Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

ON EQUALIZERS IN THE CATEGORY OF FRAMES WITH WEAKLY OPEN HOMOMORPHISMS

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(Received April 11, 1991, in revised form January 9, 1992)

The category of frames with weakly open homomorphisms, we will denote it by $\mathrm{Frm_{wo}}$, was introduced and investigated by *B. Banaschewski* and *A. Pultr* (cf. [1]) in connection with the study of booleanization. The term "weakly open" is motivated by the fact that a frame homomorphism associated with a continuous mapping f of a topological space possesses this property if and only if for each non-empty open set U in this space we have $\operatorname{int} f(U) \neq \emptyset$. As the category of frames with weakly open homomorphisms contains the category of Boolean frames as a reflective subcategory (cf. [1]), it cannot be cocomplete. There is no obvious obstruction to completeness. The existence of products is easily seen. In this paper we investigate the structure of equalizers in the category $\mathrm{Frm_{wo}}$ and show that there are couples of morphisms which fail to have them.

For the fundamental properties of frames the reader is referred to [2].

Recall that a *frame* is a complete lattice L in which $a \wedge \bigvee\{a_i \mid i \in I\} = \bigvee\{a \wedge a_i \mid i \in I\}$ holds for any elements $a \in L$, $a_i \in L$ $(i \in I)$. Every frame is relatively pseudocomplemented and so pseudocomplemented. For the sake of clarity, when denoting frames we add an associated pseudocomplementation symbol, for example $(F, *)$, whenever it is necessary. Here is a list of some properties of pseudocomplementation:

(P1) $$a^{***} = a^*$$

(P2) $$(a \vee b)^* = a^* \wedge b^*$$

(P3) $$(a \wedge b)^{**} = a^{**} \wedge b^{**}$$

(P4) $$(a \vee a^*)^{**} = 1$$

(P5) $$(a \vee a^*)^* = 0$$

(P6) $$a^* = b^{**} \iff a^{**} = b^*$$

(P7) $$a^{**} \wedge a^* = 0$$

(P8) $$0^* = 1,\ 1^* = 0,\ 0^{**} = 0,\ 1^{**} = 1$$

(P9) $$a \leqslant b \Longrightarrow b^* \leqslant a^*.$$

A lattice homomorphism $f\colon E \to F$ of a frame E to a frame F is said to be a *frame homomorphism* if $f\left(\bigvee\{a_i \mid i \in I\}\right) = \bigvee\left\{f(a_i) \mid i \in I\right\}$ holds for any elements $a_i \in E$ $(i \in I)$. A frame homomorphism $f\colon E \to F$ of a frame $(E, \bullet)$ to a frame $(F, *)$ is *weakly open* if $f(a^{\bullet\bullet}) \leqslant f(a)^{**}$ for any element $a \in E$. A subframe $(A, \bullet)$ of a frame $(F, *)$ is *weakly open* in F if the canonical embedding is a weakly open homomorphism, i.e. $a^{\bullet\bullet} \leqslant a^{**}$ for any element $a \in A$.

Proposition 1. *A subframe A of a frame F is weakly open in F if and only if for any $a \in A$ there exists $b \in A$ such that $a^* = b^{**}$.*

Proof. Assume that $(A, \bullet)$ is weakly open in $(F, *)$. Let a be an element of A. We have $1 = (a \vee a^\bullet)^{\bullet\bullet} \leqslant (a \vee a^\bullet)^{**}$ by (P4), hence $0 = (a \vee a^\bullet)^* = a^* \wedge (a^\bullet)^*$ by (P8), (P6) and (P2). Since $a^\bullet \leqslant a^*$, we have $a^{**} \leqslant (a^\bullet)^*$ according to (P9). Summing up, $a^{**} = (a^\bullet)^*$. Put $b := a^\bullet$ and apply (P6). Conversely, assume that for any $a \in A$ there exists $b \in A$ such that $a^* = b^{**}$. Let a be an element of A and b the associated element such that $a^* = b^{**}$. Then $0 = a \wedge a^* = a \wedge b^{**} \geqslant a \wedge b$, hence $b \leqslant a^\bullet$ and therefore $a^{\bullet\bullet} \leqslant b^\bullet \leqslant b^* = a^{**}$. □

For a sublattice A of a frame $(F, *)$ denote $M(A) := \{a \in A \mid \exists b \in A,\ a^* = b^{**}\}$.

