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Vassiliki Farmaki

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CONVERGENCE OF SERIES AND ISOMORPHIC EMBEDDINGS IN BANACH SPACES

VASSILIKI FARMAKI, Athens

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INTRODUCTION – PRELIMINARIES

An important question for a Banach space is whether it contains an isomorphic copy of some classical Banach spaces. The usual method of proving such results is via basic sequences. For example, James [2] has shown that if a Banach space contains an unconditional basis, then it contains an isomorphic copy of c_0, ℓ_1 or it is reflexive. This gives a partial answer to the famous open problem, namely the existence of a subspace isomorphic to c_0, ℓ_1 or to a reflexive space in any Banach space and shows that this problem is closely connected with the open problem, whether every Banach space contain an unconditional basic sequence.

We say a sequence (x_n) of non-zero vectors in a Banach space X basic iff (x_n) is a Shauder basis for its closed linear span $[x_n]_{n=1}^{\infty}$. This means for every $x \in [x_n]_{n=1}^{\infty}$ there exists a unique sequence (λ_n) of scalars such that $x = \sum\limits_{n=1}^{\infty} \lambda_n x_n$. Moreover, (x_n) is said unconditional if the series $\sum\limits_n \varepsilon_n \lambda_n x_n$ converges for every $(\varepsilon_n) \in \{-1, 1\}^{\mathbf{N}}$. Two basic sequences are called equivalent iff the convergence of the series $\sum\limits_n \lambda_n x_n$ is equivalent to that of $\sum\limits_n \lambda_n y_n$. In this case there is an isomorphism between the spans $[x_n]_{n=1}^{\infty}$ and $[y_n]_{n=1}^{\infty}$ which carries x_n to y_n. This notion gives the ability to recognize the existence of a classical Banach space via its usual base. Of course every sequence (x_n) in a Banach space X doesn't contain a basic subsequence but we can select a basic subsequence if (x_n) is seminormalized (i.e. $0 < \inf\limits_n \|x_n\|$) and weakly null using the Bessaga-Pelczynski selection principle [1].

In this paper, we extend notions and results related to basic sequences (such as equivalence, unconditionality, e.t.c.) to arbitrary sequences in a Banach space. We prove that if a Banaçh space X has an unconditional (not necessarily basic) sequence

(x_n) with $0 < \inf_n \|x_n\|$ and $X = [x_n]_{n=1}^\infty$ then X contains an isomorphic copy of c_0, ℓ_1 or it is somewhat reflexive (Theorem 22). A Banach space X is called somewhat reflexive if any subspace of X has a reflexive subspace with a basis. We also answer the second problem affirmatively for the case of a Banach space with an unconditional sequence (Theorem 19).

We give a criterion (Corollary 11) for a Banach space to contain ℓ_p isomorphically, using the notion of an ℓ_p-sequence (a sequence (x_n) such that the series $\sum_n a_n e_n$ converges if and only if $(a_n) \in \ell_p$ and $0 < \inf_n \|x_n\|$). The same result is proved by Bessaga and Pelczynski [1] for the case of c_0. We also characterize the class of ℓ_p- or c_0-sequences as the class of those bounded sequences which are equivalent to their bounded blocks (Theorem 13) extending M. Zippin's Theorem [7] which gives an analogous characterization for the basic sequences.

The central idea for proving such results is the close relation between a semi-normalized sequence (x_n) and the basic sequence (e_n) of unit vectors in the space $\Sigma^{(x_n)}$.

Let $(X, \|.\|)$ be a Banach space and (x_n) a sequence of non zero vectors in X. The vector space

$$\Sigma^{(x_n)} = \left\{(\lambda_n) \in \mathbf{R}^{\mathbf{N}} : \sup_n \left\| \sum_{i=1}^n \lambda_i x_i \right\| < \infty \right\}$$

is a Banach space with respect to the norm $|\!|\!|(\lambda_n)|\!|\!| = \sup_n \left\| \sum_{i=1}^n \lambda_i x_i \right\|$. For every sequence (x_n) such that $0 < A \leqslant \|x_n\| \leqslant B$ for $n \in \mathbf{N}$ we have that $|\!|\!|(\lambda_n)|\!|\!| \leqslant B \sum_{i=1}^\infty |\lambda_n|$ for every $(\lambda_n) \in \ell^1$ and $\sup_n |\lambda_n| \leqslant \frac{2}{A} |\!|\!|(\lambda_n)|\!|\!|$ for every $(\lambda_n) \in \Sigma^{(x_n)}$. Hence the set $\Sigma^{(x_n)}$ is contained in ℓ^∞ and contains ℓ_1. As we prove in Propositions 8 and 9 if $\Sigma^{(x_n)} = \ell^1$ (resp. $\Sigma^{(x_n)} = \ell^\infty$) then (x_n) has a basic subsequence equivalent to the usual basis of ℓ_1 (resp. to the usual basis of c_0). Also if $\Sigma^{(x_n)} = \ell_p$ for some $1 < p < \infty$ then (x_n) has a basic subsequence equivalent to the usual basis of ℓ^p (Proposition 10). From a result of Odell [5] if a normalized weakly null sequence (x_n) has no subsequence equivalent to the unit vector basis of c_0 then there exists a subsequence (x_{n_k}) so that $\Sigma^{(x_{n_k})} \subset c_0$ and if a bounded sequence (x_n) has no subsequence equivalent to the unit vector basis of ℓ_1 then for every subsequence (x_{n_k}) of (x_n) we have $\Sigma^{(x_{n_k})} \supsetneqq \ell_1$.

