Petr Habala

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Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 2, 347–356

Persistent URL: http://dml.cz/dmlcz/128461

Czechoslovak Mathematical Journal, 44 (119) 1994, Praha

STATIONARY INCOMPRESSIBLE BIPOLAR FLUIDS

PETR HABALA, Praha

(Received August 28, 1992)

The work is a contribution to the theory of multipolar fluids. While the equations are usually solved via a priori estimates and Galerkin method within Orlicz spaces, in this paper the problem is approached using the notion of pseudomonotonicity. After introducing the physical foundation for the problem, the weak formulation is stated and justified in the second part. In the third chapter the main results are collected, namely the existence of a solution to the weak problem is proven and some regularity properties are shown. For sake of simplicity only the three-dimensional case is treated with a parameter $p > \frac{9}{8}$. The same results can be obtained for n dimensions and $p > \frac{3n}{n+5}$.

1. PHYSICAL BACKGROUND

Let us suppose that all particles of the fluid remain in a bounded domain $\Omega \subseteq \mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$, i.e. no partial flux through $\partial\Omega$ is allowed. Let M, K be natural numbers. The multipolar fluid of type (M, K) is a material described by a collection of $(8+M)$ functions of time t and Euler coordinates $x = (x_1, x_2, x_3) \in \Omega$:

- $v = (v_1, v_2, v_3)$ = the velocity vector field,
- θ = the positive absolute temperature,
- ϱ = the density,
- r = the rate of external heat sources,
- $f = (f_1, f_2, f_3)$ = the specific external body force,
- e = the specific internal energy,
- η = the specific entropy,
- $q = (q_1, q_2, q_3)$ = the heat flux vector,
- $\tau_{ij}, \dots, \tau_{ii_1\dots i_{M-1}j}$ = the spatial multipolar stress tensors, $i, i_k, j = 1, 2, 3$

(we will use the notation $\tau^{(m)} = \tau_{ii_1\dots i_m j}$).

They satisfy the constitutive relations

$$\begin{aligned}
e &= e(\varrho, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta),\\
\eta &= \eta(\varrho, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta),\\
q &= q(\varrho, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta),\\
\tau^{(m)} &= \tau^{(m)}(\varrho, \nabla v, \dots, \nabla^K v, \theta, \nabla\theta), \quad m = 0, \dots, M-1
\end{aligned}$$

and the following physical laws in local form:

1) (conservation of mass)

$$\frac{\partial \varrho}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial \varrho}{\partial x_j} + \sum_{j=1}^{3} \varrho \frac{\partial v_j}{\partial x_j} = 0,$$

2) (balance of linear momentum)

$$\forall i = 1,2,3: \ \varrho \frac{\partial v_i}{\partial t} + \varrho \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} + \varrho f_i,$$

3) (the second law of thermodynamics, Clausius-Duhem inequality, [C-D])

$$\varrho \frac{\partial \eta}{\partial t} + \varrho \sum_{j=1}^{3} v_j \frac{\partial \eta}{\partial x_j} \geqslant - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(\frac{q_j}{\theta}\Big) + \varrho \frac{r}{\theta},$$

4) (balance of angular momentum)

$$\begin{aligned}
\forall i = 1,2,3: \ &\varrho \frac{\partial}{\partial t}\Big(\sum_{k,p=1}^{3} \varepsilon_{ikp} x_k v_p\Big) + \varrho \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j}\Big(\sum_{k,p=1}^{3} \varepsilon_{ikp} x_k v_p\Big)\\
&= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(\sum_{k,p=1}^{3} \varepsilon_{ikp} x_k \tau_{pj} + \varepsilon_{ikp} \tau_{pkj}\Big) + \varrho \sum_{k,p=1}^{3} \varepsilon_{ikp} x_k f_p,
\end{aligned}$$

