Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 2, 373–382

Persistent URL: http://dml.cz/dmlcz/128406

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

ON COMMUTING ISOMETRIES

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(Received January 6, 1992)

INTRODUCTION

Reflexive algebras have been studied intensively by many authors interested in invariant subspace problem. Among the most interesting results in this direction the reflexivity of the algebra generated by a single isometry, by a normal, and by a subnormal operator was proved by J. A. Deddens ([2]), D. Sarason ([9]), and by R. Olin and J. E. Thomson ([5]), respectively.

The reflexivity of algebra generated by two isometries was studied by M. Ptak in [6], [7]. He obtained the positive result for certain class of shifts which were called compatible (for definition see [4]) and which include doubly commuting shifts. The present authors conjecture that algebra generated by any family of commuting isometries is reflexive. The aim of this paper is to present some partial results in that direction.

Let $\mathbf{T} = (T_\alpha)_{\alpha \in A}$ be a family of operators on a Hilbert space $\mathcal{H}$. As usual we denote by $\operatorname{Lat}\mathbf{T}$ the lattice of all subspaces of $\mathcal{H}$ invariant with respect to any T_α ($\alpha \in A$). Further $\operatorname{AlgLat}\mathbf{T}$ denotes the (weakly closed) algebra of operators leaving invariant every subspace from $\operatorname{Lat}\mathbf{T}$.

A weakly closed algebra generated by T_α is called reflexive if it is equal to $\operatorname{AlgLat}\mathbf{T}$. The commutant of family $\mathbf{T}$ is the set of all operators which commute with every T_α ($\alpha \in A$) and is denoted by $\mathbf{T}'$. The commutant $\mathbf{T}''$ of $\mathbf{T}'$ is called the double commutant of $\mathbf{T}$.

The main result of this paper is that $\operatorname{AlgLat}\mathbf{V}$ is contained in the double commutant $\mathbf{V}'$ for any family $\mathbf{V} = (V_\alpha)_{\alpha \in A}$ of commuting isometries. Moreover, it is proved that $\widetilde{T}$ belongs to the double commutant of $(\widetilde{V}_\alpha)_{\alpha \in A}$ where $(\widetilde{V}_\alpha)_{\alpha \in A}$, $V_\alpha \in B(\mathcal{K})$, are unitary extensions of given isometries V_α and $\widetilde{T} \in B(\mathcal{K})$ the corresponding extension of $T \in \operatorname{AlgLat}\mathbf{V}$. This means that $\widetilde{T}$ is a function of $\widetilde{\mathbf{V}}$.

Throughout the paper we shall use the following well-known Wold decomposition: Is V is an isometry on a Hilbert space $\mathcal{H}$ then $\mathcal{H}$ can be decomposed into two parts $M_+(\mathcal{L})\oplus\mathcal{R}_V$ where $V|M_+(\mathcal{L})$ is the unilateral shift on $M_+(\mathcal{L}) = \mathcal{L}\oplus V\mathcal{L}\oplus V^2\mathcal{L}\oplus\ldots$, $\mathcal{L} = \mathcal{H}\ominus V\mathcal{H}$ being the corresponding wandering subspace, and $V|\mathcal{R}_V$ is unitary on the (residue) subspace $\mathcal{R}_V = \bigcap_{n\geqslant 0} V^n\mathcal{H}$.

We begin with a simple lemma.