The unit vectors e_n, $n \in \mathbf{N}$ form a basic sequence in $\Sigma^{(x_n)}$ with basic constant 1. It is easy to see that $\Sigma^{(x_n)} = \Sigma^{(e_n)}$. Also $\Sigma^{(x_n)} = [e_n]_{n=1}^\infty$ if and only if (e_n) is boundedly complete and that the function $T\colon [e_n]^{**} \to \Sigma^{(x_n)}$ with $T(x^{**}) = T\big(x^{**}(e_n^*)\big)$ is an isometry onto $\Sigma^{(x_n)}$ if and only if (e_n) is shrinking. Hence $\Sigma^{(x_n)}$ is reflexive if and only if (e_n) is shrinking and boundedly complete. As we prove in Proposition 15, if

$\Sigma^{(x_n)}$ is reflexive for some sequence (x_n) with $0 < \inf_n \|x_n\|$, then (x_n) has a basic subsequence (x_{n_k}) which is shrinking and boundedly complete and also the space $[x_n]_{n=1}^{\infty}$ is somewhat reflexive (Proposition 17). We don't kn ›w whether $X = [x_n]_{n=1}^{\infty}$ is reflexive in general, but it is easy to see that X is reflexive if (x_n) is dense in X.

Extending R. C. James' Theorem [2] we prove that if (x_n) is an unconditional sequence with $0 < \inf \|x_n\|$, then $[x_n]_{n=1}^{\infty}$ has no subspace isomorphic to ℓ^1 if and only if (e_n) is shrinking, and $[x_n]_{n=1}^{\infty}$ has no subspace isomorphic to c_0 if and only if (e_n) is boundedly complete (Proposition 21). Hence, in this case we have that $\Sigma^{(x_n)}$ is reflexive if $[x_n]_{n=1}^{\infty}$ has no subspace isomorphic to ℓ_1 or c_0.

We call two sequences (x_n) in X, (y_n) in Y equivalent if $0 < \inf_n \|x_n\|$, $0 < \inf_n \|y_n\|$ and $\Sigma^{(x_n)} = \Sigma^{(y_n)}$. As we prove in Proposition 2 the sequences (x_n) and (y_n) are equivalent if and only if the series $\sum_n a_n x_n$ converges in X, iff the series $\sum_n a_n y_n$ converges in Y. Hence this notion of equivalence extends the usual notion for basic sequences. Since $\Sigma^{(x_n)} = \Sigma^{(e_n)}$ we have that every sequence (x_n) with $0 < \inf_n \|x_n\|$ is equivalent to the basic sequence (e_n).

Our notation generally follows that of [4] where many notion and unproved statements may be found. In particular we write (x_n) for a sequence, $\sum_n x_n$ for a series and $[x_n]_{n=1}^{\infty}$ for the closed linear span of a sequence (x_n) in a Banach space.

Many of the notions of basic sequences can be defined in a meaninful way also for arbitrary sequences in Banach spaces.

Definition 1. Let X, Y be Banach spaces and (x_n), (y_n) sequences in X, Y respectively such that $0 < \inf_n \|x_n\|$ and $0 < \inf_n \|y_n\|$. The sequences (x_n) and (y_n) are called *equivalent* if $\Sigma^{(x_n)} = \Sigma^{(y_n)}$.

We show that this notion of equivalence extends the usual notion for basic sequences.

Proposition 2. *Let (x_n), (y_n) be sequences in the Banach spaces X, Y respectively, such that $0 < \inf_n \|x_n\|$ and $0 < \inf_n \|y_n\|$. The following are equivalent:*

(i) *The sequences (x_n), (y_n) are equivalent.*

(ii) *The identity mapping $I\colon \Sigma^{(x_n)} \to \Sigma^{(y_n)}$ is an isomorphism.*

(iii) *The unit vector basic sequence (e_n) in $\Sigma^{(x_n)}$ is equivalent (using the usual notion) to the same basic sequence in $\Sigma^{(y_n)}$.*

(iv) *The series $\sum_n a_n x_n$, $a_n \in \mathbf{R}$, converges in X if and only if $\sum_n a_n y_n$ converges in Y.*

Proof. (i) $\Rightarrow$ (ii) Let $\Sigma^{(x_n)} = \Sigma^{(y_n)}$ and let $I\colon \Sigma^{(x_n)} \to \Sigma^{(y_n)}$ be the identity mapping. But I has closed graph; this is easy to see from the inequality

$$(*) \qquad \sup_n |\lambda_n| \leqslant \frac{2|\!|\!|(\lambda_n)|\!|\!|}{\inf_n \|x_n\|} \text{ which holds for every } (\lambda_n) \in \Sigma^{(x_n)}.$$

Therefore I is an isomorphism.