5) (conservation of energy)

$$\begin{aligned}
&\varrho \frac{\partial}{\partial t}\Big(e + \frac{1}{2}\sum_{i=1}^{3} v_i^2\Big) + \varrho \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j}\Big(e + \frac{1}{2}\sum_{i=1}^{3} v_i^2\Big)\\
&= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(-q_j + \sum_{k=0}^{M-1} \sum_{i=1}^{3} \sum_{m=1}^{k} \sum_{i_m=1}^{3} \tau_{ii_1\dots i_k j} \frac{\partial^k v_i}{\partial x_{i_1} \dots \partial x_{i_k}}\Big) + \varrho \sum_{j=1}^{3} f_j v_j + \varrho r.
\end{aligned}$$

We will also suppose that the principle of material frame indifference is satisfied, i.e. if we consider a change of frame of the form $\bar{x}_i(t) = \sum_{j=1}^{3} Q_{ij}(t)x_j(t) + c_i(t)$, $i = 1,2,3$ (where $\{Q_{ij}(t)\}_{i,j=1}^{3}$ is an orthogonal matrix), the scalars θ, ϱ, e, r, η are invariant and the tensors q, $\tau^{(m)}$ as well as Dv, $\nabla^2 v$, ..., $\nabla^K v$, $\nabla\theta$ change in the usual tensorial fashion, whereas the changes of f and v are described as follows:

$$\bar{f}_i = \sum_{j=1}^{3} Q_{ij}f_j - \ddot{c}_i - 2\sum_{j=1}^{3} \dot{Q}_{ij}v_j - \sum_{j=1}^{3} \ddot{Q}_{ij}x_j,$$

$$\bar{v}_i = \sum_{j=1}^{3} Q_{ij}v_j + \dot{c}_i + \sum_{j,k=1}^{3} \dot{Q}_{ij}Q_{kj}\bar{x}_k.$$

Next, for $m = 0, \dots, M-1$ let us define

$$\begin{aligned} \tau^{(m,E)} &= \tau^{(m)}(\varrho, 0, \dots, 0, \theta, 0) \qquad &&\text{(the equilibrium part)},\\ \tau^{(m,V)} &= \tau^{(m)} - \tau^{(m,E)} &&\text{(the viscous part)}. \end{aligned}$$

We will write τ^E, τ^V instead of $\tau^{(m,E)}$, $\tau^{(m,V)}$ as the order is usally clear from the context. The indepth analysis of this general setting can be found in [NŠ] and [No].

Restricting ourselves to the stationary incompressible case and a bipolar (i.e. 2,2-polar) fluid, the flow system can be described by the velocity vector field $v = (v_1, v_2, v_3)$, the general pressure q and the constant density ϱ, none of which depends on time. The fluid is governed by the equations

(Ph1) ([C-D])

$$\sum_{i,j=1}^{3} \left(\tau_{ij}^V + \sum_{k=1}^{3} \frac{\partial \tau_{ijk}^V}{\partial x_k} \right) \frac{\partial v_i}{\partial x_j} + \sum_{i,j,k=1}^{3} \tau_{ijk}^V \frac{\partial^2 v_i}{\partial x_j \partial x_k} \geqslant 0,$$

(Ph2) (Navier-Stokes equations)

$$\forall i = 1,2,3: \ \varrho \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} - \sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} = \varrho f_i,$$

(Ph3) (continuity equation)

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = 0,$$

(Ph4) (no-slip boundary condition)

$$\forall x \in \partial\Omega: \ v(x) = 0.$$

Moreover, $\tau_{ij}^E = -q\delta_{ij}$ holds.

Let numbers $p > \frac{9}{8}$ (see further) and $\varrho > 0$ be fixed throughout the rest of the paper. We will consider only the special class of the stress tensors. In order to simplify the definition, let us first introduce the notation.