Remark. We can rewrite Proposition 1 using $M(A)$: A *subframe A of a frame F is weakly open in F if and only if $A = M(A)$.*

Lemma 1. *Let A be a sublattice of a frame F. Then $M(A)$ is a sublattice of F.*

Proof. Let $a_1, a_2 \in M(A)$, let $b_1, b_2 \in A$ be those elements for which $a_k^* = b_k^{**}$ $(k = 1, 2)$. In view of (P6) also $b_1, b_2 \in M(A)$. Then $(a_1 \wedge a_2)^* = (a_1 \wedge a_2)^{***} = (a_1^{**} \wedge a_2^{**})^* = (b_1^* \wedge b_2^*)^* = (b_1 \vee b_2)^{**}$. We have used (P1), (P3) and (P2). Similarly, $(b_1 \wedge b_2)^* = (a_1 \vee a_2)^{**}$. Inasmuch as $a_1 \wedge a_2,\ a_1 \vee a_2,\ b_1 \wedge b_2,\ b_1 \vee b_2 \in A$, we obtain $a_1 \wedge a_2,\ a_1 \vee a_2 \in M(A)$. □

Lemma 2. *The operator M is order-preserving and idempotent on the set of all sublattices of the frame F ordered by inclusion, that is $A \subseteq B \Longrightarrow M(A) \subseteq M(B)$ and $MM(A) = M(A)$.*

Proof. The proof is straightforward. □

As an immediate consequence of this lemma and Proposition 1 we obtain

Lemma 3. (a) *Let A be a subframe of a frame F. If $M(A)$ is a subframe of F, then it is weakly open in F.*

(b) *Let A be a subframe of a frame F, and let B be a subframe of A weakly open in F. Then $B \subseteq M(A)$.*

(c) *Let A be a finite subframe of a frame F. Then $M(A)$ is a weakly open subframe of F.*

R e m a r k . If A is an infinite subframe of F, then $M(A)$ need not be a subframe of F.

Lemma 4. *Let A be a subframe of a frame F. Then the largest subframe of A weakly open in F, if it exists, equals $M(A)$.*

P r o o f . Let A be a subframe of a frame F, and let B be the largest subframe of A weakly open in F. According to Lemma 3, $B \subseteq M(A)$. Now let $a \in M(A)$. By definition, there is an element $b \in A$ such that $a^* = b^{**}$. By (P6) we obtain $a^{**} = b^*$, and so $b \in M(A)$. $C := \{a,\ b,\ a \vee b,\ 0,\ 1\}$ is a subframe of A weakly open in F since $a^{**} = b^*$, $b^{**} = a^*$, $0^{**} = 1^*$, $1^{**} = 0^*$ and $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^* \wedge a^{**})^* = 0^*$ by (P8), (P2) and (P7). Hence $a \in C \subseteq B$. Consequently, $M(A) \subseteq B$. □

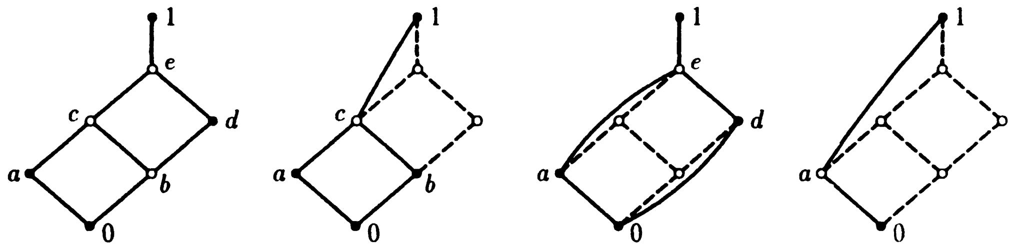
Proposition 2. *Let A be a subframe of a frame F. Then the following conditions are equivalent:*
(i) *there exists the largest subframe of A weakly open in F;*
(ii) *$M(A)$ is a subframe F;*
(iii) *$M(A)$ is the largest subframe of A weakly open in F.*

P r o o f . (i) $\Longrightarrow$ (iii) has been just proved.
(iii) $\Longrightarrow$ (ii) follows a fortiori.
(ii) $\Longrightarrow$ (i) by Lemma 3. □

It is well-known that substructures of an algebraic structure (subalgebras, sublattices, subframes) form a topped intersection structure. This is not the case for weakly open subframes.