It is clear that (ii) implies (iii) and it is easy to see that (iii) implies (i) from the equality $\Sigma^{(e_n)} = \Sigma^{(x_n)}$. The equivalence of (iii) and (iv) is a consequence of the following observation. The series $\sum_n a_n x_n$ converges in X if and only if $\sum_n a_n e_n$ converges in $\Sigma^{(x_n)}$. Indeed, if $\sum_n a_n e_n$ converges in $\Sigma^{(x_n)}$ then $\sum_n a_n x_n$ converges in X, because $\left\| \sum_{i=n}^m a_i x_i \right\| \leqslant \left|\!\left|\!\left| \sum_{i=n}^m a_i e_i \right|\!\right|\!\right|$ for every $n, m \in \mathbf{N}$. On the other hand, if $\sum_n a_n x_n$ converges in X, then for every $\varepsilon > 0$ there is $n_0 \in \mathbf{N}$ such that $\left\| \sum_{i=n}^m a_i x_i \right\| < \varepsilon$ for every $n, m \in \mathbf{N}$ with $n_0 \leqslant n \leqslant m$, hence $\left|\!\left|\!\left| \sum_{i=n}^m a_i e_i \right|\!\right|\!\right| = \sup_{n \leqslant k \leqslant m} \left\| \sum_{i=n}^k a_i x_i \right\| < \varepsilon$ for every $n, m \in \mathbf{N}$ with $n_0 \leqslant n \leqslant m$. It follows that $\sum_n a_n e_n$ converges in $\Sigma^{(x_n)}$. $\square$

In the next proposition we observe that if we perturb each element of a sequence by a sufficiently small vector, then we get an equivalent sequence.

Proposition 3. *Let X be a Banach space and (x_n), (y_n) two sequences in X such that $0 < \inf_n \|x_n\|$ and $0 < \inf_n \|y_n\|$. If $\sum_n \|x_n - y_n\| < \infty$ then (x_n) is equivalent to (y_n).*

Proof. Let $(\lambda_n) \in \Sigma^{(x_n)}$. Then $\left\| \sum_{i=1}^n \lambda_i y_i \right\| \leqslant \left\| \sum_{i=1}^n \lambda_i x_i \right\| + \left\| \sum_{i=1}^n \lambda_i (x_i - y_i) \right\| \leqslant \sup_n \left\| \sum_{i=1}^n \lambda_i x_i \right\| + (\sup_n |\lambda_n|) \cdot \sum_{i=1}^\infty \|x_i - y_i\|$ holds for every $n \in \mathbf{N}$. Hence $(\lambda_n) \in \Sigma^{(y_n)}$, since (λ_n) is bounded. $\square$

Lemma 4. *Let X be a Banach space and (x_n) a sequence in X such that $0 < \inf_n \|x_n\|$. If (u_k) is a block sequence of (x_n) with $u_k = \sum_{i=p_k}^{q_k} a_i x_i$, $a_i \in \mathbf{R}$ and $p_k \leqslant q_k < p_{k+1}$ for every $k \in \mathbf{N}$, such that $0 < \inf_k \|u_k\|$ then the sequence (u_k) is equivalent to the corresponding basic sequence (v_k) in $\Sigma^{(x_n)}$ where $v_k = \sum_{i=p_k}^{q_k} a_i e_i$ for every $k \in \mathbf{N}$.*

Proof. It is easy to see that $\|u_k\| \leqslant |\!|\!| v_k |\!|\!|$ for every $k \in \mathbf{N}$, hence $0 < \inf_k |\!|\!| v_k |\!|\!|$. The sequence (x_n) is equivalent to the unit vector sequence (e_n) in $\Sigma^{(x_n)}$, because

$\Sigma^{(e_n)} = \Sigma^{(x_n)}$. Thus, from Proposition 2 the series $\sum_k \lambda_k u_k$ converges in X if and only if the series $\sum_k \lambda_k v_k$ converges in $\Sigma^{(x_n)}$. Hence, the sequence (u_k) is equivalent to the sequence (v_k) in $\Sigma^{(x_n)}$. □

The existence in a Banach space of subspaces isomorphic to the classical Banach spaces plays a central role in the study of the space. The next results give criteria for the embedding of c_0 of ℓ_p for $1 \leqslant p < \infty$ in a Banach space.

Definition 5. A sequence (x_n) in a Banach space X such that $0 < \inf_n \|x_n\|$ is called an *ℓ_p-sequence* for some $1 \leqslant p < \infty$ (resp. *c_0-sequence*) iff it is equivalent to the usual basis of ℓ_p (resp. of c_0).