Lemma 1. *Let $V \in B(\mathcal{H})$ be an isometry, let $x \in \mathcal{H}\ominus V\mathcal{H}$ and let α be a complex number such that $|\alpha| < 1$. Then*

$$(I - \bar{\alpha}V)^{-1}x \perp (\alpha - V)\mathcal{H}.$$

Proof. The equality

$$\alpha - V = \alpha V^*V - V = (\alpha V^* - I)V$$

gives for $|\alpha| < 1$

$$V = (\alpha V^* - I)^{-1}(\alpha - V).$$

Let $h \in \mathcal{H}$. Then

$$\begin{aligned}0 = \langle x, Vh\rangle &= \left\langle x, (\alpha V^* - I)^{-1}(\alpha - V)h\right\rangle = \\ &= \left\langle(\bar{\alpha}V - I)^{-1}x, (\alpha - V)h\right\rangle,\end{aligned}$$

hence $(I - \bar{\alpha}V)^{-1}x \perp (\alpha - V)\mathcal{H}$. □

Lemma 2. *Let $V \in B(\mathcal{H})$ be an isometry, and let $T \in B(\mathcal{H})$ be an operator that leaves invariant every subspace $(\alpha - V)\mathcal{H}$ with $|\alpha| < 1$. Then $(VT - TV)\mathcal{H} \subset \bigcap_{n\geqslant 0} V^n\mathcal{H}$.*

Proof. For $h \in \mathcal{H}$ let us denote $m_0 = Th$. As T leaves $V\mathcal{H}$ invariant, $TVh \in V\mathcal{H}$, which means that $TVh = Vm_1$ for some $m_1 \in \mathcal{H}$. If $x \in \mathcal{H}\ominus V\mathcal{H}$, and $|\alpha| < 1$, then $T(\alpha - V)h \in (\alpha - V)\mathcal{H}$ and by the preceding Lemma

$$\begin{aligned}0 &= \left\langle T(\alpha - V)h, (I - \bar{\alpha}V)^{-1}x\right\rangle = \left\langle \alpha m_0 - Vm_1, \sum_{j=0}^{\infty} \bar{\alpha}^j V^j x\right\rangle = \\ &= \sum_{j=0}^{\infty} \alpha^{j+1}\left\langle m_0, V^j x\right\rangle - \sum_{j=0}^{\infty} \alpha^j \left\langle Vm_1, V^j x\right\rangle = \\ &= \sum_{j=0}^{\infty} \alpha^{j+1}\left\langle m_0 - m_1, V^j x\right\rangle.\end{aligned}$$

Hence $\langle m_0 - m_1, V^j x\rangle = 0$ for any integer $j \geqslant 0$ and $x \in \mathcal{H} \ominus V\mathcal{H}$, which implies that $m_0 - m_1 \in \bigcap_{j\geqslant 0} V^j\mathcal{H}$, and $(VT-TV)h = V(m_0 - m_1) \in \bigcap_{j\geqslant 0} V^j\mathcal{H}$. The inclusion is proved. □

Theorem 1. *Let $\mathbf{V} = (V_\alpha)_{\alpha\in A}$ be a commuting system of isometries on a Hilbert space $\mathcal{H}$. If $T \in \operatorname{AlgLat}\mathbf{V}$ then $TV_\alpha = V_\alpha T$ $(\alpha \in A)$.*

Proof. For any finite subset $F = \{\alpha_1, \ldots, \alpha_k\} \subset A$ let us denote

$$V_F = V_{\alpha_1} \ldots V_{\alpha_k}, \quad \mathcal{R}_F = \bigcap_{n\geqslant 0} V_F^n \mathcal{H}.$$

The subspace $\mathcal{R}_F$ reduces V_F and is invariant for any isometry V_α, $\alpha \in A$.

Further the common residue subspace $\mathcal{R}$ of all V_α is defined by

$$\mathcal{R} = \bigcap_{\substack{F\subset A\\ |F|<\infty}} \mathcal{R}_F.$$

Clearly, $\mathcal{R}$ is invariant for any V_α $(\alpha \in A)$.