The symbol ξ^* will denote the "triangular matrix", $\xi^* = \{\xi_{ij}\}_{i=1,\dots,3}^{j=1,\dots,i} \in \mathbb{R}^6$, and the following norms will be used:

$$|\xi^*|^p = \sum_{i,j=1}^{3,i} |\xi^*_{ij}|^p \text{ for } \xi^* \in \mathbb{R}^6,$$

$$|\eta|^p = \sum_{m,n,o=1}^{3} |\eta^m_{no}|^p \text{ for } \eta \in \mathbb{R}^{27}.$$

Definition. Assume that functions
$A_{ij}\colon \mathbb{R}^6 \to \mathbb{R}$,
$A^k_{ij}\colon \mathbb{R}^6 \times \mathbb{R}^{27} \to \mathbb{R}$,
$A_{ijk}\colon \mathbb{R}^6 \times \mathbb{R}^{27} \to \mathbb{R}$, $i,j,k = 1,2,3$,
satisfy the following conditions:
(i) (symmetry condition)
$A_{ij} = A_{ji}$,
$A^k_{ij} = A^k_{ji}$,
$A_{ijk} = A_{jik}$,
(ii) (Caratheodory conditions)
α) $A_{ij} \in C(\mathbb{R}^6)$,
$A^k_{ij} \in C(\mathbb{R}^6 \times \mathbb{R}^{27})$,
$A_{ijk} \in C(\mathbb{R}^6 \times \mathbb{R}^{27})$,
β) there exists $c > 0$ such that
$\forall \xi^* \in \mathbb{R}^6\colon |A_{ij}(\xi^*)| \leqslant c|\xi^*|^{p-1}$,
$\forall \xi^* \in \mathbb{R}^6\ \forall \eta \in \mathbb{R}^{27}\colon |A^k_{ij}(\xi^*,\eta)| \leqslant c(|\xi^*|^{p-1} + |\eta|^{p-1})$.
We define

$$\tau^V_{ij}(Dv, \nabla^2 v, \nabla^3 v) = A_{ij}(Dv) - \sum_{k=1}^{3} \frac{\partial}{\partial x_k} A^k_{ij}(Dv, \nabla^2 v),$$

$$\tau_{ij} = -q\delta_{ij} + \tau^V_{ij},$$

$$\tau^V_{ijk} = \tau_{ijk} = A_{ijk}(Dv, \nabla^2 v).$$

Now the problem has turned into the classical boundary problem for PDE of the fourth order, which will be solved in the weak sense. There is another boundary condition needed, so let us state

(Ph5) (unstable boundary condition)

$$\forall x \in \partial\Omega \ \forall i = 1,2,3: \ \sum_{j,k=1}^{3} A_{ij}^{k}(Dv, \nabla^2 v)\nu_j \nu_k = 0.$$

This condition does not follow from the physical laws stated above, however, it seems that it is not also meaningless from the physical point of view, for the left hand side term is related to the power of higher order gradients of velocity on the boundary.

2. Weak approach

We will start defining our "working space":

$$V_p = \overline{\left\{ \varphi \in C^\infty(\Omega, \mathbb{R}^3); \ \operatorname{div}(\varphi) = \sum_{i=1}^{3} \frac{\partial \varphi_i}{\partial x_i} = 0 \right\}}^{W^{2,p}(\Omega,\mathbb{R}^3)} \cap W_0^{1,p}(\Omega, \mathbb{R}^3).$$

Let $\| \ \|$ denote the $W^{2,p}(\Omega, \mathbb{R}^3)$-norm. It is clear that $(V_p, \| \ \|)$ is a reflexive separable Banach space containing finctions satisfying (Ph3) and (Ph4) in the weak sense.

Definition. Let $u, v, w \colon \Omega \to \mathbb{R}^3$ be the functions for which the following integrals exist. We will denote

$$\mathrm{a}(u,v) = \int_\Omega \varrho^{-1} \Big(\sum_{i,j=1}^{3} A_{ij}(Du) \frac{\partial v_i}{\partial x_j} + \sum_{i,j,k=1}^{3} A_{ij}^{k}(Du, \nabla^2 u) \frac{\partial^2 v_i}{\partial x_j \partial x_k} \Big)(x) \, \mathrm{d}x,$$

$$\mathrm{b}(u,v,w) = \int_\Omega \sum_{i,j=1}^{3} \Big(u_j v_i \frac{\partial w_i}{\partial x_j} \Big)(x) \, \mathrm{d}x \, .$$

Using the Hölder inequality and the imbedding theorem we obtain

Statement 2.1. *If $p > 1$, then $\mathrm{a}(.,.)$ is a mapping $[W^{2,p}(\Omega, \mathbb{R}^3)]^2 \to \mathbb{R}$ which is linear and continuous in the second variable.*
If $p > \frac{9}{8}$, then $b(.,.,.)$ is a trilinear continuous mapping $[W^{2,p}(\Omega, \mathbb{R}^3)]^3 \to \mathbb{R}$.