As corollaries of the previous results we have the next two propositions.

Proposition 6. *Let X be a Banach space, (x_n) a sequence in X such that $0 < \inf_n \|x_n\|$ and $1 \leqslant p < \infty$. The following are equivalent:*

(i) *The sequence (x_n) is an ℓ_p-sequence (resp. c_0-sequence).*

(ii) *$\Sigma^{(x_n)} = \ell_p$ (resp. $\Sigma^{(x_n)} = \ell^\infty$).*

(iii) *The identity mapping $I\colon \Sigma^{(x_n)} \to \ell^p$ (resp. $I\colon \Sigma^{(x_n)} \to \ell^\infty$) is an isomorphism.*

(iv) *The unit vector basic sequence (e_n) in $\Sigma^{(x_n)}$ is equivalent to the usual basis of ℓ_p (resp. of c_0).*

(v) *The series $\sum_n a_n x_n$ converges in X if and only if $(a_n) \in \ell_p$ (resp. $(a_n) \in c_0$).*

Proposition 7. *Let (x_n) be a sequence in a Banach space X. If (x_n) is an ℓ_p-sequence for some $1 \leqslant p < \infty$ (resp. a c_0-sequence) then every block sequence (u_k) of (x_n) with $u_k = \sum_{i=p_k}^{q_k} a_i x_i$, $a_i \in \mathbf{R}$, $p_k \leqslant q_k < p_{k+1}$ for every $k \in \mathbf{N}$, such that $0 < \inf_k \|u_k\|$ and $\sup_k \left\|\left\| \sum_{i=p_k}^{q_k} a_i e_i \right\|\right\| < \infty$ is also an ℓ_p-sequence (resp. a c_0-sequence).*

Proof. Applying Lemma 4 we have that the sequence (u_k) in X is equivalent to the basic sequence (v_k) in $\Sigma^{(x_n)}$, where $v_k = \sum_{i=p_k}^{q_k} a_i e_i$ for every $k \in \mathbf{N}$. Since (x_n) is an ℓ_p-sequence, the basic sequence (e_n) in $\Sigma^{(x_n)}$ is equivalent to the usual basis of ℓ_p. Since (v_k) is bounded it is also equivalent to the usual basis of ℓ_p. Therefore, (u_k) is an ℓ_p-sequence.

The proof for c_0-sequences is similar. □

We shall prove in Theorem 13 that the property of ℓ_p- and c_0-sequences which is described in the previous Proposition characterizes the class of these sequences.

R e m a r k . Bessaga and Pelczynski proved in [1] that a sequence (x_n) in a Banach space X such that $0 < \inf_n \|x_n\|$ is a c_0-sequence if and only if it is *weakly unconditionally Cauchy* (i.e. $\sum_{i=1}^{\infty} |f(x_n)| < \infty$ for every $f \in X^*$).

Since a c_0-sequence (x_n) is weakly null, it follows from the Bessaga-Pelczynski selection principle [1] that it has a basic sequence (x_{n_k}). The subsequence (x_{n_k}) is also a c_0-sequence and hence equivalent to the usual basis of c_0. Thus we have:

Proposition 8. *Let X be a Banach space. Every c_0-sequence in X has a basic subsequence equivalent to the usual basis of c_0.*

We shall prove the same result in the situation of ℓ_p-sequences.

Proposition 9. *Let X be a Banach space. Every ℓ_1-sequence in X has a basis subsequence equivalent to the usual basis of ℓ_1.*

P r o o f . Let (x_n) be an ℓ_1-sequence in X. Then (x_n) is bounded and it doesn't converges in X. Indeed, from Proposition 6 the unit vector basic sequence (e_n) in $\Sigma^{(x_n)}$ is equivalent to the usual basic of ℓ_1, hence there are $A, B > 0$ such that

$$A \cdot \sum_{n=1}^{k} |a_n| \leqslant \left|\!\left|\!\left| \sum_{n=1}^{k} a_n e_n \right|\!\right|\!\right| \leqslant B \cdot \sum_{n=1}^{k} |a_n|$$

holds for every $k \in \mathbf{N}$ and $a_1, \ldots, a_k \in \mathbf{R}$. It follows that $\|x_n\| = |\!|\!| e_n |\!|\!| \leqslant B$ for every $n \in \mathbf{N}$. Suppose now that (x_n) converges in X. Take $k_0 \in \mathbf{N}$ such that $B + 1 < k_0 A$ and $n_0 \in \mathbf{N}$ such that $\|x_n - x_m\| < \frac{1}{k_0}$ for every $n, m \in \mathbf{N}$ with $n_0 \leqslant n \leqslant m$. Then we have,

$$B + 1 < k_0 A \leqslant \left|\!\left|\!\left| \sum_{n=n_0+1}^{n_0+k_0} (-1)^n e_n \right|\!\right|\!\right| = \left\| \sum_{n=n_0+1}^{n_0+\lambda} (-1)^n x_n \right\| \leqslant B + \frac{\lambda}{2k_0} < B + 1$$

a contradiction. Hence (x_n) does not converge in X.