Proposition 3. *Weakly open subframes of a frame fail to form an intersection structure. Even finite intersections of finite weakly open subframes need not be weakly open.*

P r o o f . Here is a counterexample:



$(F, *)$ $(A, \bullet)$ $(B, \circ)$ $(A \cap B, \square)$

$$
\begin{array}{lll}
0^{\bullet\bullet} = 0 \leqslant 0 = 0^{**}, & 0^{\circ\circ} = 0 \leqslant 0 = 0^{**}, & \\
a^{\bullet\bullet} = a \leqslant a = a^{**}, & a^{\circ\circ} = a \leqslant a = a^{**}, & a^{\Box\Box} = 1 \not\leqslant a = a^{**}, \\
b^{\bullet\bullet} = b \leqslant d = b^{**}, & d^{\circ\circ} = d \leqslant d = d^{**}, & \\
c^{\bullet\bullet} = 1 \leqslant 1 = c^{**}, & e^{\circ\circ} = 1 \leqslant 1 = e^{**}, & \\
1^{\bullet\bullet} = 1 \leqslant 1 = 1^{**}, & 1^{\circ\circ} = 1 \leqslant 1 = 1^{**}. &
\end{array}
$$

$\Box$

The problem of finding equalizers is a bit more complicated. We will take advantage of the following decomposition lemma, formulated also in [1]. As this article is not yet published, we present a proof.

Lemma 5. (Decomposition lemma.) *Let $h\colon E \to F$ be a weakly open homomorphism of a frame E to a frame F. Then $h(E)$ is a weakly open subframe in F and the induced homomorphism of E onto $h(E)$ is weakly open.*

Proof. Since h is a frame homomorphism, $h(E)$ is a subframe of F. First, we have to prove that $\big(h(E), \Box\big)$ is weakly open in $(F, *)$. Let $x \in h(E)$, for instance $x = h(a)$, $a \in E$. Define $b := \bigvee \{y \in E \mid h(a \wedge y) = 0\}$. Clearly $h(a) \wedge h(b) = h(a \wedge b) = 0$, hence $h(b) \leqslant h(a)^*$, and $h(b)^* \geqslant h(a)^{**}$ by (P9). Furthermore, we have $(a \vee b)^\bullet = 0$ because $(a \vee b) \wedge c = 0$ implies $a \wedge c = 0$, which yields $h(a \wedge c) = 0$, hence $c \leqslant b \leqslant a \vee b$, and finally $c = 0$. Therefore, by (P8), $(a \vee b)^{\bullet\bullet} = 1$ and so $h(a \vee b)^{**} \geqslant h\big((a \vee b)^{\bullet\bullet}\big) = h(1) = 1$. Consequently, $0 = h(a \vee b)^* = \big(h(a) \vee h(b)\big)^* = h(a)^* \wedge h(b)^*$ by (P8) and (P2). Hence $h(b)^* \leqslant h(a)^{**}$. Summing up, $h(a)^* = h(b)^{**}$, and also so $h(E)$ is weakly open in F. Now we need to check that $h(a^{\bullet\bullet}) \leqslant h(a)^{\Box\Box}$. Since $h(a)^\Box \leqslant h(a)^*$, we obtain $h(a^{\bullet\bullet}) \wedge h(a)^\Box \leqslant h(a)^{**} \wedge h(a)^* = 0$, and consequently $h(a^{\bullet\bullet}) \leqslant h(a)^{\Box\Box}$. $\Box$

Let $f\colon F \to G$, $g\colon F \to G$ be weakly open homomorphism of a frame $(F, *)$ to a frame $(G, \circ)$. We define $E(f,g) := \{x \in F \mid f(x) = g(x)\}$.