Let $\alpha_0 \in A_0$. For any finite subset $F' \subset A$ containing α_0, $F' = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$, $\mathcal{R}_{F'}$ also reduces V_{α_0} as

$$V_{\alpha_0}^*\mathcal{R}_{F'} = V_{\alpha_0}^*(V_{\alpha_1}^* \cdots V_{\alpha_n}^*)(V_{\alpha_1} \cdots V_{\alpha_n})\mathcal{R}_{F'} \subset V_{F'}^*\mathcal{R}_{F'} \subset \mathcal{R}_{F'}.$$

Thus for any finite subset $F \subset A$ we have

$$V_{\alpha_0}^*\mathcal{R} \subset V_{\alpha_0}^*\mathcal{R}_{F\cup\{\alpha_0\}} \subset \mathcal{R}_{F\cup\{\alpha_0\}} \subset \mathcal{R}_F,$$
$$V_{\alpha_0}^*\mathcal{R} \subset \bigcap_{\substack{F\subset A\\ |F|<\infty}} \mathcal{R}_F = \mathcal{R}$$

which proves that $\mathcal{R}$ reduces every V_α $(\alpha \in A)$ and $V_\alpha|\mathcal{R}$ are unitary operators.

By our assumption on T, it follows that $\mathcal{R}$ reduces T, and by an application of the von Neumann double commutant theorem (see e.g. [3] or [1]) its restriction $T|\mathcal{R}$ commutes with every $V_\alpha|\mathcal{R}$, i.e. $TV_\alpha|\mathcal{R} = V_\alpha T|\mathcal{R}$ for any $\alpha \in A$.

Suppose now that $h \in \mathcal{R}^\perp = \mathcal{H} \ominus \mathcal{R}$, and $\alpha \in A$ are given. For any finite subset $F \subset A$ we have by Lemma 2 (for the isometry V_F)

$$(TV_F - V_F T)\mathcal{H} \subset \mathcal{R}_F,$$

and for the isometry $V_\alpha V_F$ we obtain analogously

$$\mathcal{R}_{F\cup\{\alpha\}} = \bigcap_{k\geqslant 0}(V_F V_\alpha)^k\mathcal{H} \subset \bigcap_{k\geqslant 0} V_F^k\mathcal{H} \subset \mathcal{R}_F,$$

which gives the inclusion

$$(V_F V_\alpha T - T V_F V_\alpha)\mathcal{H} \subset \mathcal{R}_F.$$

Then

$$(V_F V_\alpha T - V_F T V_\alpha)h = (V_F V_\alpha T - T V_F V_\alpha)h + (T V_F - V_F T)V_\alpha h \in \mathcal{R}_F$$

which implies that

$$(V_\alpha T - T V_\alpha)h \in V_F^*\mathcal{R}_F = \mathcal{R}_F$$

for any finite subset $F \subset A$, $h \in \mathcal{R}^\perp$, $\alpha \in A$. Hence

$$(V_\alpha T - T V_\alpha)h \in \bigcap_{\substack{F\subset A \\ |F|<\infty}} \mathcal{R}_F = \mathcal{R}.$$

On the other hand, as $\mathcal{R}$ reduces any V_α and T leaves invariant every subspace from $\bigcap_{\alpha\in A}$ Lat V_α, we have $T\mathcal{R}^\perp \subset \mathcal{R}^\perp$ and $V_\alpha\mathcal{R}^\perp \subset \mathcal{R}^\perp$ which implies that $(V_\alpha T - T V_\alpha)h \in \mathcal{R}^\perp$, hence $V_\alpha T h = T V_\alpha h$ for any $h \in \mathcal{R}^\perp$, $\alpha \in A$. This finishes the proof. □

Lemma 3. *Let $V, T_\alpha \in B(\mathcal{H})$ $(\alpha \in A)$ be commuting operators, V an isometry. Then there exist commuting extensions $\widetilde{V}$, $\widetilde{T}_\alpha$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\widetilde{V}|\mathcal{H} = V$, $\widetilde{T}_\alpha|\mathcal{H} = T_\alpha$, $\widetilde{V}$ is unitary, $\|\widetilde{T}_i\| = \|T_i\|$, and $\widetilde{T}_i$ is an isometry (unitary, resp.) if T_i is.*