Now we claim that (x_n) has no weak Cauchy subsequence. The result will follow from the claim by H. P. Rosenthal's fundamental result [6]. It need only be proved that (x_n) is not weak Cauchy, because from Proposition 7 every subsequence of (x_n) is also an ℓ_1-sequence. Let (x_n) be weak Cauchy. Since (x_n) does not converge in X, there is an $\varepsilon > 0$ and a subsequence (y_λ) of (x_n) such that $\|y_{2\lambda-1} - y_{2\lambda}\| > \varepsilon$ for every $\lambda \in \mathbf{N}$. Let $u_\lambda = y_{2\lambda-1} - y_{2\lambda}$ for every $\lambda \in \mathbf{N}$. Then the sequence (u_λ) is weakly null and $\|u_\lambda\| > \varepsilon$ for every $\lambda \in \mathbf{N}$. Hence from the Bessaga-Pelczynski selection principle [1], (u_λ) has a basic subsequence (u_{λ_μ}) which according to Proposition 7 is also an ℓ_1-sequence. Thus (u_{λ_μ}) is equivalent (using the usual notion) to the usual basis of ℓ_1 and weakly null, a contradiction. □

Proposition 10. *Let X be a Banach space and $1 < p < \infty$. every ℓ_p-sequence in X has a basic subsequence equivalent to the usual basis of ℓ_p.*

Proof. Let (x_n) be an ℓ_p-sequence in X for some $1 < p < \infty$. Then (x_n) is bounded and weakly null. Indeed, from Proposition 6 the identity mapping $I\colon \Sigma^{(x_n)} \to \ell_p$ is an isomorphism, hence (e_n) is bounded, weakly null in $\Sigma^{(x_n)}$ and a basis of $\Sigma^{(x_n)}$. Let $T\colon \Sigma^{(x_n)} \to X$ be the linear map defined by $T\Big(\sum\limits_{n=1}^{\infty} a_n e_n\Big) = \sum\limits_{n=1}^{\infty} a_n x_n$. Then T is well defined and bounded. Hence (x_n) is bounded and weakly null sequence in X.

From the Bessaga-Pełczynski selection principle [1], (x_n) has a basic subsequence (x_{n_k}), which from Proposition 7 is also an ℓ_p-sequence. The sequence (x_{n_k}) is therefore equivalent to the usual basis of ℓ_p. □

Corollary 11. *A Banach space X has a subspace isomorphic to ℓ_p for some $1 \leqslant p < \infty$ (resp. to c_0) if and only if it contains an ℓ_p-sequence (resp. c_0-sequence).*

Proof. If X has a subspace isomorphic to ℓ_p (resp. c_0) then it has a basic sequence (x_n) equivalent to the usual basis of ℓ_p (resp. of c_0). Thus, (x_n) is an ℓ_p-(c_0-) sequence.

Let (x_n) be an ℓ_p-sequence (resp. c_0-sequence) in X. According to Proposition 8, 9 and 10 there is a basic subsequence (x_{n_k}) of (x_n) which is equivalent to the usual basis of ℓ_p (resp. of c_0).

Using the notions that we have defined, we are able to extend to arbitrary sequences M. Zippin's Theorem [7], which characterizes in a very strong sense the basic sequences which are equivalent to the usual basis of c_0 or ℓ_p. □

Lemma 12. *Let X be a Banach space and (x_n) a sequence of a non zero vectors in X. For every $(a_n) \in \Sigma^{(x_n)}$ there is a sequence of signs (∂_n) such that*

$$\left|\!\left|\!\left| \sum_{i=1}^{n} \partial_i a_i e_i \right|\!\right|\!\right| = \left\| \sum_{i=1}^{n} \partial_i a_i x_i \right\|$$

holds for every $n \in \mathbb{N}$.

Proof. Firstly, $|\!|\!| a_1 e_1 |\!|\!| = \|a_1 x_1\|$, hence we may set $\partial_1 = 1$. Suppose that the signs $\partial_1, \partial_2, \ldots, \partial_k$ have been chosen such that

$$\left|\!\left|\!\left| \sum_{i=1}^{n} \partial_i a_i e_i \right|\!\right|\!\right| = \left\| \sum_{i=1}^{n} \partial_i a_i x_i \right\| \qquad \text{holds for every } n = 1, 2, \ldots, k.$$

