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DEPENDENCES BETWEEN DEFINITIONS OF FINITENESS II

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1. INTRODUCTION

This paper is a continuation of [11] and presents results contained in the author's works within the Student Research Competition and in his diploma paper; they were all prepared and written under the quidance of P. Vojtáš. The notation used here is similar to that in [11]. Again we work in the set theory without any form of the Axiom of Choice (AC) and every consideration of the set theory in the paper works both in the ZF and ZFU theories. We consider eight definitions of finiteness systematically studied already in [7], [9] and [11] (later we add some new ones). Before introducing the definition let us recall that a set X is called reflexive iff there exists a bijection of X onto its proper subset.

Definition. A set A is

- I-finite iff every nonempty system of its subsets has a $\subseteq$-maximal element;
- Ia-finite iff for its every subset B either B is I-finite or $A - B$ is I-finite;
- II-finite iff every nonempty system of its subsets linearly ordered by $\subseteq$ has a $\subseteq$-maximal element;
- III-finite iff its power set is not reflexive;
- IV-finite iff it is not reflexive;
- V-finite iff $2 \cdot |A| > |A|$ or $|A| = 0$;
- VI-finite iff $|A|^2 > |A|$ or $|A| \leqslant 1$;
- VII-finite iff it cannot be well-ordered or the type of its well-ordering is a natural number.

IV-finiteness is often called Dedekind finiteness (and I-finiteness is sometimes called Tarski finiteness). Without any further comments we (tacitly) use the well-known statements provable in the set theory (without any choice axiom):

- a set X is reflexive iff $\aleph_0 \leqslant |X|$ (see e.g. [8]);

– a set X is I-finite iff $(\exists n \in \omega)(n = |X|)$ (e.g. [8]).

It is an immediate observation that the equivalence of I- and VII-finiteness is equivalent to the AC.

Let us briefly introduce some notation and recall something from [11]:

For F-finiteness, F''-finiteness of A means F-finiteness of $\mathscr{P}(A)$; for F_1- and F_2-finiteness, $F_1 \rightarrow F_2$ denotes the formula

$$(\forall A)(A \text{ is } F_1\text{-finite} \Rightarrow A \text{ is } F_2\text{-finite}),$$

$F_1 \leftrightarrow F_2$, $F_1 \nrightarrow F_2$, $F_1 \nleftrightarrow F_2$ denote the formulas $(F_1 \rightarrow F_2) \,\&\, (F_2 \rightarrow F_1)$, $\neg(F_1 \rightarrow F_2)$, $\neg(F_1 \leftrightarrow F_2)$, respectively; $\mathscr{J}_F$ is the class $\{A;\, A \text{ is } F\text{-finite}\}$. Further, $\psi_1 \equiv (I \leftrightarrow Ia)$, $\psi_2 \equiv (Ia \leftrightarrow II)$, $\psi_3 \equiv (II \leftrightarrow III)$, $\psi_4 \equiv (III \leftrightarrow IV)$, $\psi_5 \equiv (IV \leftrightarrow V)$, $\psi_6 \equiv (V \leftrightarrow VI), \psi_7 \equiv (VI \leftrightarrow VII)$; $\psi_i^1 \equiv \neg\psi_i$, $\psi_i^0 \equiv \psi_i$. Hence for $\delta \in \{0,1\}$, ψ_i^δ says whether there is any set in the assigned difference of classes of finite sets or not; if for $\varepsilon = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7) \in {}^7\{0,1\}$ we define $\Phi_\varepsilon = \mathop{\&}\limits_{i=1}^{7} \psi_i^{\delta_i}$, we obtain the formula saying in which difference of classes of finiteness there is or is not a set. If in a model $\mathscr{M}$ of the set theory Φ_ε holds, we say also that $\mathscr{M}$ has type ε. The list of "important" ε's from [11]:

$$\begin{aligned}
\varepsilon_1 &= (0,0,0,0,0,0,0),\\
\varepsilon_2 &= (0,0,0,0,0,1,1),\\
\varepsilon_3 &= (0,0,0,0,1,1,1),\\
\varepsilon_4 &= (0,0,0,1,1,1,1),\\
\varepsilon_5 &= (0,0,1,1,1,1,1),\\
\varepsilon_6 &= (0,1,0,1,1,1,1),\\
\varepsilon_7 &= (0,1,1,1,1,1,1),\\
\varepsilon_8 &= (1,1,0,1,1,1,1),\\
\varepsilon_9 &= (1,1,1,1,1,1,1).
\end{aligned}$$

Now let us mention some results connected with the definitions of finiteness. The statement $I \rightarrow Ia \rightarrow II \rightarrow III \rightarrow IV \rightarrow V \rightarrow VI \rightarrow VII$ as well as the fact that all these definition are independent in the ZF and ZFU theories are well known (precise reference see in [11]). The main results of [11] which are needed in the following are Theorems 1.1 and 1.2 of [11]. They state that the AC, $V \leftrightarrow VI$, $VI \leftrightarrow VII$ are equivalent in the set theory and that $\bigvee\limits_{i=1}^{9} \Phi_{\varepsilon_i}$ holds in the set theory, i.e. there are at most 9 possibilities of the simultaneous existence of various finite sets (there are at most 9 types of models of the set theory with respect to the definitions of finiteness).