Proof. Taking $\widetilde{V}$ the minimal unitary extension of V on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ (where the condition of minimality gives $\mathcal{K} = \bigvee_{k\geqslant 0} \widetilde{V}^{*k}(\mathcal{H})$) we define

$$\widetilde{T}_\alpha\left(\sum_{k=0}^{m} \widetilde{V}^{*k}h_k\right) = \sum_{k=0}^{m} \widetilde{V}^{*k}T_\alpha h_k,$$

for $\alpha \in A$, $m \geqslant 0$, $h_0, \ldots, h_m \in \mathcal{H}$. The definition is correct as

$$\begin{aligned} \left\|\sum_{k=0}^{m} \widetilde{V}^{*k}T_\alpha h_k\right\| &= \left\|\sum_{k=0}^{m} \widetilde{V}^{m-k}T_\alpha h_k\right\| = \left\|\sum_{k=0}^{m} V^{m-k}T_\alpha h_k\right\| = \\ &= \left\|T_\alpha \sum_{k=0}^{m} V^{m-k}h_k\right\| \leqslant \|T_\alpha\| \left\|\sum_{k=0}^{m} \widetilde{V}^{m-k}h_k\right\| \\ &\leqslant \|T_\alpha\| \left\|\sum_{k=0}^{m} \widetilde{V}^{*k}h_k\right\|. \end{aligned} \tag{1}$$

Clearly, $\widetilde{T}_\alpha$ can be extended to $\mathcal{K}$ in such a way that

$$||\widetilde{T}_\alpha|| = ||T_\alpha||, \quad \widetilde{T}_\alpha \widetilde{V} = \widetilde{V}\widetilde{T}_\alpha, \quad \widetilde{T}_\alpha \widetilde{T}_\beta = \widetilde{T}_\beta \widetilde{T}_\alpha \quad (\alpha, \beta \in A).$$

If T_α is an isometry then the equality in (1) holds and $\widetilde{T}_\alpha$ is also an isometry. Analogously, if T_α is unitary then $\widetilde{T}_\alpha$ is an isometry with range dense in $\mathcal{K}$ as $\widetilde{T}_\alpha \mathcal{K} \supset \widetilde{T}_\alpha \widetilde{V}^{*k}\mathcal{H} = \widetilde{V}^{*k} T_\alpha \mathcal{H} = \widetilde{V}^{*k}\mathcal{H}$ for any nonnegative integer k. Therefore $\widetilde{T}_\alpha \mathcal{K} = \mathcal{K}$ and $\widetilde{T}_\alpha$ is unitary. □

Corollary. *Let V_α, $T \in B(\mathcal{H})$ be a commuting system of operators on a Hilbert space $\mathcal{H}$, V_α isometries ($\alpha \in A$). Then there exist commuting extensions $\widetilde{V}_\alpha$, $\widetilde{T} \in B(\mathcal{K})$ on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\widetilde{V}_\alpha|\mathcal{H} = V_\alpha$, $\widetilde{T}|\mathcal{H} = T$, and $\widetilde{V}_\alpha$ are unitary for any $\alpha \in A$.*

Proof. Take a good ordering $\{\alpha_1, \alpha_2, \ldots\}$ of A. Using the previous Lemma we construct the space $\mathcal{K}$ and operators $\widetilde{V}_\alpha$, $\widetilde{T} \in B(\mathcal{K})$ by the transfinite induction. □

Note that the Hilbert space $\mathcal{K}$ constructed above depends on isometries V_α only, not on T.

Let $\mathbf{V} = (V_\alpha)_{\alpha \in A}$ be a commuting system of isometries on a Hilbert space $\mathcal{H}$, let $T \in \operatorname{AlgLat} \mathbf{V}$ and let $\widetilde{\mathbf{V}} = (\widetilde{V}_\alpha)_{\alpha \in A}$, $\widetilde{T}$ be the extensions to the Hilbert space $\mathcal{K} \supset \mathcal{H}$ constructed above. Let $E(\cdot)$ be the spectral measure of the commuting system of unitary operators $\widetilde{V}_\alpha$ ($\alpha \in A$) (E is the projection-valued function on the Borel subsets of $\mathbf{T}^A$). For $x \in \mathcal{H}$ let us denote by $\mathcal{Z}_+(x)$ ($\mathcal{Z}(x)$) the smallest subspace containing x which is invariant (reducing) with respect to all $\widetilde{V}_\alpha$ ($\alpha \in A$).

