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CONTRACTIONS SIMILAR TO ISOMETRIES

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In the present paper we investigate properties of contractions on a Hilbert space which are similar to isometries. A contraction is considered to be a compression of an isometry V to a V^*-invariant subspace $\mathcal{H}$. A criterion for similarity of this compression to an isometry is given in terms of the space $\mathcal{H}$. As a consequence, we obtain some known criteria for a contraction to be similar to an isometry as well as some new results.

PRELIMINARY

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ a closed subspace of $\mathcal{H}$. As usual, we denote by $\mathcal{A}^\perp$ the orthocomplement of $\mathcal{A}$ in $\mathcal{H}$ and by $P(\mathcal{A})$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{A}$. The algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$.

An operator $T \in B(\mathcal{H})$ is said to be similar to an isometry if there exists a boundedly invertible operator $C \in B(\mathcal{H})$ and an isometry $Z \in B(\mathcal{H})$ such that

$$T = C^{-1}ZC.$$

It is easy to see that the fact that the isometry Z acts on the same space $\mathcal{H}$ as T is not important.

Indeed, if $T = D^{-1}Z'D$ with $D\colon \mathcal{H} \to \mathcal{H}'$ invertible, Z' an isometry on $\mathcal{H}'$, then the operator $\Phi = (D^*D)^{1/2}D^{-1} \in B(\mathcal{H}')$ is a unitary operator, and consequently, T is similar to the isometry $\Phi Z'\Phi^{-1} \in B(\mathcal{H})$.

Let $T \in B(\mathcal{H})$ be a contraction (i.e. $|T| \leqslant 1$). Since the sequence of operators $\{T^{*n}T^n\}$ is nonincreasing, the limit $\lim\limits_{n\to\infty} T^{*n}T^n$ in the strong operator topology exists and we can define a nonnegative operator

$$A(T) = (s - \lim_{n\to\infty} T^{*n}T^n)^{1/2}.$$

Since

$$|A(T)x| = \lim_{n\to\infty} |T^n x|,$$

we have

$$|A(T)Tx| = |A(T)x|.$$

It follows that there exists an isometry $W(T)$ on $\overline{\operatorname{Ran} A(T)}$ such that

$$A(T)T = W(T)A(T).$$

The just introduced concept allows us to give a simple generally known characterization of a contraction which is similar to an isometry.

Lemma 1. *A contraction $T \in B(\mathcal{H})$ is similar to an isometry if and only if $A(T)$ is invertible.*

Proof. If T is similar to an isometry, i.e. if $CTC^{-1} = Z$ is an isometry for some invertible operator C we have

$$|T^n x| = |C^{-1}Z^n Cx| \geqslant |C|^{-1} \cdot |Z^n Cx| = |C|^{-1} \cdot |Cx| \geqslant |C|^{-1} \cdot |C^{-1}|^{-1} \cdot |x|.$$

Hence

$$|A(T)x| \geqslant |C|^{-1} \cdot |C^{-1}|^{-1} \cdot |x|.$$

Since the operator $A(T)$ is nonnegative and bounded from below, we deduce that $A(T)$ is invertible.

On the other hand, if $A(T)$ is invertible then the isometry $W(T)$ acts on the whole of $\mathcal{H}$ and T is similar to $W(T)$. $\square$

Contractions similar to isometries

Let us consider a contraction $T \in B(\mathcal{H})$ which is similar to an isometry $Z \in B(\mathcal{H})$; $Z = CTC^{-1}$. Let $\mathcal{H} = \mathcal{R}_Z \oplus \mathcal{S}_Z$ be the Wold decomposition of the isometry Z, i.e. both spaces $\mathcal{R}_Z, \mathcal{S}_Z$ are Z-reducing, $Z|\mathcal{R}_Z$ is a unitary operator, $Z|\mathcal{S}_Z$ is a unilateral shift. The spaces $\mathcal{R}_Z, \mathcal{S}_Z$ can be expressed as $\mathcal{R}_Z = \bigcap_{n=1}^{\infty} Z^n\mathcal{H}, \mathcal{S}_Z = \{x; \lim_{n\to\infty} Z^{*n}x = 0\}$.

Since the isometry Z is the direct sum of $Z|\mathcal{R}_Z$ and $Z|\mathcal{S}_Z$, we will be interested in the structure of the two parts $Z|\mathcal{R}_Z$, $Z|\mathcal{S}_Z$.

