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PARTIALLY ORDERED SETS WITH NONDISTRIBUTIVE LATTICES OF MAXIMAL ANTICHAINS

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All partially ordered sets which are dealt with in the present paper are assumed to be finite.

For a partially ordered set X we denote by $MA(X)$ the system of all maximal antichains in X; this system is considered to be partially ordered (cf. Section 1 below). Then $MA(X)$ is a lattice (cf. [1]).

A convex subset of X which is isomorphic to the partially ordered set on Fig. 1 or Fig. 2 will be called a serpentine set or a serpentine cycle in X, respectively.

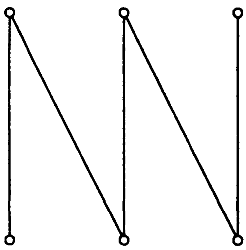

Fig. 1

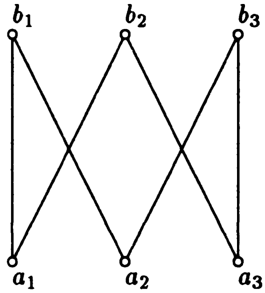

Fig. 2

In [1] the question was proposed to find an internal characterization of those partially ordered sets for which the lattice $MA(X)$ is distributive or modular, respectively.

In [3] it was shown that if $MA(X)$ is nonmodular, then X possesses a serpentine subset. Next, the notion of a regular serpentine subset was introduced and it was shown that $MA(X)$ is nonmodular iff X has a regular serpentine subset.

In this paper the notion of a regular serpentine cycle will be defined. The following result will be established:

(α) Let the lattice $MA(X)$ be modular. Then $MA(X)$ is nondistributive iff X possesses a regular serpentine cycle.

1. Preliminaries

Let X be a partially ordered set. We denote by $A(X)$ the system of all antichains in X. For $B_1, B_2 \in A(X)$ we put $B_1 \leqslant B_2$ if for each $b_1 \in B_1$ there exists $b_2 \in B_2$ with $b_1 \leqslant b_2$. Then $\leqslant$ is a partial order on $A(X)$.

Next, we denote by $MA(X)$ the set of all $B \in A(X)$ having the property that for each $C \in A(X)$ with $B \subseteq C$ the relation $B = C$ is valid. The elements of $MA(X)$ are called maximal antichains in X.

The system $MA(X)$ is partially ordered by the relation $\leqslant$ inherited from $A(X)$. In [1] it was proved that $MA(X)$ is a lattice.

Let $a, b \in X$. If a is covered by b, then we write $a \prec b$ or $b \succ a$. The same symbols are applied for denoting the covering relation in the lattice $MA(X)$. The notation $a \mid b$ means that the elements a and b are incomparable. For A, B in $A(X)$ we write $A <_1 B$ if $A < B$ and if $a \in A$, $b \in B$, $a < b$ implies $a \prec b$.

A convex subset $X_1 \neq \emptyset$ of X will be called to be a short subset of X if there exist A and B in $MA(X)$ with $A <_1 B$ having the property that $X_1 = \{x_1 \in X :$ there are $a \in A$ and $b \in B$ such that $a \leqslant x_1 \leqslant b\}$. Hence whenever x and x' are elements of X_1 with $x < x'$, then $x \prec x'$.

The following result will be proved:

(β) The lattice $MA(X)$ is distributive iff for each short subset X_1 of X the lattice $MA(X_1)$ is distributive.

For an analogous result concerning modularity cf. [3]. From [3] we also recall the following notion.

We denote by $N(X)$ the set of all triples (P_1, P_2, P_3) of mutually disjoint subsets of X such that

(i) $P_2 \neq \emptyset \neq P_3$ and each element of P_2 is covered by each element of P_3;

(ii) both sets $P_1 \cup P_2$ and $P_1 \cup P_3$ belong to $MA(X)$.

A serpentine cycle S of X will be said to be regular if there exist (B_1, B_2, A_2), (B_1', B_2', A_2') and (B_1'', B_2'', A_2'') in $N(X)$ such that (under the notation as in Fig. 2) we have

(i) $A_2 \cup A_2' \cup A_2'' \in A(X)$;

(ii) $B_1 \cup B_2 = B_1' \cup B_2' = B_1'' \cup B_2''$;

(iii) $a_1 \in A_2''$, $a_2 \in A_2'$, $a_3 \in A_2'$, $b_1 \in B_1, b_2 \in B_1'$, $b_3 \in B_1''$.