Some further relations between the considered definitions are given in this paper, namely operations constructing from a I-infinite, F-finite set an example of a G-infinite but G'-finite set for certain F, G and G'. They are presented in

Theorem 2 A.

i) *If* $A \in \mathcal{J}_{Ia} - \mathcal{J}_I$ *then* $2 \times A \in \mathcal{J}_{II} - \mathcal{J}_{Ia}$.

ii) *If* $A \in \mathcal{J}_{III} - \mathcal{J}_I$ *then* $\mathcal{P}(A) \in \mathcal{J}_{IV} - \mathcal{J}_{III}$.

iii) *If* $A \in \mathcal{J}_{IV} - \mathcal{J}_I$ *then* $A \cup \omega \in \mathcal{J}_V - \mathcal{J}_{IV}$.

iv) *If* $A \in \mathcal{J}_{III} - \mathcal{J}_I$ *then* $A \times \omega \in \mathcal{J}_{VI} - \mathcal{J}_V$.

v) *If* $A \in \mathcal{J}_{VII} - \mathcal{J}_I$ *then* $A \times \omega \in \mathcal{J}_{VII} - \mathcal{J}_V$.

vi) *If* $A \in \mathcal{J}_{VII} - \mathcal{J}_V$ *then* $(\{0\} \times A) \cup A^+ \in \mathcal{J}_{VI} - \mathcal{J}_V$.

vii) *If* $A \in \mathcal{J}_{VII} - \mathcal{J}_I$ *then* $(A \times \omega) \cup (A \times \omega)^+ \in \mathcal{J}_{VI} - \mathcal{J}_V$.

viii) *If* $A \in \mathcal{J}_{VII} - \mathcal{J}_I$ *then* ${}^{\omega}A \in \mathcal{J}_{VII} - \mathcal{J}_{VI}$.

(X^+ is the so-called Hartogs' aleph of the set X).

Note that these operations form another proof of Theorem 1.1 and 1.2 of [11] (they exclude all ε's except of $\varepsilon_1 \dots, \varepsilon_9$). They also enable us to say something concerning one natural question investigated e.g. by T. Jech ([4], [5]) and A. Tarski (the references see in [7])—the behaviour of sets of real numbers with respect to the definitions of finiteness. If we denote by $(F_1 \leftrightarrow F_2)'$ the formula

$$(\forall A \subseteq \mathbf{R})(A \text{ is } F_1\text{-finite } \Leftrightarrow A \text{ is } F_2\text{-finite})$$

and form ψ_i', $\psi_i'^{\delta}$, Φ_ε' just as ψ_i, ψ_i^δ and Φ_ε above, we obtain

Corollary 2 B. $\bigvee_{i=1}^{7} \Phi'_{\eta_i}$ *is provable in the set theory, where*

$$\eta_1 = (0,0,0,0,0,0,0),$$
$$\eta_2 = (0,0,0,0,0,0,1),$$
$$\eta_3 = (0,0,0,0,0,1,1),$$
$$\eta_4 = (0,0,0,0,1,0,1),$$
$$\eta_5 = (0,0,0,0,1,1,1),$$
$$\eta_6 = (0,0,0,1,1,0,1),$$
$$\eta_7 = (0,0,0,1,1,1,1).$$

We deal also with closedness of classes of finite sets under certain operations, summarizing several properties in

Theorem 2 C. i) *$\mathcal{J}_{IV}$ and $\mathcal{J}_{VIII}$ are closed under subsets iff AC holds, $\mathcal{J}_V$ is closed under subsets iff $IV \leftrightarrow V$.*

ii) *$\mathcal{J}_{Ia}$ is an ideal iff $I \leftrightarrow Ia$.*

iii) *$\mathcal{J}_I$, $\mathcal{J}_{II}$, $\mathcal{J}_{III}$, $\mathcal{J}_{IV}$ are ideals.*

The next results show certain strange properties of finiteness weaker than the Dedekind one: assigned classes of finite sets contain "arbitrarily large" finite sets according to

Theorem 2 D. i) *If $A \in \mathcal{J}_V - \mathcal{J}_I$ then $A \cup \alpha \in \mathcal{J}_V - \mathcal{J}_{IV}$ holds for every ordinal $\alpha \geqslant \aleph_0$.*

ii) *If $A \in \mathcal{J}_{VII} - \mathcal{J}_I$ then for every set X there exists $Y \in \mathcal{J}_{VII} - \mathcal{J}_{VI}$ such that $X \cup A \subseteq Y$.*

iii) *If $A \in \mathcal{J}_{VII} - \mathcal{J}_I$ then for every set X there exists $Z \in \mathcal{J}_{VI} - \mathcal{J}_V$ such that $X \cup A \subseteq Z$.*

A partial answer to the problem of relative consistency of the remaining nine possible $\Phi\varepsilon$'s with the ZFU and ZF theories is given by

Theorem 3 A. *Each of the formulas $\Phi\varepsilon_9$, $\Phi\varepsilon_5$, $\Phi\varepsilon_3$, $\Phi\varepsilon_2$, $\Phi\varepsilon_1$ is relatively consistent with the ZFU theory.*

Theorem 3 B. *Each of the formulas $\Phi\varepsilon_9$, $\Phi\varepsilon_4$, $\Phi\varepsilon_3$, $\Phi\varepsilon_2$, $\Phi\varepsilon_1$ is relatively consistent with the ZF theory.*

Finally we deal with some new definitions of finiteness. We start with the following ones:

Definition. A set A is

- A-finite iff every Boolean algebra $\mathcal{B}$ of its subsets (i.e. every B. a. $[\mathcal{B}, \cup, \cap, -, \emptyset, A]$ with $\mathcal{B} \subseteq \mathcal{P}(A)$) is atomic;
- C-finite iff every Boolean algebra $\mathcal{B}$ of its subsets is complete;
- P-finite iff every filter on A can be extended to a principal filter;
- H-finite iff every Hausdorff topology on A is discrete.

Comparison of them with the old definitions is contained in

Theorem 4 A. i) $I \leftrightarrow C \leftrightarrow P \leftrightarrow H$.

ii) $I \rightarrow A'' \rightarrow A \rightarrow IV$ (and $A'' \rightarrow III$).

A more interesting way to obtain a new definition (VIII) has been suggested by Professor L. Bukovský: to convert the definition of the Dedekind finiteness.