A unilateral shift is up to unitary equivalence determined by its multiplicity which is equal to the dimension of the kernel of its adjoint. The multiplicity of the unilateral shift $Z|\mathcal{S}_Z$ is equal to

$$\dim\ker(Z|\mathcal{S}_Z)^* = \dim\ker Z^*|\mathcal{S}_Z = \dim\ker Z^* = \dim\ker T^*.$$

That is why we will be interested above all in the structure of the operator $Z|\mathcal{R}_Z$. Using the fact that C is invertible we can express the operator $Z|\mathcal{R}_Z$ as

$$Z|\mathcal{R}_Z = CTC^{-1}|\bigcap_{n=1}^{\infty} Z^n\mathcal{H} = CTC^{-1}|\bigcap_{n=1}^{\infty} Z^nC\mathcal{H} = CTC^{-1}|C\bigcap_{n=1}^{\infty} T^n\mathcal{H} =$$

$$= (C|\bigcap_{n=1}^{\infty} T^n\mathcal{H})(T|\bigcap_{n=1}^{\infty} T^n\mathcal{H})(C|\bigcap_{n=1}^{\infty} T^n\mathcal{H})^{-1}.$$

As a consequence we obtain

Lemma 2. *Let a contraction T be similar to an isometry Z. Then $Z|\mathcal{R}_Z$ is similar to $T|\bigcap\limits_{n=1}^{\infty} T^n\mathcal{H}$.*

We will use the notation

$$\mathcal{A} = \bigcap_{n=1}^{\infty} T^n\mathcal{H},$$

$$\mathcal{B} = \{x;\ \lim_{n\to\infty} T^{*n}x = 0\}.$$

Lemma 3. *Let T be a contraction similar to an isometry $Z = CTC^{-1}$, then*

$$\mathcal{H} = \mathcal{A} \oplus \mathcal{B}.$$

Proof. Take an $a \in \mathcal{A}$. Let c_n be such that $a = T^nc_n (n \in \mathbb{N})$. For $b \in \mathcal{B}$ we have

$$|(a,b)| = |(T^nc_n, b)| = |(c_n, T^{*n}b)| \leqslant |c_n| \cdot |T^{*n}b| \leqslant |C^{-1}| \cdot |Cc_n| \cdot |T^{*n}b| =$$
$$= |C^{-1}| \cdot |Z^nCc_n| \cdot |T^{*n}b| \leqslant |C^{-1}| \cdot |C| \cdot |C^{-1}Z^nCc_n| \cdot |T^{*n}b| =$$
$$= |C^{-1}| \cdot |C| \cdot |a| \cdot |T^{*n}b|.$$

Since $T^{*n} \to 0$ as n tends to infinity we obtain $(a,b) = 0$ and, consequently, the spaces $\mathcal{A}$ and $\mathcal{B}$ are orthogonal.

On the other hand, the orthocomplement $\mathcal{A}^{\perp}$ can be expressed as

$$\mathcal{A}^{\perp} = \bigvee_{n=1}^{\infty} \ker T^{*n} \subset \mathcal{B},$$

thus

$$\mathcal{A} \oplus B = \mathcal{H}.$$

□

In order to obtain a criterion for a contraction to be similar to an isometry we will apply some facts from the dilation theory (cf. [1]).

Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an isometry $V \in \mathcal{B}(\mathcal{K})$ such that

$$T^n = P(\mathcal{H})V^n\big|\mathcal{H} \qquad (n \geqslant 0),$$

$$\mathcal{K} = \bigvee_{n=0}^{\infty} V^n\mathcal{H}.$$

The isometry V is uniquely determined up to unitary equivalence and is called the minimal isometric dilation of T. The above conditions imply

$$V^*\mathcal{H} \subset \mathcal{H},$$

or equivalently,

$$TP(\mathcal{H}) = P(\mathcal{H})V.$$

Our aim now is to obtain some relation between $T\big|\mathcal{A}$ and V. Let $\mathcal{K} = \mathcal{R} \oplus \mathcal{S}$ be the Wold decomposition of the minimal isometric dilation V of T.

Lemma 4. *Let T be a contraction similar to an isometry. Then $T\big|\bigcap_{n=1}^{\infty} T^n\mathcal{H}$ is similar to the unitary operator $V\big|\mathcal{R}$.*

Proof. Let T be a contraction similar to an isometry $Z = CTC^{-1}$. Since $TP(\mathcal{H}) = P(\mathcal{H})V$ we have $T^* = V^*\big|\mathcal{H}$ and

$$T^*P(\mathcal{A}) = V^*P(\mathcal{A}).$$

According to Lemma 3 we have

$$P(\mathcal{B})T^*P(\mathcal{A}) = V^*P(\mathcal{A}) - P(\mathcal{A})T^*P(\mathcal{A}).$$

Applying $P(\mathcal{B}^{\perp})$ we obtain

$$P(B^{\perp})(V^* P(\mathcal{A}) - (T|\mathcal{A})^* P(\mathcal{A})) = 0.$$

Since $\mathcal{S} = \{x \in \mathcal{K}; V^{*n}x \to 0\}$ we have $\mathcal{B} = \mathcal{S} \bigcap \mathcal{H}$ and $\mathcal{R} \subset \mathcal{B}^{\perp}$. In particular,

$$P(\mathcal{R})(V^* P(\mathcal{A}) - (T|\mathcal{A})^* P(\mathcal{A})) = 0.$$

Using the fact that the space $\mathcal{R}$ is V-reducing we obtain

$$(V^*|\mathcal{R})(P(\mathcal{R})|\mathcal{A}) = (P(\mathcal{R})|\mathcal{A})(T|\mathcal{A})^*,$$

or equivalently,

$$(P(\mathcal{R})|\mathcal{A})^*(V|\mathcal{R}) = (T|\mathcal{A})(P(\mathcal{R})|\mathcal{A})^*.$$

It remains to prove that the operator $Q = P(\mathcal{R})|\mathcal{A}$ is invertible.