1.1. Lemma. *Let $B_1, B_2 \in MA(X)$. The following conditions are equivalent:*

(i) $B_1 \leqslant B_2$.

(ii) *For each $b_2 \in B_2$ there exists $b_1 \in B_1$ such that $b_1 \leqslant b_2$.*

The proof is easy; it is omitted.

1.2. Lemma. *Let $A, B \in MA(X)$, $A < B$, and let $X_1 = A \cup B$ be a short subset of X. Then the set $MA(X_1)$ coincides with the interval $[A, B]$ of the lattice $MA(X)$.*

Proof. Let $C \in MA(X_1)$. First we shall verify that C belongs to $MA(X)$. By way of contradiction, suppose that C does not belong to $MA(X)$. Hence there is $C' \in MA(X)$ such that $C \subset C'$. Thus there is $c' \in C' \setminus C$. Then clearly $a' \notin A \cup B$.

Since $c' \notin A$ there exists $a \in A$ such that a and c' are comparable. Hence a cannot belong to C; thus a is comparable with an element c of C. Suppose that $c' < a$. If $a < c$, then $c' < c$, which is impossible. Thus $c < a$. Hence $c \notin A$ and then $c \in B$. By virtue of $A < B$ there is $b_1 \in B$ with $a < b_1$; we obtain that $c < b_1$. This cannot hold since both b_1 and c belong to B. Therefore $a < c'$.

An analogous consideration (applying 1.1) leads to the existence of $b \in B$ such that $c' < b$. From $a < c' < b$ and from the convexity of X_1 we infer that $c' \in X_1$, which is a contradiction. Thus $C \in MA(X)$.

Let $c \in C$. Then either $c \in B_2$ or $c \in A$. In the latter case there is $b' \in B$ with $c \leqslant b'$. Hence $C \leqslant B$. Analogously we obtain that $A \leqslant C$. Hence C belongs to the interval $[A, B]$ of $MA(X)$.

Conversely, let C belong to the interval $[A, B]$ of $MA(X)$. Let $c \in C$. There are $a \in A$ and $b \in B$ such that $a \leqslant c \leqslant b$. The relation $a < c < b$ is impossible, since $A \cup B$ is a short subset of X. Therefore $c \in A \cup B$. Now it is clear that $C \in MA(X_1)$. □

2. Short subsets

We denote by M the modular nondistributive lattice with five elements. A sublattice L_1 of a lattice L is said to be saturated if, whenever x and y are elements of L_1 such that x is covered by y in L_1, then x is covered by y in L. The following result is well-known (cf. [2], p. 151).

2.1. Proposition. *Let L be a finite modular lattice. Then the following conditions are equivalent:*

(i) *L is nondistributive.*

(ii) *There exists a saturated sublattice M_1 of L such that M_1 is isomorphic to M.*

Let X be a partially ordered set.

2.2. Lemma. *Let A, A' and B be elements of $MA(X)$ such that $A \prec B$, $A' \prec B$ and $A \neq A'$ Then there exists a short subset X_1 of X such that $B \in X_1$ and $A \wedge A' \in X_1$.*

Proof. This is a consequence of [3], Lemma 3.6. □

Proof of (β). Let X_1 be a short subset of X and let A, B be as in Section 1 (with respect to the given X_1). Then $MA(X_1)$ is a convex sublattice of $MA(X)$ with the least element A and the greatest element B. Hence if $MA(X)$ is distributive, then $MA(X_1)$ is distributive as well.

Conversely, suppose that $MA(X)$ fails to be distributive. First assume that $MA(X)$ is nonmodular. Thus in view of [3], Theorem 3.11, there exists a short subset X_1 of X having the property that $MA(X_1)$ is nonmodular, and so $MA(X_1)$ is nondistributive. Next, assume that $MA(X)$ is modular. Then according to 2.1, there exists a five-element saturated sublattice $M_1 = \{B, A, A', A'', C\}$ of $MA(X)$ such that M_1 is isomorphic to M, B is the greatest element of M_1 and C is the least element of M_1. Lemma 2.2 yields that there exists a short subset X_1 of X such that $M_1 \subseteq X_1$. Hence according to 1.2, $MA(X_1)$ is nondistributive. □

From (β) and from 3.11 in [3] we obtain as a corollary:

2.3. Proposition. *The following conditions are equivalent:*

(i) *$MA(X)$ is modular and non-distributive.*

(ii) *There exists a short subset X_1 of X such that $MA(X_1)$ is nondistributive, and $MA(X_2)$ is modular for each short subset X_2 of X.*

3. Nondistributivity

In this section we suppose that X is a partially ordered set such that the lattice $MA(X)$ is modular and non-distributive. Thus there exists a saturated sublattice M_1 of $MA(X)$ with the properties as in the proof of (β) in Section 2. Denote

$$B_2 = B \setminus A, \quad B_1 = B \setminus B_2, \quad A_2 = A \setminus B_1,$$

and let B_2', B_1', A_2', B_2'', B_1'' and A_2'' be defined analogously.