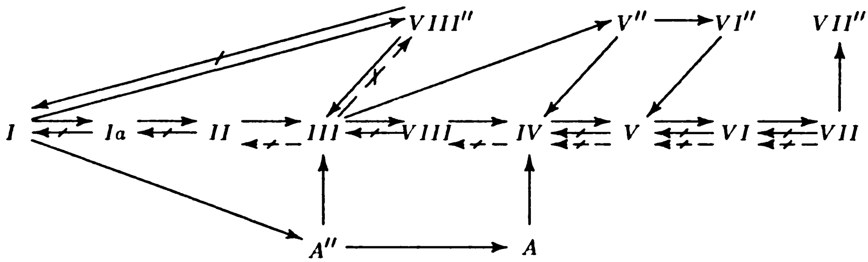
Definition. A set A is VIII-finite iff every $f\colon A \to A$ which is onto A is also one-to-one.

We have also some independence results concerning VIII-finiteness (one different from that in Theorem 4B ii) is in the first lemma of Section 4).

Theorem 4 B. i) *$III \to VIII \to IV$ (and $I \to VIII'' \to III$).*

ii) *$\Phi\varepsilon_9 \,\&\, (IV \nrightarrow VIII \nrightarrow III \nrightarrow VIII'' \nrightarrow I)$ is relatively consistent both with the ZFU and ZF theories.*

Summarizing the information from [11] (Theorem 2.1) and from this paper (Section 3—the properties of models $\mathcal{M}_1$, $\mathcal{M}^+$, $\mathcal{M}_1^+$, $\mathcal{N}_1^+$ and Section 4—the definition VIII in these models) we obtain the diagram presented below; $\longrightarrow$ denote formulas provable in the set theory, $\not\longrightarrow$ and $- \not- \rightarrow$ denote nonprovable formulas true in $\mathcal{M}_1$ and $\mathcal{M}^+$, respectively; moreover, all these formulas hold both in $\mathcal{M}_1^+$ and $\mathcal{N}_1^+$ models (hence $\mathcal{M}_1^+$ and $\mathcal{N}_1^+$ show the simultaneous relative consistency of all nonprovable formulas occurring in the diagram). Provability of every missing implication in the diagram (up to the "composition" of implications and except of $VII'' \nrightarrow VII$ known in every model with $\mathcal{P}(\omega) = \mathbf{R}$ not well-orderable) is an open problem.



The rest of the paper is organized as follows: Section i is devoted to the proof of Theorems iX and to the presentation of some more detailed information, remarks, comments and problems concerning the given topic.

2. Operations constructing sets with various relations to definitions of finiteness. Properties of $\mathcal{J}_F$'s

Proof of Theorem 2 A. i) Folklore.

ii) $\mathcal{P}(A) \in \mathcal{J}_{IV}$ holds by the definition and $\mathcal{P}(A) \notin \mathcal{J}_{III}$ holds by the well-known fact that $|\mathcal{P}(\mathcal{P}(A))| \geqslant \aleph_0$ holds for every I-infinite set A ([8]).

iii) Clearly without loss of generality one can assume $A \cap \omega = \emptyset$. IV-infiniteness is evident. Let $A \cup \omega$ be V-infinite. Then

$$|A \cup \omega| = 2 \cdot |A \cup \omega| = |(2 \times A) \cup \omega|,$$

and let

$$f\colon (2 \times A) \cup \omega \to A \cup \omega$$

be one-to-one. There are only I-finitely many elements of $f[\omega]$ which may belong to A, I-finitely many (n) elements of $f[\{0\} \times A]$ and I-finitely many (m) elements of $f[\{1\} \times A]$ which may belong to ω. Therefore one can construct a one-to-one mapping

$$g\colon 2 \times A \to A \cup (n+m);$$

hence $2 \times A$ is IV-infinite. This implies IV-infiniteness of A (take any one-to-one sequence of size $\aleph_0$ in $2 \times A$).

iv) V-infiniteness is obvious. Let $A \times \omega$ be VI-infinite, i.e. there exists an injection

$$g\colon A \times \omega \times A \times \omega \to A \times \omega$$

and therefore also an injective

$$f\colon A \times A \to A \times \omega.$$

If $(\exists a \in A)(\forall n_0 \in \omega)(\exists n > n_0)(\exists [b,c] \in A \times A)(f([b,c]) = [a,n])$ then one has a one-to-one sequence of size $\aleph_0$ in $A \times A$; either from the first or from the second components of the pairs one can form a one-to-one sequence of size $\aleph_0$ in A, which gives a contradiction. Therefore $[(\forall a \in A)(\exists n_a \in \omega)((\forall [b,c] \in A \times A)(f([b,c]) = [a,n] \Rightarrow n < n_a) \,\&\, ([a, n_a - 1] \in \operatorname{rng} f \vee n_a = 0))]$. Now if there exists $n_0 \in \omega$ such that

$$(\forall a \in A)(n_a \leqslant n_0)$$

then $|A \times A| \leqslant n_0 \cdot |A|$ and thus $A \times A$ is IV-infinite, which leads to a contradiction like above. Therefore n_a's are not bounded in ω. For $a \in A$ put

$$A_a = \{b \in A \colon n_a = n_b\}$$

and define a mapping

$$h\colon \{n_a \in \omega;\, a \in A\} \to \mathscr{P}(A)$$

by $h(n_a) = A_a$. Clearly the size of $\operatorname{dom} h$ is $\aleph_0$ and h is one-to-one, which means that $\mathscr{P}(A)$ is IV-infinite—a contradiction.

viii) VI-infiniteness is obvious. A can be mapped one-to-one into ${}^{\omega}A$ and therefore ${}^{\omega}A$ is not well-orderable, hence it is VII-finite.