If $\left\| \sum_{i=1}^{k} a_i \partial_i x_i \right\| \leqslant \left\| \sum_{i=1}^{k} a_i \partial_i x_i + a_{k+1} x_{k+1} \right\|$, then set $\partial_{k+1} = 1$. In the other case set $\partial_{k+1} = -1$, because if $\left\| \sum_{i=1}^{k} a_i \partial_i x_i + a_{k+1} x_{k+1} \right\| < \left\| \sum_{i=1}^{k} a_i \partial_i x_i \right\|$ then for every $f \in X^*$ such that $\|f\| = 1$ and $\left\| \sum_{i=1}^{k} a_i \partial_i x_i \right\| = \sum_{i=1}^{k} a_i \partial_i f(x_i)$ we have $f(a_{k+1} x_{k+1}) < 0$ and so

$$\left\| \sum_{i=1}^{k} a_i \partial_i x_i - a_{k+1} x_{k+1} \right\| \geqslant \sum_{i=1}^{k} a_i \partial_i f(x_i) - a_{k+1} x_{k+1} f(x_{k+1}) > \left\| \sum_{i=1}^{k} a_i \partial_i x_i \right\|$$

$$= \left\| \sum_{i=1}^{k} a_i \partial_i e_i \right\|. \quad \text{Hence we have the equality} \quad \left\| \sum_{i=1}^{k+1} a_i \partial_i x_i \right\| = \left\| \sum_{i=1}^{k+1} a_i \partial_i x_i \right\|$$

□

Theorem 13. *Let X be a Banach space. A normalized sequence (x_n) in X is an ℓ_p-sequence for some $1 \leqslant p < \infty$ or a c_0-sequence if and only if (x_n) is equivalent to any of its blocks (u_k) with $u_k = \sum_{i=p_k}^{q_k} a_i x_i$, $a_i \in \mathbf{R}$ and $p_k \leqslant q_k < p_{k+1}$ for every $k \in \mathbf{N}$, such that $0 < \inf_k \|u_k\|$ and $\sup_k \left\| \sum_{i=p_k}^{q_k} a_i e_i \right\| < \infty$.*

P r o o f. That an ℓ_p-sequence or a c_0-sequence is equivalent to any such block sequence has been shown in Proposition 7.

For the converse, according to Proposition 6 it need be proved that the unit vector sequence (e_n) in $\Sigma^{(x_n)}$ is equivalent either to the usual basis of ℓ_p for some $1 \leqslant p < \infty$ or the usual basis of c_0. Using M. Zippin's Theorem [7] is need only be proved that (e_n) is equivalent to each of its normalized blocks.

First notice that for every choice of signs (∂_n) the sequence (x_n) is equivalent to $(\partial_n x_n)$. Hence the sequence (e_n) is equivalent to $(\partial_n e_n)$ in $\Sigma^{(x_n)}$ according to Proposition 7. Thus (e_n) is an unconditional basic sequence in $\Sigma^{(x_n)}$.

Let $v_k = \sum_{i=p_k}^{q_k} a_i e_i$, where $a_i \in \mathbf{R}$, $p_k \leqslant q_k < p_{k+1}$ and $\|v_k\| = 1$ for every $k \in \mathbf{N}$. From Lemma 12 there are signs ∂_i such that $\left\| \sum_{i=p_k}^{q_k} a_i \partial_i e_i \right\| = \left\| \sum_{i=p_k}^{q_k} a_i \partial_i x_i \right\|$ holds for every $k \in \mathbf{N}$. Set $u_k = \sum_{i=p_k}^{q_k} a_i x_i$, $v'_k = \sum_{i=p_k}^{q_k} a_i \partial_i e_i$ and $u'_k = \sum_{i=p_k}^{q_k} a_i \partial_i x_i$ for every $k \in \mathbf{N}$. Since (e_n) is an unconditional basic sequence in $\Sigma^{(x_n)}$, there is $M > 0$ such that $\frac{1}{M} \leqslant \|u'_k\| = \|v'_k\| \leqslant M$ holds for every $k \in \mathbf{N}$. By the hypothesis we get that (u'_k) is equivalent to (x_n). According to Lemma 4 the sequence (u'_k) is also equivalent to (v'_k). Hence (v'_k) is equivalent to (e_n) in $\Sigma^{(x_n)}$. By the unconditionality of (e_n) we have that (v_k) is equivalent to (e_n). This completes the proof. □

Corollary 14. *Let X be a Banach space and (x_n) a normalized sequence in X. If (x_n) is equivalent to all its normalized blocks then (x_n) has a basic subsequence equivalent to the usual basis of c_0 or of some ℓ_p for $1 \leqslant p < \infty$.*

For the next two Propositions, we are concerned with the case where $\Sigma^{(x_n)}$ is reflexive.

Proposition 15. *Let X be a Banach space and (x_n) a sequence in X such that $0 < \inf_n \|x_n\|$. If $\Sigma^{(x_n)}$ is reflexive, then (x_n) has a basic subsequence (x_{n_k}) such that $[x_{n_k}]_{k=1}^{\infty}$ is a reflexive subspace of X.*

Proof. If $\Sigma^{(x_n)}$ is reflexive, then (e_n) is weakly null in $\Sigma^{(x_n)}$ and a basis of $\Sigma^{(x_n)}$. The linear map $T\colon \Sigma^{(x_n)} \to X$ defined by $T\Big(\sum_{n=1}^{\infty} a_n e_n\Big) = \sum_{n=1}^{\infty} a_n x_n$ is bounded, hence (x_n) is weakly null sequence in X.