In the proof of the second part of the statement we have to use the imbedding twice and the need of estimating the term $\int_\Omega uv \frac{\partial w}{\partial x_i} \, \mathrm{d}x$ forces us to restrict ourselves to the case $p > \frac{9}{8}$ throughout the rest of the paper.

Now we are ready to introduce

Definition. Let us define the following operators $V_p \to V_p'$:

$$\begin{aligned}
&\mathrm{A}\colon V_p \to V_p';\ \langle \mathrm{A}(u), v\rangle = \mathrm{a}(u,v),\\
&\mathrm{B}\colon V_p \to V_p';\ \langle \mathrm{B}(u), v\rangle = \mathrm{b}(u,u,v),\\
&\mathrm{P}\colon V_p \to V_p';\ \mathrm{P} = \mathrm{A} - \mathrm{B}.
\end{aligned}$$

It is clear that $\mathrm{P}(u) = f$ is the weak form of the equation (Ph2). Let $f \in V_p'$. The functions u from V_p satisfying $\mathrm{P}(u) = f$ will be called "weak solutions".

With help of Green's identity we get the following result justifying this definition:

Theorem 2.2. *Let $A_{ij} \in C^1(\mathbb{R}^6)$, $A_{ij}^k \in C^2(\mathbb{R}^6 \times \mathbb{R}^{27})$, $f \in C(\Omega, \mathbb{R}^3)$. Let $q \in C^1(\Omega)$, $u \in C_0^4(\Omega, \mathbb{R}^3)$. If u, q satisfy* (Ph2)–(Ph5) *then u is a weak solution.*

Now we will start investigating the operators defined above.

Statement 2.3. *The operators* A, B, *and* P *are continuous $V_p \to V_p'$. The operator* B *is continous $(V_p, \text{weak}) \to (V_p', \|\ \|')$.*

The weak continuity of B can be easily obtained from the compactness of the imbedding. The most difficult part of the proof is the continuity of A. The difference $|\langle \mathrm{A}(u^n) - \mathrm{A}(u), v\rangle|$ is estimated by Hölder's inequality. The crucial point of the proof is the use of Vitali's L^1-convergence theorem: If $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\Omega)$, $f \in L^1(\Omega)$, $f_n \to f$ almost everywhere in Ω and $\forall \varepsilon > 0\ \exists \delta > 0\ \forall E \subseteq \Omega[\lambda(E) < \delta]\ \forall n \in \mathbb{N}$: $\int_\Omega f_n \, d\lambda < \varepsilon$ then $f_n \to f$ in $L^1(\Omega)$.

Using this statement one can easily prove

Theorem 2.4. *Let* A′ *be an operator $V_p \to V_p'$. Then* (A′ − B) *is coercive, pseudomonotone iff* A′ *is coercive, pseudomonotone, respectively.*

We can see that if these important solvability conditions are satisfied for our operator A, they are also satisfied for the operator P = A − B. Unfortunately, this is no longer true if we consider the strict monotonicity (i.e. the unicity condition).

