Ján Jakubík
On convexities of lattices

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 2, 325–330

Persistent URL: http://dml.cz/dmlcz/128333

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

ON CONVEXITIES OF LATTICES

JÁN JAKUBÍK, Košice*)

(Received January 7, 1991)

At the problem Session of the Conference on General Algebra (Krems, 1988) E. Fried proposed a problem concerning the "number" of convexities of lattices (cf. [7], p. 255). In the present paper a solution of this problem is given.

Convex sublattices of lattices were investigated by M. Kolibiar [6]. Systems of convex subsets of partially ordered sets were study by M. K. Bennet and G. Birkhoff (cf. [1], [2], [3], [5]).

1.

A nonempty class of lattices will be said to be a *convexity* if it is closed under homomorphic images, convex sublattices and direct products.

This notion was introduced by Fried [7]. He proposed the following question:

What is the "number" of convexities?

Next, he expressed the conjecture that there is no such cardinal. The validity of this conjecture will be proved below.

Let us denote by $\mathcal{C}$ the collection of all convexities. This collection will be considered to be partially ordered by inclusion.

For a subclass X of the class $\mathcal{L}$ of all lattices we denote by

HX – the class of all homomorphic images of elements of X,

CX – the class of all convex sublattices of elements of X,

PX – the class of all direct products of elements of X.

Lemma 1.1. (Cf. Fried [7].) *Let $\emptyset \neq X \subseteq \mathcal{L}$. Then $HCPX$ is the least convexity containing X.*

*) Supported by SAV grant 362/91

In view of 1.1 the convexity $H\ C\ P\ X$ will be said to be generated by X. If X is a one-element class, then $H\ C\ P\ X$ will be called principal. A convexity is said to be modular if all lattices belonging to it are modular. We denote by C_{pm} the collection of all convexities which are principal and modular.

If Ω is a congruence relation on a lattice L and $x \in L$, then we denote by L/Ω the corresponding factor lattice and by $x(\Omega)$ the element of L/Ω which contains x.

For a lattice L we denote by $P(L)$ the set of all pairs (u, v) of elements of L having the property that there exist distinct mutually incomparable elements $x_i \in L$ $(i = 1, 2, 3)$ such that $u < x_i < v$ for $i = 1, 2, 3$ and $\{u, v, x_1, x_2, x_3\}$ is a sublattice of L.

Lemma 1.2. *Let L be a modular lattice and let Ω be a congruence relation on L such that $u(\Omega) = v(\Omega)$ whenever $(u, v) \in P(L)$. Then the lattice L/Ω is distributive.*

Proof. By way of contradiction, assume that the lattice L/Ω fails to be distributive. Then there are elements x_1, x_2 and x_3 in L such that

(i) $x_1(\Omega)$, $x_2(\Omega)$ and $x_3(\Omega)$ are distinct,

(ii) $x_1(\Omega) \wedge x_2(\Omega) = x_1(\Omega) \wedge x_3(\Omega) = x_2(\Omega) \wedge x_3(\Omega)$,

(iii) $x_1(\Omega) \vee x_2(\Omega) = x_1(\Omega) \vee x_3(\Omega) = x_2(\Omega) \vee x_3(\Omega)$.

Let L_1 be the sublattice of L generated by the elements x_1, x_2 and x_3. Further, let Ω_1 be the congruence relation on L_1 which is induced by Ω. Let $L^{(3)}$ be the free modular lattice generated by the free generators x_1^0, x_2^0 and x_3^0. There exists a homomorphism φ of $L^{(3)}$ onto L_1 such that $\varphi(x_i^0) = x_i$ for $i = 1, 2, 3$. Hence there exists a congruence relation Ω_2 on $L^{(3)}$ having the property that there is an isomorphism ψ of $L^{(3)}/\Omega_2$ onto L_1/Ω_1 such that $\psi(x_i^0(\Omega_2)) = x_i(\Omega_1)$ for $i = 1, 2, 3$.