Since V^n is an isometry the projection onto the space $V^n\mathcal{K}$ is equal to V^nV^{*n}. Since $\mathcal{R} = \bigcap_{n=1}^{\infty} V^n\mathcal{K}$ we have, for $h \in \mathcal{H}$,

$$P(\mathcal{R})h = \lim_{n\to\infty} V^nV^{*n}h = \lim_{n\to\infty} V^nT^{*n}h$$

and

$$|Qa|^2 = \lim_{n\to\infty} |V^nT^{*n}a| = \lim_{n\to\infty} |T^{*n}a|^2$$

for $a \in \mathcal{A}$. Further, let c_n be such that $a = T^nc_n$ $(n \in \mathbf{N})$. Using the fact that $|(T^{*n}T^n)^{1/2}x|^2 = |T^nx|^2$ and Lemma 1 we have

$$\begin{aligned}
&|T^{*n}a| = |T^{*n}T^nc_n| = |[(T^{*n}T^n)^{1/2}]^2c_n| = |T^n(T^{*n}T^n)^{1/2}c_n| \geqslant \\
&\geqslant |C|^{-1} \cdot |C^{-1}|^{-1} \cdot |(T^{*n}T^n)^{1/2}c_n| = \\
&= |C|^{-1} \cdot |C^{-1}|^{-1} \cdot |T^nc_n| = |C|^{-1} \cdot |C^{-1}|^{-1} \cdot |a|.
\end{aligned}$$

Thus

$$|Qa| \geqslant |C|^{-1} \cdot |C^{-1}|^{-1} \cdot |a|,$$

in other words, Q is bounded from below. It suffices to prove that

$$\mathcal{R} \ominus \operatorname{Ran} Q = \ker Q^* = \ker P(\mathcal{A})|\mathcal{R} = \{0\}.$$

Let us consider $r \in R$ such that $P(\mathcal{A})r = 0$. Since $\mathcal{R} \subset \mathcal{B}^{\perp}$ we have also

$$P(\mathcal{H})r = P(\mathcal{A})r + P(\mathcal{B})r = 0,$$

or equivalently, $r \in \mathcal{H}^\perp$.

Using the fact that $V|\mathcal{R}$ is a unitary operator and $V\mathcal{H}^\perp \subset \mathcal{H}^\perp$ we obtain, for $n \in \mathbf{N}$,

$$0 = P(\mathcal{H})r = P(\mathcal{H})V^n V^{*n} r = P(\mathcal{H})V^n(P(\mathcal{H}) + P(\mathcal{H}^\perp))V^{*n} r = \\ = P(\mathcal{H})V^n P(\mathcal{H})V^{*n} r = T^n P(\mathcal{H})V^{*n} r.$$

Since T^n is injective we have $P(\mathcal{H})V^{*n}r = 0$ so that

$$|r| = |V^{*n} r| = |P(\mathcal{H}^\perp)V^{*n} r| = |(V|\mathcal{H}^\perp)^{*n} r|.$$

The fact that $V|\mathcal{H}^\perp$ is a shift operator (i.e. $(V|\mathcal{H}^\perp)^{*n} \to 0$) completes the proof.

□

Putting together Lemma 2 and Lemma 4 we conclude that operators $Z|\mathcal{R}_Z$ and $V|\mathcal{R}$ are similar. Since both operators are unitary they are even unitarily equivalent. In particular, in view of Lemma 4 we obtain that two isometries are similar if and only if they are unitarily equivalent. Further, since $\dim\ker T^* \leqslant \dim\ker V^*$ there exists a V-invariant subspace $\mathcal{M}$ of $\mathcal{K}$ such that $V|\mathcal{M}$ is unitarily equivalent to Z and hence $V|\mathcal{M}$ is similar to T. Let $\mathcal{N}$ be a V-invariant subspace of $\mathcal{K}$. It follows from $P(\mathcal{H})VP(\mathcal{H}) = P(\mathcal{H})V$ that

$$T(P(\mathcal{H})|\mathcal{N}) = (P(\mathcal{H})|\mathcal{N})(V|\mathcal{N}).$$

We know that there exists a V-invariant subspace $\mathcal{M}$ of $\mathcal{K}$ such that T is similar to $V|\mathcal{M}$. We will show now that there exists a V-invariant subspace $\mathcal{N}$ of $\mathcal{K}$ such that $P(\mathcal{H})|\mathcal{N}$ is invertible. First, we will prove the following simple lemma.

Lemma 5. *Let $\mathcal{E}, \mathcal{F}$ be closed subspaces of a Hilbert space $\mathcal{K}$. Then $\mathcal{E} \dotplus \mathcal{F} = \mathcal{K}$ (i.e. $\mathcal{K}$ is the direct sum of $\mathcal{E}$ and $\mathcal{F}$) if and only if $P(\mathcal{E}^\perp)|\mathcal{F}$ is an invertible operator from $\mathcal{F}$ onto $\mathcal{E}^\perp$.*

Proof. It is sufficient to observe that

$$\mathcal{E} \cap \mathcal{F} = \{0\} \Leftrightarrow P(\mathcal{E}^\perp)|\mathcal{F} \text{ is injective,}$$
$$\mathcal{E} + \mathcal{F} = \mathcal{K} \Leftrightarrow P(\mathcal{E}^\perp)|\mathcal{F} \text{ is surjective.}$$

□

Proposition 6. *Let T be a contraction, let V be its minimal isometric dilation. Then T is similar to an isometry if and only if there exists a V-invariant subspace $\mathcal{N}$ of $\mathcal{K}$ such that*

$$\mathcal{N} \dot{+} \mathcal{H}^{\perp} = \mathcal{K},$$

or equivalently, the operator $P(\mathcal{H})|\mathcal{N}$ is invertible.