Clearly, $\mathcal{Z}_+(x) \subset \mathcal{Z}(x)$, and $\mathcal{Z}_+(x)$ is the closure in $\mathcal{H}$ of all $p(\mathbf{V})x$, $p \in \mathcal{P}$, where $\mathcal{P}$ is set of all polynomials with $|A|$ commuting variables.

As $T \in \operatorname{AlgLat} \mathbf{V}$, $T\mathcal{Z}_+(x) \subset \mathcal{Z}_+(x)$. The extensions $\widetilde{V}_\alpha$ are unitary on $\mathcal{K}$, hence $\widetilde{T}$ commutes with all $\widetilde{V}_\alpha, \widetilde{V}_\alpha^*$ ($\alpha \in A$). It follows that

$$\widetilde{T}\mathcal{Z}(x) \subset \mathcal{Z}(Tx) \subset \mathcal{Z}(x).$$

Further let us denote $\mu_x = ||E(\cdot)x||^2$ the positive scalar measure (spectral measure) corresponding to $x \in \mathcal{H}$.

Lemma 4. *If x, $y \in \mathcal{H}$ then there exists a complex number λ such that the measures $\mu_x \vee \mu_y$ and $\mu_{x+\lambda y}$ are equivalent (i.e. absolutely continuous with respect to each other).*

Proof. Let us denote $\mu = \mu_x \vee \mu_y$. As

$$\mu_{x+\lambda y}(B) = \|E(B)(x+\lambda y)\|^2 = \|E(B)x + \lambda E(B)y\|^2$$

for any complex λ and a Borel subset $B \subset \mathbf{T}^A$, $\mu_{x+\lambda y} \prec \mu$. Hence there exists a measurable function (the spectral density) $f_\lambda \in L^1(\mu)$ such that $d\mu_{x+\lambda y} = f_\lambda \, d\mu$.

If $C_\lambda = \{z \in \mathbf{T}^A : f_\lambda(z) = 0\}$ denotes the set of zeros of f_λ then

$$\mu_{x+\lambda y}(C_\lambda) = \int_{C_\lambda} f_\lambda \, d\mu = 0.$$

To obtain the equivalence $\mu \sim \mu_{x+\lambda y}$ for some $\lambda \in \mathbf{C}$, it is sufficient to prove that $\mu(C_\lambda) = 0$. If C_λ, C_κ are the corresponding zero sets for two different complex numbers $\lambda \neq \kappa$, for $C = C_\lambda \cap C_\kappa$ it holds

$$0 \leqslant \mu_{x+\lambda y}(C) \leqslant \mu_{x+\lambda y}(C_\lambda) = 0,$$

and analogously $\mu_{x+\kappa y}(C) = 0$, which implies that

$$E(C)(x+\lambda y) = E(C)(x+\kappa y) = 0,$$
$$(\lambda - \kappa)E(C)y = 0,$$

and as $\lambda - \kappa \neq 0$ we get $E(C)y = E(C)x = 0$, i.e. $\mu_x(C) = \mu_y(C) = 0$, hence $\mu(C) = 0$. Summing up these equalities we obtain that

$$\mu(C_\lambda \cup C_\kappa) = \mu(C_\lambda) + \mu(C_\kappa).$$

The last equality implies that there could be only countable number of those $\lambda \in \mathbf{C}$ for which $\mu(C_\lambda) \neq 0$, which proves the existence of the desired complex λ. □