v) The same as in viii).

vi) Clearly $(\{0\} \times A) \cup A^+$ is V-infinite. Let it be VI-infinite, i.e.

$$|A|^2 + 2 \cdot |A| \cdot A^+ + A^+ = |A| + A^+,$$

$$|A| \cdot A^+ \leqslant |A| + A^+.$$

By Tarski's lemma mentioned also in [11] (see [8]) $|A|$ and A^+ are comparable, hence $|A| \leqslant A^+$, therefore A is well-orderable, which is a contradiction.

vii) It follows immediately from v) and vi). □

Part iv) of the theorem may seem to be not useful because of the stronger result of part vii) but the set in iv) is simpler (smaller) and so there is some interesting information in iv), too. Several of these operations (namely $2 \times A$, $\mathscr{P}(A)$, $A \cup \omega$, $A \times \omega$, ${}^{\omega}A$) have been used in the proofs of some independence results, but in specific models of the ZFU theory; the argument of the operations was the set of all atoms (see [7]). Parts i) and iii) have been already proved in [7] and [2] in the form $(II \leftrightarrow Ia) \Rightarrow (Ia \leftrightarrow I)$ and $(IV \leftrightarrow V) \Rightarrow (IV \leftrightarrow I)$, respectively. The proofs in [7] and [2] proceed by way of contradiction and the sets which make those contradictions are $2 \times A$ and $A \cup \omega$, respectively (A is a I-infinite set which is Ia- and IV-finite, respectively). The proofs of ii) and viii) are from [7], too, because they work not only in the models used there (on the other hand, the proofs of the properties of $A \cup \omega$ and $A \times \omega$ in [7] essentially use the properties of the model). Part vi) is the one in which the original Tarski's idea used in the proof of his nontrivial statement $AC \Leftrightarrow (I \leftrightarrow VI)$ is hidden.

Problems. 2.1) To find an operation φ such that $\varphi(A) \in \mathscr{I}_{III} - \mathscr{I}_{II}$ holds for every $A \in \mathscr{I}_{II} - \mathscr{I}_{I}$ or to prove that it does not exist.

2.2) Does not $A \times \omega \in \mathscr{I}_{VI} - \mathscr{I}_{V}$ hold already for $A \in \mathscr{I}_{IV} - \mathscr{I}_{I}$ or even for $A \in \mathscr{I}_{V} - \mathscr{I}_{I}$?

Considering the set of all real numbers $\mathbf{R}$, one observes at once that if $\mathbf{R}$ is well-ordered then each of its subsets is such and thus $I \leftrightarrow VII$ holds for sets of real numbers. Therefore the single interesting case is $\mathbf{R}$ not well-orderable. Recall that the definitions I and III are equivalent for subsets of $\mathbf{R}$ ([4] and a footnote in [7]) and if $\mathbf{R}$ is not well-orderable then $\mathbf{R}$ itself is VII-finite, VI-infinite. Note that if $A \subseteq \mathbf{R}$ then also $A \cup \omega$, $A \times \omega$ and ${}^{\omega}A$ can be embedded into $\mathbf{R}$ ($A \subseteq \mathbf{R}$, hence ${}^{\omega}A \subseteq {}^{\omega}\mathbf{R}$; clearly $|{}^{\omega}\mathbf{R}| = |\mathbf{R}|$ without AC).

Proof of Corollary 2 B. It follows from the above remarks that under the assumption **R** not well-orderable, $\bigvee_{\eta\in\Theta} \Phi'\eta$ holds for sets of real numbers, where Θ is the set of all vectors of the form $(0,0,0,\varrho_4,\varrho_5,\varrho_6,1)$ with $\varrho_i \in \{0,1\}$. The possibilities $\eta = (0,0,0,1,0,0,1)$ and $\eta = (0,0,0,1,0,1,1)$ are excluded by the operation $A \cup \omega$. □

Problem. 2.3) To eliminate as many possibilities as possible and prove the consistency of the rest.

Further let us consider the classes $\mathcal{J}_F$ and investigate several of their properties. There are some natural demands which the concept "finiteness" should satisfy, like "the power set of a finite set is finite", "a subset of a finite set is finite", "the union of two finite sets is finite" etc. It follows easily from the fact that the definitions $I,\ldots,VII$ are independent and from Theorem 2.1 of [11] that only $\mathcal{J}_I$ and $\mathcal{J}_{VII}$ are closed under the power set operation, the others need not be such. The last two requirements mentioned above form together with "$\emptyset$ is finite" and "not every set is finite" just the definition of an ideal. So we will answer the question which from the classes $\mathcal{J}_F$ must be ideals and which ones need not. The first step is to solve the question about subsets.

Lemma. *For every set A the following holds:*

i) *A is F-finite iff every proper subset of A is F-finite for $F \in \{I, Ia, II, III, IV\}$.*

ii) *A is IV-finite iff every proper subset of A is V-finite iff every proper subset of A is VI-finite iff every proper subset of A is VII-finite.*

Proof. i) Not difficult, verified in [9]. ii) Since we have $IV \to V \to VI \to VII$ and i) it suffices to show that if A is IV-infinite then there exists a well-ordered infinite proper subset of A; and this clearly holds. □

Consequently, it is interesting to deal with the requirement concerning the union of two finite sets only for the classes $\mathcal{J}_I$, $\mathcal{J}_{Ia}$, $\mathcal{J}_{II}$, $\mathcal{J}_{III}$, $\mathcal{J}_{IV}$. The result is in our Theorem 2C.