Proof. If there exists a subspace $\mathcal{N}$ such that $V\mathcal{N} \subset \mathcal{N}$ and $P(\mathcal{H})|\mathcal{N}$ is invertible then T is similar to an isometry as a consequence of the relation

$$T(P(\mathcal{H})|\mathcal{N}) = (P(\mathcal{H})|\mathcal{N})(V|\mathcal{N}).$$

Conversely, let a contraction $T \in B(\mathcal{H})$ be similar to an isometry $U \in B(\mathcal{H}), T = C^{-1}UC$. Suppose that $\mathcal{L}$ is a subspace of $\mathcal{K}$ such that $V\mathcal{L} \subset \mathcal{L}$ and $\mathcal{H}^{\perp} \dot{+} \mathcal{L} = \mathcal{K}$. According to Lemma 5 the operator $X = P(\mathcal{H})|\mathcal{L} \colon \mathcal{L} \to \mathcal{H}$ is invertible and so

$$X^{-1}T = VX^{-1}.$$

Using the fact that

$$C^{-1}U = TC^{-1}$$

we obtain

$$(X^{-1}C^{-1})U = V(X^{-1}C^{-1}),$$

so the operator $X^{-1}C^{-1}$ intertwines U and the minimal isometric dilation V of T. Let us observe that $P(\mathcal{H})X^{-1}C^{-1} = C^{-1}$ and $\mathcal{L} = \operatorname{Ran} X^{-1}$.

On the other hand, by virtue of the Lifting Theorem the relation $C^{-1}U = TC^{-1}$ implies that there exists an operator $Y : \mathcal{H} \to \mathcal{K}$ such that

$$YU = VY,$$
$$P(\mathcal{H})Y = C^{-1}.$$

That is why we set

$$\mathcal{N} = \operatorname{Ran} YC.$$

First, $VYC = YUC = YCT$, hence $V\mathcal{N} \subset \mathcal{N}$. Since

$$YCP(\mathcal{H})YCP(\mathcal{H}) = YCC^{-1}CP(\mathcal{H}) = YCP(\mathcal{H})$$

the operator $YCP(\mathcal{H})$ is an idempotent and $\mathcal{N} = \operatorname{Ran} YCP(\mathcal{H})$. It remains to show that

$$\ker YCP(\mathcal{H}) = \mathcal{H}^{\perp}.$$

Indeed,

$$\mathcal{H}^{\perp} \subset \ker YCP(\mathcal{H}) \subset \ker P(\mathcal{H})YCP(\mathcal{H}) = \ker P(\mathcal{H}) = \mathcal{H}^{\perp}.$$

The proof is complete. □

The just proved condition for a contraction to be similar to an isometry has the following equivalent form.

Lemma 7. *The following conditions are equivalent.*
1^0 *There exists a subspace* $\mathcal{N}$ *of* $\mathcal{K}$ *such that*

$$V\mathcal{N} \subset \mathcal{N},$$
$$\mathcal{H}^\perp \dotplus \mathcal{N} = \mathcal{K}.$$

2^0 *There exists an operator* $L\colon \mathcal{S} \to \mathcal{H}^\perp$ *such that*

$$LP(\mathcal{S})|\mathcal{H}^\perp = I_{\mathcal{H}^\perp},$$
$$L(V|\mathcal{S}) = (V|\mathcal{H}^\perp)L.$$

Moreover, condition 1^0 *implies* $\mathcal{R} \subset \mathcal{N}$.

Proof. Assume that 1^0 holds. We set

$$X = P(\mathcal{H})|\mathcal{N},$$
$$\widetilde{L} = (1 - X^{-1}P(\mathcal{H}))$$

and

$$L = \widetilde{L}|\mathcal{S}\colon \mathcal{S} \to \mathcal{H}^\perp.$$

The operator $\widetilde{L}$ is idempotent, equals the identity on $\mathcal{H}^\perp$ and vanishes just on $\mathcal{N}$, hence $\operatorname{Ran}\widetilde{L} = \mathcal{H}^\perp$. Take a $k \in \mathcal{K}$ and its decomposition with respect to $\mathcal{K} = \mathcal{H}^\perp \dotplus \mathcal{N}$,

$$k = h^\perp + n, \quad (h^\perp \in \mathcal{H}^\perp, n \in \mathcal{N}).$$

It follows that

$$\widetilde{L}Vk = \widetilde{L}Vh^\perp + \widetilde{L}Vn = Vh^\perp = V\widetilde{L}h^\perp + V\widetilde{L}n = V\widetilde{L}k.$$