It follows from [1] that (x_n) has a basic subsequence (x_{n_k}), which from Lemma 4 is equivalent to the basic sequence (e_{n_k}) in $\Sigma^{(x_n)}$. Since $[e_{n_k}]_{k=1}^{\infty}$ is a reflexive subspace of $\Sigma^{(x_n)}$, the vector space $[x_{n_k}]_{k=1}^{\infty}$ is also reflexive. □

Lemma 16. *Let X be an infinite dimensional Banach space, (x_n) a sequence in X such that $0 < \inf_n \|x_n\|$ and $X = [x_n]_{n=1}^{\infty}$ and Y an infinite dimensional closed subspace of X with a basis (y_n). Then there is a basic block sequence (u_k) of (x_n) which is equivalent to a bounded block sequence (w_k) of (y_n).*

Proof. We construct the sequences (u_k), (w_k) inductively. Pick $z_1 \in Y$ with $\|z_1\| = 1$. Let $w_1 = \sum_{i=1}^{\tau_1} \mu_i y_i$ with $\tau_1 \in \mathbf{N}$ and $\mu_i \in \mathbf{R}$, such that $\|z_1 - w_1\| < \frac{1}{8K}$, where K is the basis constant of (y_n). Let now $u_1 = \sum_{i=p_1}^{q_1} \lambda_i x_i$ with $p_1 \leqslant q_1 \in \mathbf{N}$ and $\lambda_i \in \mathbf{R}$, such that $\|z_1 - u_1\| < \frac{1}{8K}$. Thus we have $\|w_1 - u_1\| < \frac{1}{4K}$.

Now $Y \cap [x_n]_{n=q_1+1}^{\infty}$ is an infinite dimensional subspace of Y, hence $[x_n]_{n=q_1+1}^{\infty} \cap [y_n]_{n=\tau_1+1}^{\infty}$ is also an infinite dimensional subspace of Y. We pick $z_2 \in [x_n]_{n=q_1+1}^{\infty} \cap [y_n]_{n=\tau_1+1}^{\infty}$ with $\|z_2\| = 1$. Let $w_2 = \sum_{i=\tau_1+1}^{\tau_2} \mu_i y_i$ with $\tau_1 < \tau_2 \in \mathbf{N}$ and $\mu_i \in \mathbf{R}$, such that $\|z_2 - w_2\| < \frac{1}{4^2 \cdot K \cdot 2}$. Also let $u_2 = \sum_{i=p_2}^{q_2} \lambda_i x_i$ with $q_1 < p_2 \leqslant q_2 \in \mathbf{N}$ and $\lambda_i \in \mathbf{R}$, such that $\|z_2 - u_2\| < \frac{1}{4^2 \cdot K \cdot 2}$. Then we have $\|u_2 - w_2\| < \frac{1}{4^2 \cdot K}$. We continue in an obvious manner.

The sequence (w_k) obtained in this way is a block basis of (y_n) and $\frac{3}{4} < \|w_k\| < \frac{5}{4}$ holds for every $k \in \mathbf{N}$. Since $\sum_{k=1}^{\infty} \|w_k - u_k\| < \frac{1}{3K}$ it follows by [3] that (u_k) is a basic sequence and equivalent to (w_k). □

Proposition 17. *Let X be a Banach space and (x_n) a sequence in X such that $0 < \inf_n \|x_n\|$ and $X = [x_n]_{n=1}^{\infty}$. If $\Sigma^{(x_n)}$ is reflexive then X is somewhat reflexive space.*

P r o o f. Let Z be a nonreflexive subspace of X. Then Z contains a basic sequence (y_n) and let $Y = [y_n]_{n=1}^{\infty}$. According to Lemma 16 there is a block sequence (w_k) of (y_n) which is equivalent to a basic block sequence (u_k) of (x_n). If $u_k = \sum_{i=p_k}^{q_k} a_i x_i$ with $a_i \in \mathbf{R}$ and $p_k \leqslant q_k < p_{k+1}$ for every $k \in \mathbf{N}$, then by Lemma 4 the basic sequence (u_k) is equivalent to the basic sequence (v_k) in $\Sigma^{(x_n)}$, where $v_k = \sum_{i=p_k}^{q_k} a_i e_i$ for every $k \in \mathbf{N}$. Since $[v_k]_{k=1}^{\infty}$ is a reflexive subspace of $\Sigma^{(x_n)}$, the subspaces $[u_k]_{k=1}^{\infty}$ and $[w_k]_{k=1}^{\infty}$ of X are also reflexive. Hence Z contains a reflexive subspace with a basis. □

Definition 18. A sequence (x_n) in a Banach space X is called *unconditional* iff whenever a series $\sum_n a_n x_n$, $a_n \in \mathbf{R}$ converges in X the convergence is unconditional.