Proof of Theorem 2 C. i) Immediate consequence of Theorem 1.1 of [11] and the lemma ii).

ii) It follows easily from lemma i) and Theorem 2 A i).

iii) We only show that if $A \cap B = \emptyset$ and $A \cup B \notin \mathcal{J}_{III}$ then $A \notin \mathcal{J}_{III}$ or $B \notin \mathcal{J}_{III}$, everything else (about $\mathcal{J}_I$, $\mathcal{J}_{II}$, $\mathcal{J}_{IV}$) being easy. Take a one-to-one sequence $\mathcal{D} = \{D_n \in \mathcal{P}(A \cup B);\, n \in \omega\}$ and construct sequences $\mathcal{A} = \{A \cap D_n;\, n \in \omega\}$, $\mathcal{B} = \{B \cap D_n;\, n \in \omega\}$. If

$$(\exists n_0 \in \omega)(\forall n > n_0)(\exists n' \leqslant n_0)(A_{n'} = A_n)$$

then

$$(\exists k \leqslant n_0)(\forall m_0 \in \omega)(\exists m > m_0)(A_m = A_k).$$

Therefore

$$|\{m \in \omega;\ A_m = A_k\}| = \aleph_0$$

and clearly

$$(m \neq n \,\&\, A_m = A_n) \Rightarrow B_m \neq B_n,$$

thus B is III-infinite. If n_0 from the above consideration does not exist then one can easily construct a one-to-one sequence of size $\aleph_0$ in $\mathscr{P}(A)$, which proves III-infiniteness of A. □

Proof of Theorem 2 D. i) $A \cup \alpha \notin \mathscr{J}_{IV}$ is obvious. $A \cup \alpha \in \mathscr{J}_V$:

Case $\alpha < A^+$. There exists $B \subseteq A$ such that $|B| = |\alpha|$. Then obviously $|A \cup \alpha| = |A|$ and hence $A \cup \alpha \in \mathscr{J}_V$.

Case $\alpha \geqslant A^+$. One can use similar argument as in the proof of Theorem 2 A iii). Put $B = A - \alpha$. If B is IV-finite then the proof is just the same as that of Theorem 2 A iii). Suppose now $|B| \geqslant \alpha_0$. We prove by contradiction that $2 \cdot |B \cup \alpha| > |B \cup \alpha|$ holds. Assume the contrary, i.e. there exists a one-to-one mapping

$$f\colon 2 \times (B \cup \alpha) \to B \cup \alpha.$$

Take the cardinal number

$$\lambda = |f[2 \times B] \cap \alpha|.$$

Clearly $\lambda < B^+ \leqslant A^+$. Hence one is able to construct an injective

$$g\colon 2 \times B \to B \cup \lambda$$

where $\lambda \leqslant |B|$ (and $|B| \geqslant \aleph_0$; this is important for the case $\lambda < \aleph_0$). Since $|B \cup \lambda| = |B|$ there exists a one-to-one mapping

$$h\colon 2 \times B \to B$$

and hence $2 \cdot |B| \leqslant |B|$. But there exists a cardinal number κ such that $|A| = |B| + \kappa$ and therefore

$$2 \cdot |A| = 2 \cdot (|B| + \kappa) \leqslant |B| + \kappa = |A|,$$

which contradicts V-finiteness of A.

ii) and iii) Clearly if $A \in \mathscr{J}_{VII} - \mathscr{J}_I$ then $X \cup A$ is such, too. By Theorem 2 A viii), if $A \in \mathscr{J}_{VII} - \mathscr{J}_I$ then

$$Y' =^{\omega} (X \cup A) \in \mathscr{J}_{VII} - \mathscr{J}_{VI} \text{ and } |X \cup A| \leqslant |Y'|$$

holds for every X. By Theorem 2 A vii), if $A \in \mathscr{J}_{VII} - \mathscr{J}_I$ then

$$Z' = ((X \cup A) \times \omega) \cup ((X \cup A) \times \omega)^+ \in \mathscr{J}_{VI} - \mathscr{J}_V \text{ and } |X \cup A| \leqslant |Z'|$$

holds for every X. Now it is easy to construct from Y', Z' sets Y, Z of the same size and with $X \cup A \subseteq Y, Z$. □

A result similar to Theorem 2 D i) is formulated in Corollary to Lemma 9 in [7], namely that the statement "for any given aleph there exists a V-finite set with cardinality greater than that aleph" is relatively consistent with ZFU+ "every set can be linearly ordered". The proof of V-finiteness of the set $A \cup \alpha$ in the model used there (Lemma 9) works with the specific properties of that model (hence the proof in [7] is entirely different from ours and is not sufficient for our more general result).

Problems. 2.4) Can any (sufficiently large) set X (not only every ordinal $\alpha \geqslant \aleph_0$) be also extended to $W \in \mathscr{J}_V - \mathscr{J}_{IV}$?

2.5) Cannot an upper estimation for Dedekind finite sets be found—e.g. rank α such that for every IV-finite set there exists a set of the same size with rank $\beta \leqslant \alpha$ (for F-finite, Dedekind infinite sets it is impossible, as we can see from Theorem 2 D)?

3. Consistency of some $\Phi\varepsilon$'s

(i.e. of some types of simultaneous occurrence of differently finite sets)

To prove Theorems 3 A and 3 B we recall some properties of several models of the ZFU and ZF theories and give their types or at least particular information about it.

First let us consider the ZFU theory. We will use the so-called permutation models of ZFU. They are all defined in ZFC, hence in the description of models we will not distinguish between various kinds of finiteness. Such a model is uniquely determined by a set of all atoms A, a group of permutations $\mathscr{G}$ on it and a normal filter of subgroups of that group. In every model used here the filter is constructed in the well-known way from a nontrivial ideal J on $\mathscr{P}(A)$. (Elements of J are the so-called supports of sets of the model.) The first model mentioned below is due to A. Fraenkel, the second is due to A. Mostowski; both of them together with our fourth model are described for instance in [4]; each of their properties which are needed here was known except of $2 \cdot |A| > |A|$ in $\mathscr{M}_4$.