In particular, we have also

$$L(V|\mathcal{S}) = (V|\mathcal{H}^\perp)L.$$

Since $V|\mathcal{H}^\perp$ is a shift operator we obtain

$$\widetilde{L}\mathcal{R} = \widetilde{L}\bigcap_{n=1}^{\infty} V^n\mathcal{K} \subset \bigcap_{n=1}^{\infty} V^n\widetilde{L}\mathcal{K} \subset \bigcap_{n=1}^{\infty} V^n\mathcal{H}^\perp = \{0\}.$$

Thus we obtain $\mathcal{R} \subset \mathcal{N}$. Consequently, for $h^\perp \in \mathcal{H}^\perp$ we have

$$LP(\mathcal{S})h^\perp = \widetilde{L}(P(\mathcal{S}) + P(\mathcal{R}))h^\perp = h^\perp.$$

On the other hand, assume 2^0 and set

$$\mathcal{N} = \ker LP(\mathcal{S}).$$

It is obvious that

$$V\mathcal{N} \subset N.$$

Since $LP(\mathcal{S})LP(\mathcal{S}) = LP(\mathcal{S})$ the operator $LP(\mathcal{S})$ is an idempotent and its range is equal to $\mathcal{H}^\perp$. Thus

$$\mathcal{H}^\perp \dotplus \mathcal{N} = \mathcal{K}.$$

□

Condition 2^0 as a criterion for similarity of a contraction to an isometry was proved by Sz.-Nagy and Foiaş [2]. From this criterion another criterion for similarity of a contraction to an isometry in terms of the characteristic function was derived in [2]. We also derive this criterion from Proposition 6 and Lemma 7 but in a different way.

First, let us recall one of the definitions of the Hardy space $H^2(\mathcal{E})$ (cf. [1]).

Let $\mathcal{E}$ be a separable Hilbert space. We denote by $H^2(\mathcal{E})$ the space of all functions with values in $\mathcal{E}$ which are analytic in the unit disc and their Taylor coefficients are square summable. The norm on $H^2(\mathcal{E})$ is defined as

$$\left|\sum_{k=0}^{\infty} \lambda^k a_k\right|^2 = \sum_{k=0}^{\infty} |a_k|^2 \quad (a_k \in \mathcal{E}).$$

Let $T \in B(\mathcal{H})$ be a contraction, let V be its minimal isometric dilation. As usual, we use the notation

$$D = (1 - T^*T)^{1/2}, \quad \mathcal{D} = \overline{RanD},$$
$$D_* = (1 - TT^*)^{1/2}, \quad \mathcal{D} = \overline{RanD_*},$$

$$\mathcal{L} = \overline{(V - T)\mathcal{H}},$$
$$\mathcal{L}_* = \overline{(I - VT^*)\mathcal{H}}.$$

Let $\varphi \colon \mathcal{L} \to \mathcal{D}$ be an isometry defined on the space $(V - T)\mathcal{H}$ by the formula

$$\varphi(V - T)h = Dh \quad (h \in \mathcal{H}).$$

Clearly, φ is a unitary mapping of $\mathcal{L}$ onto $\mathcal{D}$. Since

$$\mathcal{H}^{\perp} = \mathcal{L} \oplus V\mathcal{L} \oplus V^2\mathcal{L} \oplus \ldots,$$

there exists a unitary operator $\Phi\colon \mathcal{H}^{\perp} \to H^2(\mathcal{D})$ such that

$$\Phi\Big(\sum_0^n V^k \ell_k\Big) = \sum_0^n \varphi(\ell_k)\lambda^k \quad (\ell_k \in \mathcal{L}).$$

Similarly, there exist a unitary mapping $\varphi_* : \mathcal{L}_* \to \mathcal{D}_*$ such that

$$\varphi_*(I - VT^*)h = D_* h \quad (h \in \mathcal{H})$$

and a unitary operator $\Phi_* : \mathcal{S} \to H^2(\mathcal{D}_*)$ such that

$$\Phi_*\Big(\sum_0^n V^k \ell_{*k}\Big) = \sum_0^n \varphi_*(\ell_{*k})\lambda^k.$$

Let us consider the operator $P(\mathcal{S})\big|\mathcal{H}^{\perp}$ which is one of the forms of the characteristic function. The operator $\Theta_+ : H^2(\mathcal{D}) \to H^2(\mathcal{D}_*)$ defined as

$$\Theta_+ = \Phi_*(P(\mathcal{S})\big|\mathcal{H}^{\perp})\Phi^{-1}$$

is another form of the characteristic function. The bounded analytic function $\Theta(\Theta(\lambda)\colon \mathcal{D} \to \mathcal{D}_*)$ which is defined for $|\lambda| < 1$ as

$$\Theta(\lambda) = -T + D_*(\lambda^{-1} - T^*)^{-1} D,$$

is the third form of the characteristic function. The operator Θ_+ can be expressed as

$$(\Theta_+ u)(\lambda) = \Theta(\lambda)u(\lambda) \quad u \in H^2(\mathcal{D}).$$

Let $T_1 \in B(\mathcal{H}_1), T_2 \in B(\mathcal{H}_2)$ be contractions. We use the notation

$$\mathcal{I}(T_1, T_2) = \{X : \mathcal{H}_1 \to \mathcal{H}_2 \text{ a bounded linear operator; } T_2 X = X T_1\}.$$

Proposition 8. *Let T be a contraction, let V be its minimal isometric dilation, let Θ be its characteristic function. Then T is similar to an isometry if and only if there exists a bounded analytic function $\Delta(\lambda)(|\lambda| < 1)$ such that*

$$\Delta(\lambda)\Theta(\lambda) = I_{\mathcal{D}}$$

for all $|\lambda| < 1$.