For any $x \in \mathcal{H}$ the restriction $\widetilde{T}|\mathcal{Z}(x)$ is unitarily equivalent to the multiplication M_{t_x} on the space $L^2(\mu_x)$ by some function $t_x \in L^\infty(\mu_x)$. The equivalence is given by the unitary operator

$$\Phi_x : \mathcal{Z}(x) \to L^2(\mu_x),$$

with

$$\Phi_x V_\alpha = M_{z_\alpha}\Phi_x, \quad \Phi_x \widetilde{T} = M_{t_x}\Phi_x, \quad \Phi_x x = 1.$$

Hence, the operator $\widetilde{T} \in B(\mathcal{K})$ can be viewed as $t_x(\widetilde{\mathbf{V}})$ on $\mathcal{Z}(x)$. Moreover, we may suppose that

$$\|t_x\|_\infty = \sup_{z \in \mathbf{T}^A} |t_x(z)| \leqslant \|\widetilde{T}\| = \|T\|.$$

Lemma 5. *Let x, $y \in \mathcal{H}$ be any vectors in a Hilbert space $\mathcal{H}$ such that the corresponding spectral measures satisfy $\mu_x \prec \mu_y$. Then*

$$t_x = t_y \quad \mu_x\text{-a.e.}$$

Proof. Let $\varepsilon > 0$ and let $f \in L^1(\mu_y)$ satisfy $d\mu_x = f\, d\mu_y$. For given $\delta > 0$ let us denote $N_\delta = \{z\colon |f(z)| \geqslant \delta\}$. From the inclusion $T(x+y) \in \mathcal{Z}_+(x+y) = \{p(\widetilde{\mathbf{V}})(x+y)\}^-$ we can deduce that there exists a polynom p such that

$$\|(T - p(\widetilde{\mathbf{V}}))(x+y)\| < \frac{\varepsilon^3\sqrt{\delta}}{4\|T\|}.$$

Denoting by $z = \big(T - p(\widetilde{\mathbf{V}})\big)(x+y) = (t_y - p)(\widetilde{\mathbf{V}})y + (t_x - p)(\widetilde{\mathbf{V}})x$ we obtain

$$\|Tz\| = \|t_y(t_y - p)(\widetilde{\mathbf{V}})y + t_x(t_x - p)(\widetilde{\mathbf{V}})x\| < \frac{\varepsilon^3\sqrt{\delta}}{4}, \tag{1}$$

$$\|t_y(\widetilde{\mathbf{V}})z\| = \|t_y(t_y - p)(\widetilde{\mathbf{V}})y + t_y(t_x - p)(\widetilde{\mathbf{V}})x\| < \sup_{z\in\mathbf{T}^A} |t_y(z)| \frac{\varepsilon^3}{4\sqrt{\delta}\|T\|} \leqslant$$

$$\leqslant \frac{\varepsilon^3\sqrt{\delta}}{4}, \tag{2}$$

$$\|t_x(\widetilde{\mathbf{V}})z\| = \|t_x(t_y - p)(\widetilde{\mathbf{V}})y + t_x(t_x - p)(\widetilde{\mathbf{V}})x\| < \frac{\varepsilon^3\sqrt{\delta}}{4}. \tag{3}$$

By substracting (2) – (1) and (1) – (3) we have

$$\|(t_y - t_x)(t_x - p)(\widetilde{\mathbf{V}})x\| < \frac{\varepsilon^3\sqrt{\delta}}{2},$$
$$\|(t_y - t_x)(t_y - p)(\widetilde{\mathbf{V}})y\| < \frac{\varepsilon^3\sqrt{\delta}}{2},$$

which means that

$$\|(t_y - t_x)(t_x - p)\|_{L^2(\mu_x)} = \|(t_y - t_x)(t_x - p)|f|^{1/2}\|_{L^2(\mu_y)} < \frac{\varepsilon^3\sqrt{\delta}}{2},$$
$$\|(t_y - t_x)(t_y - p)\|_{L^2(\mu_y)} < \frac{\varepsilon^3\sqrt{\delta}}{2} \leqslant \frac{\varepsilon^3}{2}$$