Definition. Let us introduce the following conditions:
(PhM1) is satisfied iff $\forall \xi^* \in \mathbb{R}^6\ \forall \eta \in \mathbb{R}^{27}\ \forall \mu \in \mathbb{R}^{81}$:

$$
\begin{aligned}
2\sum_{i,j=1}^{3,i} \Big(A_{ij}(\xi^*) &+ \sum_{p=1}^{3}\Big(\frac{1}{2}\sum_{k,l=1}^{3,k} \frac{\partial(A_{ijp}-A_{ij}^{p})}{\partial \xi_{kl}^*}(\xi^*,\eta)(\eta_{lp}^{k}+\eta_{kp}^{l}) \\
&+ \sum_{m,n,o=1}^{3} \frac{\partial(A_{ijp}-A_{ij}^{p})}{\partial \eta_{no}^{m}}(\xi^*,\eta)\mu_{nop}^{m}\Big)\Big)\cdot \xi_{ij}^* \\
&- \sum_{i=1}^{3}\Big(A_{ii}(\xi^*) + \sum_{p=1}^{3}\Big(\frac{1}{2}\sum_{k,l=1}^{3,k} \frac{\partial(A_{iip}-A_{ii}^{p})}{\partial \xi_{kl}^*}(\xi^*,\eta)(\eta_{lp}^{k}+\eta_{kp}^{l}) \\
&+ \sum_{m,n,o=1}^{3} \frac{\partial(A_{iip}-A_{ii}^{p})}{\partial \eta_{no}^{m}}(\xi^*,\eta)\mu_{nop}^{m}\Big)\Big)\cdot \xi_{ii}^* \\
&+ \sum_{i,j,k=1}^{3} A_{ijk}(\xi^*,\eta)\eta_{jk}^{i} \geqslant 0
\end{aligned}
$$

(M1) is satisfied iff $\exists K \in \mathbb{R}\ \exists c_0 > 0\ \forall \xi^* \in \mathbb{R}^6\ \forall \eta \in \mathbb{R}^{27}$:

$$
\Big(2\sum_{i,j=1}^{3,i} A_{ij}(\xi^*)\xi_{ij}^* - \sum_{i=1}^{3} A_{ii}(\xi^*)\xi_{ii}^* + \sum_{i,j,k=1}^{3} A_{ij}^{k}(\xi^*,\eta)\eta_{jk}^{i}\Big) \geqslant c_0|\eta|^p - K
$$

(M2) is satisfied iff $\forall \xi^* \in \mathbb{R}^6\ \forall \eta^{(1)},\eta^{(2)} \in \mathbb{R}^{27}\ [\eta^{(1)} \neq \eta^{(2)}]$:

$$
\sum_{i,j,k=1}^{3} \Big((A_{ij}^{k}(\xi^*,\eta^{(1)}) - A_{ij}^{k}(\xi^*,\eta^{(2)}))\Big)(\eta_{jk}^{(1)i} - \eta_{jk}^{(2)i}) > 0.
$$

By the chain rule we clearly obtain

Theorem 2.5. *Let $A_{ij} \in C^1(\mathbb{R}^6)$; A_{ij}^{k}, $A_{ijk} \in C^1(\mathbb{R}^6 \times \mathbb{R}^{27})$. If these functions satisfy* (PhM1) *then for every $\Omega \subseteq \mathbb{R}^3$ and for every $v \in C^3(\Omega,\mathbb{R}^3)$ the tensors τ_{ij}, τ_{ijk} satisfy* (Ph1).

From the very definition we have

Lemma. *Let* (M1) *be satisfied. Then* A *is a coercive operator $V_p \to V_p'$.*

The following lemma is the heart of this work:

Lemma. *Let* (M1) *and* (M2) *be satisfied. Then* A *is a pseudomonotone operator.*

Proof. (sketch): Let $\{u^n\}_{n=1}^{\infty} \subseteq V_p$, $u \in V_p$,

$$u^n \overset{w}{\rightharpoonup} u \text{ in } V_p \text{ and } \limsup_{n\to\infty}(\langle \mathrm{A}(u^n), u^n - u\rangle) \leqslant 0,$$

let $v \in V_p$ be arbitrary.

We need to prove that $\liminf\limits_{n\to\infty}(\langle \mathrm{A}(u^n), u^n - v\rangle) \geqslant \langle \mathrm{A}(u), u - v\rangle$.

(i) We will show $\langle \mathrm{A}(u^n) - \mathrm{A}(u), u^n - u\rangle \to 0$ as $n \to \infty$.