P r o o f of Theorem 3 A. Model $\mathscr{M}_1$.

A is countable infinite, $\mathscr{G}$ is the group of all permutations on A, J is the ideal of all finite subsets of A. Here A is I-infinite and $B \subseteq A$ is an element of $\mathscr{M}_1$ iff it is either finite or a complement of a finite set. Hence A is Ia-finite, thus $\mathscr{M}_1$ is of type ε_8 or ε_9.

Model $\mathscr{M}^+$.

A is a countable linearly ordered set with dense ordering without the minimal and maximal elements, $\mathscr{G}$ is the group of all order-preserving permutations on A, J is as for the model $\mathscr{M}_1$. $\mathscr{M}^+$ is linearly ordered by a class-relation $\preccurlyeq$; its restriction to every $B \in \mathscr{M}^+$ gives a linear ordering of B. Since for linearly ordered sets $I \leftrightarrow II$ holds (e.g. [7]) we have that A is II-infinite. It is also known that A is III-finite, i.e. $\mathscr{M}^+$ has type ε_5.

$\mathscr{M}_1^+$.

It is a combination of $\mathscr{M}_1$ and $\mathscr{M}^+$. Let $A_1 \cap A' = \emptyset$, let A_1 be countable infinite, A' a countable linearly ordered set with dense ordering without the minimal and maximal elements, $A = A_1 \cup A'$. Let $\mathscr{G} = \{\pi;\ \pi$ is a permutation on A, $\pi[A_1] = A_1$, π preserves the ordering of $A'\}$. J is again the same as for the models $\mathscr{M}_1$, $\mathscr{M}^+$. Hence A is the union of disjoint sets A_1, A' of atoms with the same properties as the sets of atoms have in $\mathscr{M}_1$ and $\mathscr{M}^+$, respectively. Obviously, for A_1 and A' the statements about their finiteness proved above stay true. Thus $\mathscr{M}_1^+$ is of type ε_9.

Model $\mathscr{M}_4$.

A has $\aleph_1$ elements, $\mathscr{G}$ is the group of all permutations on A, J is the ideal of all subsets of A which are at most countable. Clearly a subset of A is an element of $\mathscr{M}_4$ iff it is either at most countable or a complement of an at most countable set of atoms, hence A is IV-infinite. V-finiteness of A will be proved by contradiction. Let A be V-infinite, thus there exists an injective

$$f\colon A \times 2 \to A$$

with a support $E \in J$. There exist $x, y \in A$ such that

$$f([x,0]) = a \,\&\, f([y,1]) = b \,\&\, a, b \notin E;$$

clearly $a \neq b$. Take the permutation π which exchanges a, b and maps all the other atoms identically.

It must be true that

$$b \notin f[A \times \{0\}] = \pi f[A \times \{0\}] = f[\pi A \times \{0\}].$$

But $[[x,0],a] \in f$, hence also $[[\pi x,0],\pi a] \in \pi f = f$ and thus $\pi a = b \in f[\pi A \times \{0\}]$, which is a contradiction. The countable Axiom of Choice (CAC) holds in $\mathcal{M}_4$: if g is a function in the universe and

$$\operatorname{dom} g = \omega \,\&\, (\forall n \in \omega)(g(n) \in \mathcal{M}_4)$$

then $g \in \mathcal{M}_4$ (lemma in [4], chap. 8). This implies CAC in $\mathcal{M}_4$ since we assume AC in the universe.

As is well-known, $CAC \Rightarrow (I \leftrightarrow IV)$ ([4]), hence $\mathcal{M}_4$ has type ε_3.

Model $\mathcal{M}_3$.

We denote by $\mathcal{M}_3$ the model of ZFU constructed in [3]. By [3]

$$\mathcal{M}_3 \models ((I \leftrightarrow V) \,\&\, \neg AC)$$

holds; thus $\mathcal{M}_3$ is of type ε_2. □

Problems. 3.1) What is the type of $\mathcal{M}_1$?

3.2) Let us consider the so-called second Fraenkel's model of ZFU (we will denote it by $\mathcal{M}_2$):

A is the union of $\aleph_0$ disjoint pairs P_n of atoms. $\mathcal{G}$ is the group of all permutations preserving the pairs, J is the same as for $\mathcal{M}_1$. It is easily provable that every $P_n \in \mathcal{M}_2$, $\{[n, P_n];\, n \in \omega\} \in \mathcal{M}_2$ and that A is IV-finite; clearly A is III-infinite. Hence $\mathcal{M}_2$ is of type $(\delta_1, \delta_2, \delta_3, 1, 1, 1, 1)$. What are the values of δ_1, δ_2, δ_3?

3.3) To find models of the other types or to prove that they cannot occur.

Next we turn our attention to ZF. Our last two models are made from models of ZFU by using Jech-Sochor's theorem which says, roughly speaking, that if α is an ordinal in a given model $\mathcal{M}$ of ZFU then there exists a model $\mathcal{N}$ of ZF such that the initial segment of $\mathcal{M}$ below the level α can be $\in$-isomorphically embedded into $\mathcal{N}$. This means that if we have a formula φ, the truth value of which depends only on sets with ranks bounded by a common constant, then from a model of ZFU which satisfies φ we can construct a model of ZF which satisfies φ, too. Note that "A is F-finite" is such a formula for every F considered here (and hence "A is F-infinite", too).

Proof of Theorem 3B. Model $\mathcal{N}_1$.

By $\mathcal{N}_1$ we will denote the basic Cohen's model (see [4], chap. 5). The following hold in $\mathcal{N}_1$:

i) $I \leftrightarrow III$;

ii) there exists $B \in \mathcal{N}_1$, $B \subseteq \mathbf{R}$ which is IV-finite, I-infinite.