hence

$$\|(t_y - t_x)(t_x - p)|N_\delta\|_{L^2(\mu_y)} \leqslant \|(t_y - t_x)(t_x - p)|f|^{1/2}|N_\delta\|_{L^2(\mu_y)} \cdot \delta^{-1/2} < \frac{\varepsilon^3}{2}$$
$$\|(t_y - t_x)(t_x - p)|N_\delta\|_{L^2(\mu_y)} < \frac{\varepsilon^3}{2},$$

which by subtracting results into

$$\|(t_y - t_x)^2 | N_\delta\|_{L^2(\mu_y)} < \varepsilon^3$$

and

$$\mu_y(\{z \in N_\delta : |(t_y - t_x)(z)| \geqslant \varepsilon\}) \leqslant \varepsilon.$$

As ε was arbitrary we have obtained

$$\mu_y(\{z \in N_\delta : t_y(z) \neq t_x(z)\}) = 0,$$

and

$$\mu_x(\{z \in \mathbf{T}^A : t_y(z) \neq t_x(z)\}) \leqslant$$
$$\leqslant \sum_{n=1}^{\infty} \mu_x(\{z \in N_{1/n} : t_y(z) \neq t_x(z)\}) = 0.$$

□

Theorem 2. *Let $\mathbf{V} = (V_\alpha)_{\alpha\in A}$ be a commuting system of isometries on a Hilbert space $\mathcal{H}$. Then* $\operatorname{AlgLat}\mathbf{V} \subset \mathbf{V}''$.

Proof. Let $T \in \operatorname{AlgLat}\mathbf{V}$ and let $S \in B(\mathcal{H})$ be any operator in commutant of $(V_\alpha)_{\alpha\in A}$, $SV_\alpha = V_\alpha S$ $(\alpha \in A)$. Using Corollary of Lemma 3 we can construct Hilbert space $\mathcal{K} \supset \mathcal{H}$ and operators $\widetilde{V}_\alpha$, $\widetilde{S}$ and $\widetilde{T}$ such that $\widetilde{V}_\alpha\widetilde{S} = \widetilde{S}\widetilde{V}_\alpha$, $\widetilde{V}_\alpha|\mathcal{H} = V_\alpha$, $\widetilde{S}|\mathcal{H} = S$, $T|\mathcal{H} = T$.

Let $x \in \mathcal{H}$ be arbitrary, and y be a linear combination (which exists by Lemma 4) of vectors x and Sx such that $\mu_x \prec \mu_y$, $\mu_{Sx} \prec \mu_y$. Then $\widetilde{T}|\mathcal{Z}(x) = t_x(\widetilde{\mathbf{V}})|\mathcal{Z}(x) = t_y(\widetilde{\mathbf{V}})|\mathcal{Z}(x)$ and $\widetilde{T}|\mathcal{Z}(Sx) = t_{Sx}(\widetilde{\mathbf{V}})|\mathcal{Z}(Sx) = t_y(\widetilde{\mathbf{V}})|\mathcal{Z}(Sx)$ by the previous result. It follows that

$$STx = \widetilde{S}\widetilde{T}x = \widetilde{S}t_y(\widetilde{\mathbf{V}})x = t_y(\widetilde{\mathbf{V}})\widetilde{S}x = t_y(\widetilde{\mathbf{V}})Sx = \widetilde{T}Sx = TSx,$$

hence

$$ST = TS, \qquad T \in (V_\alpha)''.$$

□

Theorem 3. *Let $(V_\alpha)_{\alpha\in A}$ be a commuting system of isometries on a Hilbert space $\mathcal{H}$ and suppose that $T \in \operatorname{AlgLat}\mathbf{V}$. Then $\widetilde{T}$, the extension of T defined in Corollary of Lemma 3, belongs to the double commutant of $\widetilde{\mathbf{V}} = (\widetilde{V}_\alpha)_{\alpha\in A}$, the minimal extension of $\mathbf{V}$.*