3.1. Lemma. $B \cap C = B_1 \cap B_1'$.

Proof. This is a consequence of 3.6 in [3]. □

3.2. Corollary. $B_1 \cap B_1' = B_1 \cap B_1'' = B_1' \cap B_1''$.

Denote $X_2 = \{x_2 \in X_1 : c \leqslant x_2 \leqslant b$ for some $c \in C \setminus B$ and some $b \in B \setminus C\}$. For each $P \in MA(X_1)$ (where X_1 is the interval $[C, B]$ of $MA(X)$) we have $B \cap C \subseteq P$ and the mapping $P \to P \setminus (B \cap C)$ is an isomorphism of the lattice $MA(X_1)$ onto the lattice $MA(X_2)$.

The above consideration shows that $MA(X_2)$ is modular and nondistributive as well; hence without loss of generality we can suppose that $B \cap C = \emptyset$. Thus in view of 3.2 we assume that

$$B_1 \cap B_1' = B_1 \cap B_1'' = B_1' \cap B_1'' = \emptyset.$$

(For an analogous procedure cf. [3], Section 4.)

Denote $Y(A, A') = C \setminus (A_2 \cup A_2')$.

3.3. Lemma. $A_2 \cap A_2' = A_2 \cap A_2'' = A_2' \cap A_2'' = \emptyset$.

From 3.3 and [3], Lemma 3.6 we infer:

3.4. Lemma. $A_2'' \subseteq Y(A, A')$.

3.5. Lemma. *Each of the sets A_2, A_2', A_2'', B_1, B_1' and B_1'' is nonempty.*

Proof. This follows from [3], Lemmas 4.2 and 4.4. □

3.6. Lemma. *Let $y \in Y(A, A')$, $b_1 \in B_1$ and $b_1' \in B_1'$. Then $y \prec b_1$ and $y \prec b_1'$.*

Proof. We have $y \in C$ and $b_1 \in A$. Next, $C \prec A$ is valid. In view of Lemma 3.6.1 in [3] there exists b_1^* in B_1 such that $y \prec b_1^*$. Also, $b_1 \in B_2'$ and according to 3.3 there is $a_2' \in A_2'$; hence $a_2' \prec b_1$. Therefore from 2.7 in [3] we infer that $y \prec b_1$ is valid. Similarly we obtain that the relation $y \prec b_1'$ holds. □

3.7. Lemma. *Let $a'' \in A_2''$, $b_1 \in B_1$ and $b_1' \in B_1'$. Then $a'' \prec b_1$ and $a'' \prec b_1'$.*

Proof. This follows immediately from 3.4 and 3.5. □

Similarly we have

3.7.1. Lemma. *Let $a \in A_2$, $a' \in A_2'$, $b_1'' \in B_1''$. Next, let b_1 and b_1' be as in 3.7. Then $a \prec b_1'$, $a \prec b_1''$, $a' \prec b_1$ and $a' \prec b_1''$.*

3.8. Proposition. *Assume that $MA(X)$ is modular and nondistributive. Then X possesses a regular serpentine cycle.*

P r o o f. In view of 2.1 there exists a saturated sublattice $\{C, A, A', A'', B\}$ of $MA(X)$ which is isomorphic to the lattice M. Let us apply the notation as above. According to 3.5 there exist elements a, a', a'', b_1, b_1' and b_1'' with the properties as in 3.7.1. Then a, a' and a'' are distinct elements belonging to C, hence they are mutually incomparable. Next, b_1, b_1' and b_1'' are distinct elements belonging to B, hence they are mutually incomparable as well. It is easy to verify that the elements a, a', a'', b_1, b_1', b_1'' are distinct. Therefore in view of 3.6.1 the set consisting of these elements is a regular serpentine cycle in X. □