Lemma 6. *$E(f,g)$ is a subframe of F.*

Proof. Clearly $0, 1 \in E(f,g)$. Let $x, y \in E(f,g)$. Then $f(x \wedge y) = f(x) \wedge f(y) = g(x) \wedge g(y) = g(x \wedge y)$. Let $x_i \in E(f,g)$ $(i \in I)$. Then $f\big(\bigvee\{x_i \mid i \in I\}\big) = \bigvee\{f(x_i) \mid i \in I\} = \bigvee\{g(x_i) \mid i \in I\} = g\big(\bigvee\{x_i \mid i \in I\}\big)$. $\Box$

Lemma 7. *Let $h\colon E \to F$ be an equalizer of f, g. Then $h(E)$ is a subframe of $E(f,g)$ weakly open in F and the canonical embedding of $h(E)$ into F is also an equalizer of f, g.*

Proof. By Lemma 5, $h(E)$ is a weakly open subframe in F. By Lemma 6, $E(f,g)$ is a subframe of F. Since h is an equalizer of f, g, we have $f\big(h(e)\big) = g\big(h(e)\big)$ for each $e \in E$, and therefore $h(e) \in E(f,g)$. To sum up, the subframe $h(E)$ of F is a subset of the subframe $E(f,g)$ of F, hence $h(E)$ is a subframe of $E(f,g)$.

According to Lemma 5, h can be decomposed into a surjective weakly open homomorphism $h'\colon E \to h(E)$ and a weakly open embedding $\hat{h}\colon h(E) \to F$. We have $h = \hat{h} \cdot h'$. Let $d\colon D \to F$ be a weakly open homomorphism such that $fd = gd$. Inasmuch as h is an equalizer of f, g, there exists a unique weakly open homomorphism $\bar{d}\colon D \to E$ such that $d = h\bar{d}$. Then also $d = \hat{h}(h'\bar{d})$. Uniqueness should be proved. Supposing $d = \hat{h}d'$, we obtain $d' = h'\bar{d}$ because $d'(x) = \hat{h}(d'(x)) = \hat{h}\big(h'(\bar{d}(x))\big) = h'\big(\bar{d}(x)\big)$. It follows that $\hat{h}$ is an equalizer of f, g. $\square$

Lemma 8. *Let the canonical embedding of $E \subseteq E(f,g)$ into F be an equalizer of f, g. Then $E = M\big(E(f,g)\big)$.*

Proof. The assumptions imply that E is the largest subframe of $E(f,g)$ weakly open in F. By Lemma 4, $E = M\big(E(f,g)\big)$. $\square$

Lemma 9. *Let $(A,\square)$ be a subframe of a frame $(F,*)$, let $h\colon (E,\bullet) \to F$ be a weakly open homomorphism such that $h(E) \subseteq A$. Then the induced homomorphism $\bar{h}\colon E \to A$ is weakly open.*

Proof. We have $h(a^{\bullet\bullet}) \leqslant h(a)^{**}$ and $h(a)^{\square} \leqslant h(a)^{*}$. Then $h(a^{\bullet\bullet}) \wedge h(a)^{\square} \leqslant h(a)^{**} \wedge h(a)^{*} = 0$, and consequently, $\bar{h}(a^{\bullet\bullet}) = h(a^{\bullet\bullet}) \leqslant h(a)^{\square\square} = \bar{h}(a)^{\square\square}$. $\square$

Proposition 4. *The following conditions are equivalent:*
(i) *there exists an equalizer of f, g;*
(ii) *$M\big(E(f,g)\big)$ is a subframe of F;*
(iii) *the canonical embedding of $M\big(E(f,g)\big)$ into F is an equalizer of f, g.*