Proof. Let $\widetilde{S} \in B(\mathcal{K})$ be any operator in commutant of $(\widetilde{V}_\alpha)_{\alpha\in A}$, $\widetilde{S}\widetilde{V}_\alpha = \widetilde{V}_\alpha\widetilde{S}$ ($\alpha \in A$), and let $u \in \mathcal{K}$ be any vector of $\mathcal{K}$. Let $\varepsilon > 0$ be given. As $\mathcal{K} = \bigvee_{x\in\mathcal{H}} \mathcal{Z}(x)$, one can find vectors $x_1, \ldots, x_n \in \mathcal{H}$, $x'_1, \ldots, x'_m \in \mathcal{H}$, and vectors $u_1, \ldots, u_n \in \mathcal{K}$, $u'_1, \ldots, u'_m \in \mathcal{K}$ such that

$$\left\| u - \sum_{i=1}^{n} u_i \right\| < \varepsilon, \quad \left\| \widetilde{S}u - \sum_{i=1}^{m} u'_i \right\| < \varepsilon,$$
$$u_i \in \mathcal{Z}(x_i)\ (1 \leqslant i \leqslant n),\ u'_i \in \mathcal{Z}(x'_i)\ (1 \leqslant i \leqslant m).$$

By Lemma 4 and 5 there exists a function $f \in L^\infty(\mu)$, where $\mu = \bigvee_{i=1}^{n} \mu_{x_i} \vee \bigvee_{i=1}^{m} \mu_{x'_i}$ such that $\widetilde{T}|\mathcal{Z}_{x_i} = f(\widetilde{\mathbf{V}})|\mathcal{Z}_{x_i}$ $(i = 1, \ldots, n)$ and $\widetilde{T}|\mathcal{Z}_{x'_i} = f(\widetilde{\mathbf{V}})|\mathcal{Z}_{x'_i}$ $(i = 1, \ldots, m)$. Then

$$\begin{aligned}
\|\widetilde{S}\widetilde{T}u - \widetilde{T}\widetilde{S}u\| &\leqslant \left\|\widetilde{S}\widetilde{T}u - \widetilde{S}\widetilde{T}\sum_{i=1}^{n} u_i\right\| + \left\|\widetilde{S}\widetilde{T}\sum_{i=1}^{n} u_i - \widetilde{T}\sum_{i=1}^{m} u'_i\right\| + \left\|\widetilde{T}\sum_{i=1}^{m} u'_i - \widetilde{T}\widetilde{S}u\right\| \\
&\leqslant \|\widetilde{S}\|\,\|\widetilde{T}\| \left\|u - \sum_{i=1}^{n} u_i\right\| + \left\|\widetilde{S}f(\widetilde{\mathbf{V}})\sum_{i=1}^{n} u_i - f(\widetilde{\mathbf{V}})\sum_{i=1}^{m} u'_i\right\| + \|\widetilde{T}\| \left\|\sum_{i=1}^{m} u'_i - \widetilde{S}u\right\| \\
&\leqslant \|\widetilde{S}\|\,\|T\|\varepsilon + \|f(\widetilde{\mathbf{V}})\| \left(\left\|\widetilde{S}\sum_{i=1}^{n} u_i - \widetilde{S}u\right\| + \left\|\widetilde{S}u - \sum_{i=1}^{m} u'_i\right\| \right) + \|T\|\varepsilon \\
&\leqslant 2\|\widetilde{S}\|\,\|T\|\varepsilon + 2\|T\|\varepsilon.
\end{aligned}$$

As ε was arbitrary we have $\widetilde{S}\widetilde{T}u = \widetilde{T}\widetilde{S}u$, i.e. $\widetilde{S}\widetilde{T} = \widetilde{T}\widetilde{S}$ and $\widetilde{T} \in (\widetilde{V}_\alpha)''$. □

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