F(t,C)\big) \leqslant \beta\big(f(t,C)\big) + \beta\big(B(t)V(t)\big).$$

Since by hypothesis $H(B)$, $B(t)$ is compact, $B(t)V(t) \in P_{kc}(X)$ and so $\beta(B(t)V(t)) = 0$. Also from hypothesis $H(f)$ (2), we get $\beta(f(t,C)) \leqslant \ell(t)\beta(C)$ a.e. Therefore we get

$$\beta(F(t,C)) \leqslant \ell(t)\beta(C) \text{ a.e.}$$

Let $v_n \colon T \to Y$, $n \geqslant 1$, be measurable functions s.t. $V(t) = \overline{\{v_n(t)\}_{n\geqslant 1}}$ for all $t \in T$. Then for every $x^* \in X^*$, $t \to \sigma(x^*, B(t)V(t)) = \sup_{n\geqslant 1} \sigma(x^*, B(t)v_n(t))$ is measurable $\Longrightarrow$ $t \to B(t)V(t)$ is measurable $\Longrightarrow$ $t \to F(t,x)$ is measurable.

Also if x, $y \in X$ and $z \in F(t,x)$, then by definition $z = f(t,x) + B(t)u$, $u \in V(t)$. So we have:

$$\begin{aligned} d(z, F(t,y)) &\leqslant \|f(t,x) - f(t,y)\| \leqslant \ell(t)\|x-y\| \\ \Longrightarrow\ & h(F(t,x), F(t,y)) \leqslant \ell(t)\|x-y\| \text{ a.e.} \end{aligned}$$

Finally because of hypothesis $H(f)$ (3) we get

$$|f(t,x)| \leqslant a(t) + kN + c(t)\|x\| \text{ a.e.}$$

So we satisfied hypothesis $H(F)_1$. Then consider the following integral inclusions

$$x(t) \in p(t) + \int_0^t U(t,s)F(s,x(x))\,\mathrm{d}s\,, \quad t \in T \tag{5}$$

and

$$x(t) \in p(t) + \int_0^t U(t,s)\operatorname{ext} F(s,x(x))\,\mathrm{d}s\,, \quad t \in T. \tag{6}$$

Let S^1 be the solution set of (5) and S_e^1 the solution set of (6). A straightforward application of Aumann's selection theorem, gives us that $S = S^1$. On the other hand, since $\operatorname{ext} B(t)V(t) \subseteq B(t)\operatorname{ext} V(t)$, we have that $S_e^1 \subseteq S_e$. From theorem 2, we get

$$\begin{aligned} &\overline{S_e^1}^{C(T,X)} = S \\ \Longrightarrow\ &\overline{S_e}^{C(T,X)} = S. \end{aligned}$$

Recalling that the evaluation map $e_t \colon C(T,X) \to X$, defined by $e_t(x) = x(t)$ is continuous, we also get that $R(t) = \overline{R_e(t)}^{\|\cdot\|}$ for all $t \in T$. □

So if $J \colon C(T,X) \to \mathbb{R}$ is a continuous cost functional and $m = \inf[J(x) \colon x \in S]$, then given $\varepsilon > 0$ we can find a bang-bang control $u \in S^1_{\operatorname{ext} V(\cdot)}$ with corresponding trajectory $z_\varepsilon(\cdot) \in S_e$ s.t. $0 \leqslant J(x_\varepsilon) - m \leqslant \varepsilon$. Bang-bang controls can be realized physically much easier than the other control functions.

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