Proof. (i) $\Longrightarrow$ (iii) by Lemmas 7 and 8, implications (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii) are obvious.
(ii) $\Longrightarrow$ (iii): Let $M\big(E(f,g)\big)$ be a subframe of F. It is weakly open in F by Lemma 3. Let $d\colon D \to F$ be a weakly open homomorphism such that $fd = gd$. Then clearly $d(D) \subseteq E(f,g)$ and $d(D)$ is weakly open in F by Lemma 5. According to Lemma 3, $d(D) \subseteq M\big(E(f,g)\big)$. By Lemma 9, the induced homomorphism $\bar{d}\colon D \to M\big(E(f,g)\big)$ is weakly open. $\square$

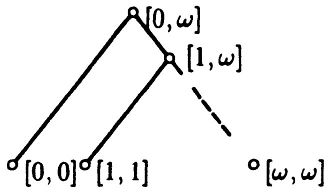
Corollary. *If $E(f,g)$ is finite, the canonical embedding of $M\big(E(f,g)\big)$ into F is an equalizer of f, g.*

Proof. The proof follows from Lemma 3. $\square$

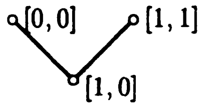
Remark. We have shown that the equalizer of f, g is exactly (up to isomorphism) the canonical embedding of the largest subframe of $E(f,g)$ weakly open in F.

Proposition 5. *The category* $\mathrm{Frm}_{\mathrm{wo}}$ *fails to have equalizers.*

Proof. Let ω be the least infinite ordinal. Let $(P, \leqslant)$ and $(Q, \leqslant)$ be the subsets of $\overline{\omega + 1} \times (\omega + 1)$ defined by $P := \{[n, \omega] \mid n \in \omega\} \cup \{[n, n] \mid n \in \omega + 1\}$ and $Q := \{[0,0], [1,1], [1,0]\}$ with the induced order.



$(P, \leqslant)$



$(Q, \leqslant)$

Let F and G be the sets of all down-sets in $(P, \leqslant)$ and $(Q, \leqslant)$, respectively. Then F and G are complete lattices of sets, and therefore frames with respect to the inclusion of sets. Denote them by $(F, *)$ and $(G, \circ)$. Notice that for any $Y \in G$, $Y \neq \emptyset$ implies $Y^{\circ\circ} = Q$. For $c \in \{0, 1\}$ and $X \in F$ put

$$f_c(X) = \begin{cases} Q & \text{if } (\exists n \in \omega)\ [n, \omega] \in X \text{ (then of course } [\omega, \omega] \in X), \\ \{[c, c], [1, 0]\} & \text{if } [\omega, \omega] \in X \ \&\ (\forall n \in \omega)\ [n, \omega] \notin X, \\ \emptyset & \text{otherwise} \quad (\text{i.e. } [\omega, \omega] \notin X). \end{cases}$$

It is obvious that f_c $(c \in \{0, 1\})$ are frame homomorphisms. Now let $X \in F$. If $[\omega, \omega] \in X$, then $f_c(X) \neq \emptyset$, hence $f_c(X^{**}) \subseteq Q = f_c(X)^{\circ\circ}$. If $[\omega, \omega] \notin X$, then $[\omega, \omega] \in X^*$, thus $[\omega, \omega] \notin X^{**}$, and consequently $f_c(X^{**}) = \emptyset \subseteq f_c(X)^{\circ\circ}$. We have just proved that f_c $(c \in \{0, 1\})$ are weakly open. Now we are able to apply Proposition 4. For any $n \in \omega$, we have $\{[n, n]\} \in M(E(f_0, f_1))$ since $\{[n, n]\}^* \in E(f_0, f_1)$ and $\{[n, n]\}^{**} = \{[n, n]\}$, but $\left(\bigcup\left\{\{[n, n]\} \mid n \in \omega\right\}\right)^* = \{[n, n] \mid n \in \omega\}^* = \{[\omega, \omega]\}$, and the only element $X \in F$ with $X^{**} = \{[\omega, \omega]\}$ is $X = \{[\omega, \omega]\} \notin E(f_0, f_1)$. □

Acknowledgement. I am grateful to *A. Pultr* for introducing me to this area.

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[1] *B. Banaschewski, A. Pultr*: Booleanization as reflection, to appear.
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