Proof. Since $\Phi \in \mathcal{I}(V|\mathcal{H}^{\perp}, S), \Phi_* \in \mathcal{I}(V|\mathcal{S}, S_*)$ the operator $P(\mathcal{S})|\mathcal{H}^{\perp}$ has a left inverse operator $L \in \mathcal{I}(V|\mathcal{S}, V|\mathcal{H}^{\perp})$ if and only if Θ_+ has a left inverse operator intertwining the shifts S and S_* on $H^2(\mathcal{D})$ and $H^2(\mathcal{D}_*)$, respectively. If the operator Θ_+ has a left inverse operator $\Delta_+ \in \mathcal{I}(S_*, S)$ then, according to Proposition V.3.2 [1], there exists a bounded analytic function Δ such that

$$(\Delta_+ u)(\lambda) = \Delta(\lambda)u(\lambda) \quad (u \in H^2(\mathcal{D})).$$

It follows that

$$\Delta(\lambda)\Theta(\lambda) = I_{\mathcal{D}} \qquad (|\lambda| < 1).$$

On the other hand, if there exists a bounded analytic function $\Delta(\lambda)$ ($|\lambda| < 1$) such that

$$\Delta(\lambda)\Theta(\lambda) = I_{\mathcal{D}}$$

then the operator Δ_+ defined as

$$(\Delta_+ u)(\lambda) = \Delta(\lambda)u(\lambda) \qquad (u \in H^2(\mathcal{D}))$$

is a left inverse operator to the operator Θ_+ and $\Delta_+ \in \mathcal{I}(S_*, S)$. It suffices now to apply Proposition 6 and Lemma 7. □

Similarity of compressions

Let $V \in B(\mathcal{K})$ be an isometry. The compression $P(\mathcal{H})V|\mathcal{H}$ on a V^*-invariant subspace $\mathcal{H}$ is a contraction. On the other hand, it follows from the existence of the minimal isometric dilation of a contraction that any contraction can be obtained in this way.

It is natural to ask when such a compression is similar to an isometry (i.e. what condition must the space $\mathcal{H}$ satisfy). In Proposition 6 and Lemma 7 two equivalent conditions concerning the space $\mathcal{H}$ are given. We will reformulate condition 1^0 from Lemma 7 in terms of an invertible operator commuting with V^* and V-reducing subspaces of $\mathcal{K}$:

Let us consider condition 1^0, i.e. there exists a closed V-invariant subspace $\mathcal{N}$ of $\mathcal{K}$ such that

$$\mathcal{H}^{\perp} \dot{+} \mathcal{N} = \mathcal{K}.$$

We will construct a Hilbert space

$$\widetilde{\mathcal{K}} = \mathcal{H}^{\perp} \oplus \mathcal{N},$$

an isometry $\widetilde{V}$ on $\widetilde{\mathcal{K}}$ and use the relation between V and $\widetilde{V}$. We obtain an interesting representation of the space $\mathcal{H}$ with the required properties.

Theorem 9. *Let $T \in B(\mathcal{H})$ be a contraction, $V \in B(\mathcal{K})$ its minimal isometric dilation with the Wold decomposition $\mathcal{K} = \mathcal{R} \oplus \mathcal{S}$. Then the contraction T is similar to an isometry if and only if there exists a closed subspace $\mathcal{A}\,(\mathcal{R} \subset \mathcal{A} \subset \mathcal{K})$ which is V-reducing, and an invertible operator $\Theta \in B(\mathcal{K})$ commuting with V^* such that*

$$\mathcal{H} = \Theta\mathcal{A}.$$

P r o o f. Assume that the contraction T is similar to an isometry. According to Proposition 6 there exists a V-invariant subspace $\mathcal{N}$ of $\mathcal{K}$ such that

$$\mathcal{N} \dotplus \mathcal{H}^{\perp} = \mathcal{K}.$$

Both spaces $\mathcal{H}^{\perp}$ and $\mathcal{N}$ are V-invariant. We now define the space

$$\widetilde{\mathcal{K}} = \{x \oplus y; x \in \mathcal{N}, y \in \mathcal{H}^{\perp}\}$$

and the operator $\widetilde{V} : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$,

$$\widetilde{V}(x \oplus y) = Vx \oplus Vy.$$

We set $\widetilde{\mathcal{N}} = \{x \oplus 0; x \in \mathcal{N}\}$. The subspace $\widetilde{\mathcal{N}}$ of $\widetilde{\mathcal{K}}$ is V-reducing and $\widetilde{\mathcal{N}}^{\perp} = \{0 \oplus y; y \in \mathcal{H}^{\perp}\}$. Further, we define an operator $\Delta : \widetilde{\mathcal{K}} \to \mathcal{K}$ as

$$\Delta(x \oplus y) = x + y.$$

If follows from the relation $\mathcal{N} \dotplus \mathcal{H}^{\perp} = \mathcal{K}$ that Δ is invertible. The spaces $\mathcal{H}^{\perp}, \mathcal{N}$ can be expressed as

$$\mathcal{H}^{\perp} = \Delta\widetilde{\mathcal{N}}^{\perp},$$
$$\mathcal{N} = \Delta\widetilde{\mathcal{N}}.$$