Theorem 19. *Let X be an infinite dimensional Banach space, (x_n) an unconditional sequence in X and $X = [x_n]_{n=1}^{\infty}$. Then every infinite dimensional closed subspace Y of X has an unconditional basic sequence.*

P r o o f. Let (y_n) be a basic sequence in Y. According to Lemma 16 there is basic block sequence (w_k) of (y_n) which is equivalent to a basic block sequence (u_k) of (x_n). Since (x_n) is unconditional, (u_k) is also unconditional. Hence, (w_k) is an unconditional basic sequence in Y. □

Proposition 20. *Let X be an infinite dimensional Banach space and (x_n) an unconditional sequence in X such that $0 < \inf_n \|x_n\|$ and $X = [x_n]_{n=1}^{\infty}$. Then X has a subspace isomorphic to ℓ_p for some $1 \leqslant p < \infty$ (resp. to c_0) if and only if the subspace $[e_n]_{n=1}^{\infty}$ of $\Sigma^{(x_n)}$ contains a subspace isomorphic to ℓ_p (resp. to c_0).*

P r o o f. Let Y is a subspace of X isomorphic to ℓ_p, then from Lemma 16 there is a basic block sequence (u_k) of (x_n) which is equivalent to the usual basis of ℓ_p. By Lemma 4 there is a block sequence (v_k) of (e_n) in $\Sigma^{(x_n)}$ which is equivalent to (u_k). Hence (v_k) is equivalent to the usual basis of ℓ_p.

Conversely, if $[e_n]_{n=1}^{\infty}$ has a subspace isomorphic to ℓ_p, then there is a normalized block sequence (v_k) of (e_n) equivalent to the usual basis of ℓ_p. Let $v_k = \sum_{i=p_k}^{q_k} a_i e_i$ with $a_i \in \mathbf{R}$ and $p_k \leqslant q_k < p_{k+1}$ for every $k \in \mathbf{N}$. By Lemma 12 there are sings ∂_i such that

$$\left\|\left|\sum_{i=p_k}^{q_k} a_i \partial_i e_i\right|\right\| = \left\|\sum_{i=p_k}^{q_k} a_i \partial_i x_i\right\|$$

holds for every $k \in \mathbf{N}$.

Set $v'_k = \sum_{i=p_k}^{q_k} a_i \partial_i e_i$ and $u_k = \sum_{i=p_k}^{q_k} a_i \partial_i x_i$ for every $k \in \mathbf{N}$. Since (e_n) is an unconditional basic sequence in $\Sigma^{(x_n)}$, the sequence (v'_k) is equivalent to (v_k) and there is $M > 0$ such that $\frac{1}{M} \leqslant \|u_k\| = \|v'_k\| \leqslant M$ holds for every $k \in \mathbf{N}$. By Lemma 4 the sequence (u_k) is equivalent to (v'_k), hence (u_k) is an ℓ_p-sequence in X. The result follows from Corollary 11.

The proof for c_0 is similar. □

From Proposition 20 we get immediately the following results which extend R. C. James Theorem [2] related to unconditional basic sequences to arbitrary unconditional sequences.

Proposition 21. *Let X be an infinite dimensional Banach space and (x_n) an unconditional sequence in X such that $0 < \inf_n \|x_n\|$ and $X = [x_n]_{n=1}^{\infty}$. Then:*

(i) *X has no subspace isomorphic to ℓ_1 if and only if the unit vector basic sequence (e_n) in $\Sigma^{(x_n)}$ is shrinking*

(ii) *X has no subspace isomorphic to c_0 if and only if the basic sequence (e_n) in $\Sigma^{(x_n)}$ is boundedly complete.*

Proof. Since (x_n) is an unconditional sequence in X the basic sequence (e_n) in $\Sigma^{(x_n)}$ is also unconditional. From [2] we have that (e_n) is shrinking if and only if the subspace $[e_n]_{n=1}^{\infty}$ of $\Sigma^{(x_n)}$ has no subspace isomorphic to ℓ_1 and (e_n) is boundedly complete if and only if $[e_n]_{n=1}^{\infty}$ has no subspace isomorphic to c_0. The result is now obvious from Proposition 20. □

Theorem 22. *Let X be an infinite dimentioanl Banach space and (x_n) an unconditional sequence in X such that $0 < \inf_n \|x_n\|$ and $X = [x_n]_{n=1}^{\infty}$. Then X contains a subspace isomorphic to ℓ_1, c_0 or X is somewhat reflexive.*

Proof. If X has no subspace isomorphic to c_0 or ℓ_1 then from Proposition 20 we have that the unit vector basic sequence (e_n) in $\Sigma^{(x_n)}$ is shrinking and boundedly complete. Hence $[e_n]_{n=1}^{\infty} = \Sigma^{(x_n)}$ and $\Sigma^{(x_n)}$ is reflexive. According to Proposition 17 X is somewhat reflexive. □

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Author's address: University of Athens, Department of Mathematics, Section of Mathematical Analysis and its Applications, Panepistemiopolis, 15784 Athens, Greece.