Denoting

$$f_n(x) = \sum_{i,j,k=1}^{3} \left(A_{ij}^k(Du^n, \nabla^2 u^n) - A_{ij}^k(Du^n, \nabla^2 u)\right)\left(\frac{\partial^2 u_i^n}{\partial x_j \partial x_k} - \frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)$$

we can write

$$\begin{aligned}
\langle \mathrm{A}(u^n) - \mathrm{A}(u), u^n - u\rangle &= \int_\Omega \varrho^{-1} f_n \,\mathrm{d}x \\
+ \int_\Omega \varrho^{-1} &\sum_{i,j=1}^{3} \left(A_{ij}(Du^n) - A_{ij}(Du)\right)\left(\frac{\partial u_i^n}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\right)\mathrm{d}x \\
+ \int_\Omega \varrho^{-1} &\sum_{i,j,k=1}^{3} \left(A_{ij}^k(Du^n, \nabla^2 u) - A_{ij}^k(Du, \nabla^2 u)\right)\left(\frac{\partial^2 u_i^n}{\partial x_j \partial x_k} - \frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)\mathrm{d}x\,.
\end{aligned}$$

Following the proof of continuity of A, we can show that the last two right hand side terms tend to zero, (M2) implies $f_n \geqslant 0$, hence $\liminf\limits_{n\to\infty}(\langle \mathrm{A}(u^n) - \mathrm{A}(u), u^n - u\rangle) \geqslant 0$.

From the limsup assumption it follows that $\limsup\limits_{n\to\infty}(\langle \mathrm{A}(u^n) - \mathrm{A}(u), u^n - u\rangle) \leqslant 0$ and the proof of part 1 is complete. Note that consequently $f_n \to 0$ in $L^1(\Omega)$.

(ii) Let us prove $\nabla^2 u^n \to \nabla^2 u$ a.e. in Ω.

The imbedding implies $u^n \to u$ in $L^p(\Omega, \mathbb{R}^3)$ and $\nabla u^n \to \nabla u$ in $L^p(\Omega, \mathbb{R}^9)$. Hence $u^n \to u$, $\nabla u^n \to \nabla u$ a.e. in Ω. By part 1 also $f_n \to 0$ a.e. in Ω. We can write

$$\begin{aligned}
f_n(x) = &\sum_{i,j,k=1}^{3} A_{ij}^k(Du^n, \nabla^2 u^n)(x)\frac{\partial^2 u_i^n}{\partial x_j \partial x_k}(x) + \sum_{i,j=1}^{3} A_{ij}(Du^n(x))\frac{\partial u_i^n}{\partial x_j}(x) \\
&- \sum_{i,j,k=1}^{3} A_{ij}^k(Du^n, \nabla^2 u^n)(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) - \sum_{i,j=1}^{3} A_{ij}(Du^n(x))\frac{\partial u_i^n}{\partial x_j}(x) \\
&- \sum_{i,j,k=1}^{3} A_{ij}^k(Du^n, \nabla^2 u)(x)\frac{\partial^2 u_i^n}{\partial x_j \partial x_k}(x) + \sum_{i,j,k=1}^{3} A_{ij}^k(Du^n, \nabla^2 u)(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k}(x).
\end{aligned}$$

Thus (M1) and the Caratheodory conditions yield

$$f_n(x) + K + c|Du^n(x)|^{p-1}(2|\nabla^2 u(x)| + |Du^n(x)|) + c|\nabla^2 u(x)|^{p-1}|\nabla^2 u(x)|$$
$$\geqslant c_0|\nabla^2 u^n(x)|^p - c|\nabla^2 u^n(x)|(|Du^n(x)|^{p-1} + |\nabla^2 u(x)|^{p-1}) - c|\nabla^2 u^n(x)|^{p-1}|\nabla^2 u(x)|.$$

Since all the terms except the powers of $|\nabla^2 u^n|$ are bounded, the sequence $\{|\nabla^2 u^n|\}_{n=1}^{\infty}$ is bounded as well. Then the (M2) condition along with the compactness of a bounded interval and the continuity of A_{ij}, A_{ij}^k imply $\nabla^2 u^n \to \nabla^2 u$ a.e. in Ω.