Moreover, ω is VII-infinite but VII''-finite (because **R** contains a IV-infinite, I-finite set, thus **R** cannot be well-ordered). Hence the type of $\mathcal{N}_1$ is ε_4.

Model $\mathcal{N}_3$.

We will denote by $\mathcal{N}_3$ the model from [10]. In $\mathcal{N}_3$ $I \leftrightarrow V$ holds but AC fails. Hence $\mathcal{N}_3$ is of type ε_2.

Model $\mathcal{N}_1^+$.

In $\mathcal{M}_1^+$ there are sets $X \in \mathcal{J}_{Ia} - \mathcal{J}_I$ and $Y \in \mathcal{J}_{III} - \mathcal{J}_{II}$. Taking α sufficiently large one can construct $\mathcal{N}_1^+$ by Jech-Sochor's theorem such that $\exists X' \in \mathcal{J}_{Ia} - \mathcal{J}_I \,\&\, \exists Y' \in \mathcal{J}_{III} - \mathcal{J}_{II}$ holds in $\mathcal{N}_1^+$. Then $\mathcal{N}_1^+$ is of type ε_9.

Model $\mathcal{N}_4$.

Using Jech-Sochor's theorem on $\mathcal{M}_4$ and the formula $A \in \mathcal{J}_V - \mathcal{J}_{IV}$ we obtain the model $\mathcal{N}_4$ in which $\exists X \in \mathcal{J}_V - \mathcal{J}_{IV}$ holds. By [4], chap. 8, Lemma 8.5 and proofs of Theorems 8.6 and 8.8 CAC holds in $\mathcal{N}_4$. (Our model $\mathcal{M}_4$ is a special case of models mentioned there for $\alpha = 0$ and $\alpha = 1$.) Since $CAC \Rightarrow (I \leftrightarrow IV)$ we know that the type of $\mathcal{N}_4$ is ε_3. $\square$

Thus we see that both in ZFU and ZF there are at least 5 and at most 9 types of models (models with AC have type ε_1).

Note that the models $\mathcal{M}_3$ and $\mathcal{N}_3$ show that the statement "I-infinite cardinals are idemmultiple" is not equivalent to AC. Hence we know that Theorem 1.1 from [11] cannot be improved.

Problems (similar as for ZFU). 3.4) To find models of the remaining types or to prove that they cannot occur.

3.5) Let us denote by $\mathcal{N}_2$ the so-called second Cohen's model of ZF (see [4]): In $\mathcal{N}_2$ there exists a set B with the same properties as A in $\mathcal{M}_2$, i.e.

i) $B = \bigcup_{n\in\omega} P_n$, P_n's disjoint, $|P_n| = 2$, P_n are sets;

ii) $\{[n, P_n]\,;\, n \in \omega\}$ is a set;

iii) B is IV-finite;

hence we know $\delta_4 = 1$ but we do not know the values of δ_1, δ_2, δ_3.

A common problem for all models:

3.6) Which $\Phi'\eta$ holds in a given model for sets of real numbers?

Our last remark is the following. It could be possible to find a model giving the negative answer to the problem 2.1). As we can see, neither this section nor Theorems 3 A and 3 B (nor the whole paper) solve this question; it remains open.

An intuitive idea of finiteness has its reflection in the properties of various mathematical concepts. Its formalization leads to some (mathematically) natural definitions of finiteness, like the definitions A, C, P, H, $VIII$ mentioned in Introduction. First we deal with our provable statements concerning them.

Proof of Theorem 4 A. i) The fact that I-finiteness implies C-, P- and H-finiteness is obvious. Converse implications: I-finite subsets of a I-infinite set and their complements form a Boolean algebra which is not complete. Complements of I-finite subsets of a I-infinite set give a filter which cannot be extended to a principal filter. Taking a fixed x from a I-infinite set A we can introduce the topology on $A - \{x\}$ discretely and define the completements of I-infinite subsets of $A - \{x\}$ to be open neighborhoods of x; we obtain a nondiscrete Hausdorff topology on A.

ii) $I \rightarrow A''$ is obvious.

$A'' \rightarrow A$: it can be easily proved that if X is A-finite then every $Y \subseteq X$ is A-finite, too. This implies $A'' \rightarrow A$.

$A \rightarrow IV$: let A be IV-infinite, let $f: \omega \rightarrow A$ be one-to-one. We will use classes of congruence modulo 2^m to construct a certain system of subsets of A. Put

$$A_{m,0} = \{f(2^m \cdot k) \in A\,;\, k \in \omega\} \cup (A - \operatorname{rng} f)$$

and for $n \in \omega, 0 < n < 2^m$ put

$$A_{m,n} = \{f(2^m \cdot k + n) \in A\,;\, k \in \omega\},$$
$$\mathscr{C} = \{A_{m,n} \in \mathscr{P}(A)\,;\, m, n \in \omega\}.$$

Now we will construct the smallest B. a. of subsets of A which contains $\mathscr{C}$: put $\mathscr{B}_0 = \mathscr{C}$ and for $k \in \omega$

$$\mathscr{B}_{k+1} = \{Y \cup Z \in \mathscr{P}(A)\,;\, Y, Z \in \mathscr{B}_k\} \cup \{Y - Z \in \mathscr{P}(A)\,;\, Y, Z \in \mathscr{B}_k\}.$$

Then $\mathscr{B} = \bigcup_{k \in \omega} \mathscr{B}_k$ is a B.a. of subsets of A which even has no atom. □

Proof of Theorem 4B i). $VIII \rightarrow IV$ is obvious. $III \rightarrow VIII$:
Let A be $VIII$-infinite and let $f: A \rightarrow A$ be onto A but not one-to-one. For $B \subseteq A$ take $g(B) = f^{-1}[B]$. g is one-to-one but not onto $\mathscr{P}(A)$ (not every singleton from A can be a value of g). □

Finally, let us deal with our independence results concerning $VIII$-finiteness. Clearly only models with $III \nleftrightarrow IV$ are interesting in this situation. The properties of $\mathscr{M}_2$ and $\mathscr{N}_2$ follow from

Lemma. *Let B be a set with the following properties:*
i) $B = \bigcup_{n\in\omega} P_n$, *$P_n$'s disjoint sets,* $|P_n| = 2$;
ii) $\{[n, P_n];\ n \in \omega\}$ *is a set;*
iii) B *is IV-finite.*
Then B is $VIII$-finite.