Since

$$x \in \mathcal{H} \Leftrightarrow (x, \Delta\widetilde{\mathcal{N}}^{\perp}) = 0 \Leftrightarrow \Delta^* x \in \widetilde{\mathcal{N}} \Leftrightarrow x \in \Delta^{*-1}\widetilde{\mathcal{N}}$$

we have

$$\mathcal{H} = \Delta^{*-1}\widetilde{\mathcal{N}}.$$

Since the operator Δ is invertible and $\Delta\widetilde{V} = V\Delta$ the isometries V and $\widetilde{V}$ are similar. Consequently, the unitary parts $V|\mathcal{R}$, $\widetilde{V}|\widetilde{\mathcal{R}}$ as well as the shift parts $V|\mathcal{S}$, $\widetilde{V}|\widetilde{\mathcal{S}}$ are unitarily equivalent. In other words, there exists a unitary operator $\Phi\colon \mathcal{K} \to \tilde{\mathcal{K}}$ such that

$$\Phi\mathcal{R} = \tilde{\mathcal{R}},$$
$$\Phi\mathcal{S} = \tilde{\mathcal{S}},$$
$$\tilde{V}\Phi = \Phi V.$$

Consequently, we have also

$$\Phi^*\tilde{V} = V\Phi^*.$$

Since $\mathcal{R} = \bigcap_{n=1}^{\infty} V^n\mathcal{K}, \tilde{\mathcal{R}} = \bigcap_{n=1}^{\infty} \tilde{V}^n\tilde{\mathcal{K}}$ the space $\tilde{\mathcal{R}}$ can be expressed as $\tilde{\mathcal{R}} = \Delta^{-1}\Delta\tilde{\mathcal{R}} = \Delta^{-1}\bigcap_{n=1}^{\infty} \Delta\tilde{V}^n\tilde{\mathcal{K}} = \Delta^{-1}\bigcap_{n=1}^{\infty} V^n\mathcal{K} = \Delta^{-1}\mathcal{R}$. If we define

$$\mathcal{A} = \Phi^*\tilde{\mathcal{N}},$$
$$\Theta = \Delta^{*-1}\Phi,$$

we obtain the required properties: $\mathcal{A}$ is a V-reducing subspace of $\mathcal{K}$, $\mathcal{R} = \Phi^*\tilde{\mathcal{R}} = \Phi^*\Delta^{-1}\mathcal{R} \subset \Phi^*\Delta^{-1}\mathcal{N} = \Phi^*\tilde{\mathcal{N}} = \mathcal{A}$ and Θ is an invertible operator on $\mathcal{K}$ commuting with V^*. According to the definition of $\mathcal{A}$ and Θ we obtain

$$\mathcal{H} = \Delta^{*-1}\tilde{\mathcal{N}} = \Delta^{*-1}\Phi\Phi^*\tilde{\mathcal{N}} = \Theta\mathcal{A}.$$

We have proved one part of the theorem.

It is easy to see that $\mathcal{N} = \Theta^{*-1}\mathcal{A}$. Since $\mathcal{H}^{\perp} \dotplus \mathcal{N} = \mathcal{K}$ the operator $P(\mathcal{H})|\mathcal{N}$ is invertible according to Lemma 5. It follows from the intertwining relation

$$(P(\mathcal{H})V|\mathcal{H})(P(\mathcal{H})|\mathcal{N}) = (P(\mathcal{H})|\mathcal{N})(V|\mathcal{N})$$

that T is similar to $V|\Theta^{*-1}\mathcal{A}$.

Suppose that there exists a closed subspace $\mathcal{A}$ of $\mathcal{K}$ which reduces V and an invertible operator Θ on $\mathcal{K}$ which commutes with V^* and such that $\mathcal{H} = \Theta\mathcal{A}$. Let us define an operator $X\colon \Theta^{*-1}\mathcal{A} \to \Theta\mathcal{A}$ as follows:

$$X = P(\Theta\mathcal{A})|\Theta^{*-1}\mathcal{A}.$$

The space $\Theta^{*-1}\mathcal{A}$ is invariant for V since

$$V\Theta^{*-1}\mathcal{A} = \Theta^{*-1}V\mathcal{A} \subset \Theta^{*-1}\mathcal{A}$$

and, consequently,

$$V(\Theta\mathcal{A})^{\perp} \subset (\Theta\mathcal{A})^{\perp}.$$

This inclusion can be rewritten as

$$P(\Theta\mathcal{A})V(I - P(\Theta\mathcal{A})) = 0.$$

Restricting this relation to the space $\Theta^{*-1}\mathcal{A}$ we obtain

$$(P(\Theta\mathcal{A})V\big|\Theta\mathcal{A})(P(\Theta\mathcal{A})\big|\Theta^{*-1}\mathcal{A}) = (P(\Theta\mathcal{A})\big|\Theta^{*-1}\mathcal{A})(V\big|\Theta^{*-1}\mathcal{A}).$$