(iii) Using the above inequality and the Vitali's theorem that was mentioned above we can prove that $\nabla^2 u^n \to \nabla^2 u$ in L^p, that is, $u^n \to u$ in V_p. The continuity of A implies $\lim\limits_{n\to\infty}(\langle \mathrm{A}(u^n), u^n - v\rangle) = \langle \mathrm{A}(u), u - v\rangle$, which is even more than we wanted to prove. □

As an immediate consequence of these lemmas and Theorem 2.4 we have

Theorem 2.6. *Let* (M1) *and* (M2) *by satisfied. Then* P *is a coercive pseudomonotone operator* $V_p \to V_p'$.

3. Main Results

From the theory of monotone operators and Theorem 2.6 we get the main existence result:

Theorem 3.1. *Let* (M1) *and* (M2) *be satisfied. Then for every* $f \in V_p'$ *there exists a weak solution.*

Now let us show several regularity properties. We will start with the theorem that implies the existence of a "presure in the weak sense".

Theorem 3.2. *Let* $A_{ij} \in C^1(\mathbb{R}^6)$, $A_{ij}^k \in C^2(\mathbb{R}^6 \times \mathbb{R}^{27})$, $f \in L^p$. *Let* $u \in C_0^4(\Omega, \mathbb{R}^3)$ *be a weak solution. Then there exists* $q \in W^{1,p}(\Omega, \mathbb{R})$ *such that* (Ph2) *holds in the sense of distributions.*

In other words, a weak solution u satisfies (Ph2)–(Ph4) in the weak sense provided the functions A_{ij}, A_{ij}^k, and u are smooth enough. We conclude this paper with the following theorem that is a natural conjugate to Theorem 2.2:

Theorem 3.3. *Let* $A_{ij} \in C^1(\mathbb{R}^6)$, $A_{ij}^k \in C^2(\mathbb{R}^6 \times \mathbb{R}^{27})$, $A_{ijk} \in C^1(\mathbb{R}^6 \times \mathbb{R}^{27})$, $f \in C(\Omega, \mathbb{R}^3)$, *and let* (PhM1) *be satisfied. Let* $u \in C_0^4(\Omega, \mathbb{R}^3)$ *be a weak solution. If* *there exists* $q \in C^1(\Omega)$ *such that* (Ph2) *holds in the sense of distributions, then* u, q *satisfy* (Ph1)–(Ph5).

A c k n o w l e d g e m e n t s. This work is based on the author's MSc Thesis prepared at Charles University, Prague under the supervision Professor Jindřich Nečas. I would like to thank Professor Nečas for his help and advice and Professor J. Málek for valuable suggestions concerning this paper.

[N] *Nečas J.*: Introduction to the Theory of Nonlinear Elliptic Equations. BSB B.G. Teubner Verlagsgesellschaft, Leipzig (1983).

[NNŠ1] *Nečas J., Novotný A., Šilhavý M.*: Global Solution to the Viscous Compressible Barotropic Fluid. to appear in J. Math. Anal. Appl..

[NNŠ2] *Nečas J., Novotný A., Šilhavý M.*: Global Solution to the Ideal Compressible Heat Conductive Multipolar Fluid. Comment. Math. Univ. Carolinae *30,3* (1989), 551-564.

[NNŠ3] *Nečas J., Novotný A., Šilhavý M.*: Some Qualitative Properties of the Viscous Compressible Heat Conductive Multipolar Fluid. Commun. in Partial Differential Equation *16* (1991), no. 2&3, 197–220.

[NŠ] *Nečas J., Šilhavý M.*: Viscous Multipolar Fluids. to appear in Quarterly for Applied Mathematics.

[No] *Novotný A.*: Viscous Multipolar Fluids. Thesis, Prague, 1990.

Author's address: Department of Mathematics, University of Alberta, Edmonton, T6G 2G1, Canada; e-mail: `phabala@vega.math.ualberta.ca`, `phabala@ualtavm.bitnet`.