Proof. Indirectly—let there exists a mapping $f\colon B \to B$ which is onto B but not one-to-one. Put

$$X'_n = \Big\{a \in P_n\,;\ f(a) \notin f\Big[\bigcup_{k<n} P_k\Big]\Big\}.$$

Clearly $X'_n \subseteq P_n$ and $\operatorname{rng} f = f[\bigcup_{n\in\omega} X'_n]$. If

$$(\forall n_0 \in \omega)(\exists n > n_0)(|X'_n| = 2\ \&\ |f[X'_n]| = 1)$$

then f defines an infinite one-to-one sequence in B, which means IV-infiniteness of B. Thus let

$$(\exists n_0 \in \omega)(\forall n \geqslant n_0)(|X'_n| = 2 \Rightarrow |f[X'_n]| = 2).$$

If for n ($\leqslant \mathcal{N}_0$)

$$|X'_n| = 2\ \&\ |f[X'_n]| = 1$$

holds then take an arbitrary a $\in X'_n$ and put $X_n = \{a\}$ (we choose only I-finitely many times). For the other n's put $X_n = X'_n$. Now $f \restriction \bigcup_{n\in\omega} X_n$ is a bijection between B and $\bigcup_{n\in\omega} X_n \subset B$, which implies IV-infiniteness of B. □

Hence $\mathcal{M}_2, \mathcal{N}_2 \models (III \nleftrightarrow VIII)$.

While we do not need (and do not use) the preceding lemma for the proof of Theorem 4 B ii), the following results are those which in fact prove this statement. The next lemma gives information about Definition VIII in $\mathcal{M}_1$, $\mathcal{M}^+$ and therefore in $\mathcal{M}_1^+$, $\mathcal{N}_1^+$, too.

Lemma. i) ([1])$\mathcal{M}_1 \models \mathcal{P}(A) \in \mathcal{J}_{VIII} - \mathcal{J}_{III}$.
ii) $\mathcal{M}^+ \models \mathcal{P}(A) \in \mathcal{J}_{IV} - \mathcal{J}_{VIII}$.
(A is the set of all atoms in the model.)

Proof. i) It is proved in [1] that in $\mathcal{M}_1$ the following holds: every $f\colon \mathcal{P}(A) \to \mathcal{P}(A)$ which is onto $\mathcal{P}(A)$ is also one-to-one. This implies $\mathcal{P}(A) \in \mathcal{J}_{VIII} - \mathcal{J}_{III}$.

ii) IV-finiteness of $\mathcal{P}(A)$ is known; it suffices to show its $VIII$-infiniteness. But the following holds for $\mathcal{M}^+$ (see for instance [6], [7]): $B \subseteq A$ is an element of $\mathcal{M}^+$

iff it is the union of finitely many intervals and isolated points of A. Thus for every $B \subseteq A$, $B \in \mathscr{M}^+$, B with isolated points there exists its unique minimal isolated point p_B. Put $f(B) = B - \{p_B\}$ for B with isolated points, $f(B) = B$ for the other $B \subseteq A$. Clearly such $f\colon \mathscr{P}(A) \to \mathscr{P}(A)$ is onto $\mathscr{P}(A)$ but not one-to-one, which means that $\mathscr{P}(A)$ is $VIII$-infinite. □

Corollary.

i) *The following statements hold in* $\mathscr{M}_1^+$*:*

$$\begin{aligned} \mathscr{P}(A') &\in \mathscr{I}_{IV} - \mathscr{I}_{VIII}; \\ \mathscr{P}(A_1) &\in \mathscr{I}_{VIII} - \mathscr{I}_{III}; \\ A' &\in \mathscr{I}_{III} - \mathscr{I}_{VIII''}; \\ A_1 &\in \mathscr{I}_{VIII''} - \mathscr{I}_{I}. \end{aligned}$$

ii) *The following formulas hold in* $\mathscr{N}_1^+$*:*

$$\begin{aligned} &\exists X \in \mathscr{I}_{IV} - \mathscr{I}_{VIII}; \\ &\exists Y \in \mathscr{I}_{VIII} - \mathscr{I}_{III}; \\ &\exists Z \in \mathscr{I}_{III} - \mathscr{I}_{VIII''}; \\ &\exists W \in \mathscr{I}_{VIII''} - \mathscr{I}_{I}. \end{aligned}$$

Proof. i) An immediate consequence of the definition of $\mathscr{M}_1^+$ and Lemma.

ii) By Corollary i) $\mathscr{M}_1^+$ is a model of ZFU satisfying the desired formulas. If we choose α in the construction of $\mathscr{N}_1^+$ large enough also with respect to them (this is possible—in fact a very small or even no change is necessary), we guarantee that $\mathscr{N}_1^+$ satisfies all of them, too. □

Proof of Theorem 4B ii) is now finished:

By the Corollary, $\mathscr{M}_1^+$ and $\mathscr{N}_1^+$ are models of

$$\Phi\varepsilon_9 \,\&\, (IV \nleftrightarrow VIII \nleftrightarrow III \nleftrightarrow VIII'' \nleftrightarrow I).$$

□

Problem. 4.1) To give complete information about the definitions A, VIII (including whether $\mathscr{I}_A$, $\mathscr{I}_{VIII}$ are ideals).

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