If we set $X = P(\Theta\mathcal{A})\big|\Theta^{*-1}\mathcal{A}$ then

$$TX = X(V\big|\Theta^{*-1}\mathcal{A}).$$

According to Lemma 5 the operator X is invertible if and only if

$$(\Theta\mathcal{A})^{\perp} \dot{+} \Theta^{*-1}\mathcal{A} = \mathcal{K}.$$

Since $(\Theta\mathcal{A})^{\perp} = \Theta^{*-1}\mathcal{A}^{\perp}$ the above condition is clearly satisfied. So we have proved that T is similar to the isometry $V\big|\Theta^{*-1}\mathcal{A}$. □

The assumption of minimality of the isometric dilation V of T does not play an important role.

Let $V \in B(\mathcal{K})$ be an isometry, let $\mathcal{H}$ be a V^*-invariant subspace of $\mathcal{K}$ such that the compression $T = P(\mathcal{H})V\big|\mathcal{H}$ is similar to an isometry. If we set

$$\mathcal{K}_0 = \bigvee_{n=0}^{\infty} V^n\mathcal{H},$$

then $\mathcal{K}_0$ is the V-reducing subspace of $\mathcal{K}$ and $V\big|\mathcal{K}_0$ is the minimal isometric dilation of T. According to Theorem 9 there exists a closed subspace $\mathcal{A}$ of $\mathcal{K}_0 \subset \mathcal{K}$ which reduces $V\big|\mathcal{K}_0$, and an invertible operator Θ_0 commuting with $(V\big|\mathcal{K}_0)^* = V^*\big|\mathcal{K}_0$ such that

$$\mathcal{H} = \Theta_0\mathcal{A}.$$

If we define an operator $\Theta\colon \mathcal{K} \to \mathcal{K}$ by the formula

$$\Theta k = \Theta_0 P(\mathcal{K}_0)k + P(\mathcal{K}_0^{\perp})k \quad (k \in \mathcal{K})$$

then Θ is an invertible operator commuting with V^* and

$$\mathcal{H} = \Theta\mathcal{A}.$$

So we have proved

Corollary 10. *Let $V \in B(\mathcal{K})$ be an isometry and $\mathcal{H}$ a V^*-invariant subspace of $\mathcal{K}$. Then an operator $T = P(\mathcal{H})V|\mathcal{H}$ is similar to an isometry if and only if there exist a closed subspace $\mathcal{A}$ of $\mathcal{K}$ which reduces V and an invertible operator $\Theta \in B(\mathcal{K})$ which commutes with V^* such that*

$$\mathcal{H} = \Theta\mathcal{A}.$$

In the preceding theorem subspaces reducing an isometry play an important role. These subspaces can be described in the following way. If a subspace $\mathcal{A}$ reduces an isometry $V \in B(\mathcal{K})$ then both operators $V|\mathcal{A}$, $V|\mathcal{A}^{\perp}$ are isometries. If $\mathcal{A} = \mathcal{S}_{\mathcal{A}} \oplus \mathcal{R}_{\mathcal{A}}, \mathcal{A}^{\perp} = \mathcal{S}_{\mathcal{A}^{\perp}} \oplus \mathcal{R}_{\mathcal{A}^{\perp}}$ are the Wold decompositions of these isometries then

$$\mathcal{K} = \mathcal{S} \oplus \mathcal{R} = (\mathcal{S}_{\mathcal{A}} \oplus \mathcal{S}_{\mathcal{A}^{\perp}}) \oplus (\mathcal{R}_{\mathcal{A}} \oplus \mathcal{R}_{\mathcal{A}^{\perp}})$$

is the Wold decomposition of the isometry V. Since $V|\mathcal{S}_{\mathcal{A}}$ is a shift operator the space $\mathcal{S}_{\mathcal{A}}$ can be expressed as

$$\mathcal{S}_{\mathcal{A}} = \mathcal{M} \oplus V\mathcal{M} \oplus V^2\mathcal{M} \oplus \ldots$$

where

$$\mathcal{M} = \ker(V|\mathcal{S}_{\mathcal{A}})^* = \ker V^*|\mathcal{S}_{\mathcal{A}} \subset \ker V^*|\mathcal{S} = V \ominus V\mathcal{S}.$$

Thus any subspace $\mathcal{A}$ which reduces the isometry V has the form

$$\mathcal{A} = (\mathcal{M} \oplus V\mathcal{M} \oplus V^2\mathcal{M} \oplus \ldots) \oplus \mathcal{R}_{\mathcal{A}},$$

where $\mathcal{M}$ is a subspace of the generating subspace $V \ominus V\mathcal{S}$ and $\mathcal{R}_{\mathcal{A}}$ is a subspace of $\mathcal{R}$ which reduces V.

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