Let $C_0 \in MA(X)$ and $A_0 \in A(X)$. Assume that $A_0 < C_0$ is valid in $A(X)$ and that, whenever $a_0 \in A_0$, $c_0 \in C_0$ and $a_0 \leqslant c_0$, then $a_0 \prec c_0$. Put $Q = \{c_0 \in C_0 \colon c_0 \mid a_0$ for each $a_0 \in A_0\}$. Next, let Q_1 be the set of all $x \in X$ such that

(i) $x \mid y$ for each $y \in A_0 \cup Q$;

(ii) there exists $c_0 \in C_0$ with $x \prec c_0$;

(iii) if $c \in C_0$ and $x \leqslant c$, then $x \prec c$.

We set $C^* = A_0 \cup Q \cup Q_1$. It is obvious that $C^* \in A(X)$ and that, whenever $t \in X \setminus C^*$, $t \leqslant c$ for some $c \in C_0$, then t is comparable with an element of C^*. Hence we obtain from Lemma 2.1 in [3]:

3.9. Lemma. *Under the above notation, C^* belongs to $MA(X)$.*

Also, from the construction of C^* we immediately conclude:

3.10. Lemma. *Let A_0, C_0 and C^* be as above. Let $D \in MA(X)$ be such that $A_0 \subseteq D$ and $D \subseteq C_0$. Then $D \leqslant C^*$.*

3.11. Proposition. *Assume that $MA(X)$ is modular and that X possesses a regular serpentine cycle. Then $MA(X)$ is nondistributive.*

P r o o f. Let us assume that X possesses a regular serpentine cycle S. Next, let (i), (ii) and (iii) be as in Section 1.

Denote $B = B_1 \cup B_2$, $A = B_1 \cup A_2$, $A' = B_1' \cup A_2'$, $A'' = B_1'' \cup A_2''$. Then B, A, A' and A'' belong to $MA(X)$. In view of (ii) and [3], Lemma 2.7 we have

$$(1) \qquad A \prec B, \quad A' \prec B, \quad A'' \prec B.$$

Since $b_1 \in B_1$, $a' \in A_2'$ and $a' \prec b_1$ we infer that a' does not belong to A_2 and clearly $a' \notin B$. Therefore $A \neq A'$. Similarly we can verify that $A \neq A''$ and $A' \neq A''$.

Put $A_0 = A_2 \cup A_2' \cup A_2''$ and $C_0 = B$. Let C^* be as in Lemma 3.9.

We have $a \in A$ and $a \in C^*$, hence $a \in A \vee C^*$. Clearly $a \notin B$, thus $A \vee C^* \neq B$. Since $A \leqslant A \vee C^* \leqslant B$, according to (1) we obtain that $A \vee C^* = A$ and therefore $C^* \leqslant A$. Similarly we obtain that $C^* \leqslant A'$ and $C^* \leqslant A''$. Hence

$$C^* \leqslant A \wedge A' \wedge A''. \tag{2}$$

The fact that $A_2 \cup A_2' \cup A_2''$ is an antichain in X and that $A_2 \subseteq A$, $A_2' \subseteq A$ and $A_2'' \subseteq A''$ implies that

$$A_2 \cup A_2' \cup A_2'' \subseteq A \wedge A' \wedge A''$$

is valid. Thus (2) and 3.10 yield

$$C^* = A \wedge A' \wedge A''.$$

Now from Lemma 3.7 in [3] and by applying the relation $A_2 \cup A_2' \cup A_2'' \in A(X)$ again we infer that $A_2'' \subseteq A \wedge A'$. Thus

$$A_2 \cup A_2' \cup A_2'' \subseteq A \wedge A'$$

and hence $C^* \leqslant A \wedge A'$. Therefore $A \wedge A' \wedge A'' = A \wedge A'$. Similarly we infer that

$$A \wedge A'' = A \wedge A' \wedge A'' = A' \wedge A''.$$

Thus the sublattice of $MA(X)$ consisting of the elements A, A', A'', $A \wedge A'$ and B is nondistributive. □

From 3.8 and 3.11 we obtain that (α) holds.

References

[1] *G. Behrendt*: Maximal antichains in partially ordered sets, Ars conbinatoria *25C* (1988), 149–157.
[2] *G. Grätzer*: Lectures on lattice theory, Vol I, San Francisco, 1971.
[3] *J. Jakubík*: Maximal antichains in a partially ordered set., Czechoslov. Math. J. *41* (1991), 75–84.

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