The conditions (i), (ii) and (iii) remain valid if Ω is replaced by Ω_1. Hence the same relations remain true if Ω and x_i are replaced by Ω_2 and x_i^0 $(i = 1, 2, 3)$; these modified conditions will also be denoted by (i), (ii) and (iii), respectively.

For the elements of $L^{(3)}$ we apply the same notation as in [4], Chap. III, §6 (with the distinction that we now have x_1^0, x_2^0 and x_3^0 instead of x, y and z).

In view of (ii) the relation

$$O(\Omega_2) = o(\Omega_2) \tag{1}$$

is valid. Analogously, (iii) implies that

$$i(\Omega_2) = I(\Omega_2) \tag{2}$$

holds.

Next, $(o,i) \in P(L^{(3)})$. If $\varphi(o) \neq \varphi(i)$, then $(\varphi(o),\varphi(i)) \in P(L_1)$ and hence $\varphi(o)(\Omega_1) = \varphi(i)(\Omega_1)$. Therefore

$$o(\Omega_2) = i(\Omega_2). \tag{3}$$

The relations (1), (2) and (3) give $O(\Omega_2) = I(\Omega_2)$, whence $x_1^0(\Omega_2) = x_2^0(\Omega_2) = x_3^0(\Omega_2)$. In view of the isomorphism ψ we obtain $x_1(\Omega_1) = x_2(\Omega_1) = x_3(\Omega_1)$, which contradicts the relation (i). □

Let α be a cardinal, $\alpha \geqslant \beta$. We denote by L_α the latice consisting of elements u, v, x_j $(j \in J)$, where $\operatorname{card} J = \alpha$, $u < x_j < v$, and $x_{j(1)}$ is incomparable with $x_{j(2)}$ whenever $j(1)$ and $j(2)$ are distinct element of J.

Lemma 1.3. *Let α and β be cardinals, $3 \leqslant \alpha < \beta$. Then L_α does not belong to $HCP\{L_\beta\}$.*

Proof. By way of contradiction, assume that L_α belongs to $HCP\{L_\beta\}$. Thus

(i) there exist lattices $A_i = \{u_i, v_i, x_k^i\}_{k\in K}$ $(i \in I)$ where $\operatorname{card} K = \beta$, $u_i < x_k^i < v_i$ for each $k \in K$, and each A_i is isomorphic to L_β;

(ii) there exist a convex sublatice B of $\prod\limits_{i\in I} A_i$ and a congruence relation Ω on B such that L_α is isomorphic to B/Ω.

Let φ be an isomorphic of L_α onto B/Ω. For each $t \in L_\alpha$ we denote $\varphi(t) = t^*$. The elements of L_α will be denoted as above.

Choose any $u' \in u^*$. There exists $v' \in v^*$ such that $u' < v'$. Let L be the interval $[u', v']$ of B and let Ω_1 be the congruence relation on L which is induced by Ω. Then L_a is isomorphic to L/Ω_1. Thus without loss of generality we can assume that $B = [u', v']$.

For $z \in \prod\limits_{i\in I} A_i$ and $i \in I$ let $z(A_i)$ be the component of z in the direct factor A_i. Similarly, for $Z \subseteq \prod\limits_{i\in I} A_i$ we denote $Z(A_i) = \{z(A_i) \colon z \in Z\}$. Since B is an interval of $\prod\limits_{i\in I} A_i$, we get

$$B = \prod_{i\in I} B(A_i).$$

If $P(B) = \emptyset$, then (since B is modular) we obtain that B is distributive, hence B/Ω is distributive as well, which is a contradiction. Thus $P(B) = \emptyset$. Hence we can choose $(a, b) \in P(B)$.

We have $[a, b] = \prod\limits_{i\in I}[a(A_i), b(A_i)]$. Put $I(1) = \{i \in I \colon \operatorname{card}[a(A_i), b(A_i)] \geqslant 2\}$. If $I(1) = \emptyset$, then the interval $[a, b]$ is distributive, which is a contradiction. Hence $I(1) \neq \emptyset$. Next, since A_i is isomorphic to L_β, for each $i \in I(1)$ the relations $a(A_i) =$

u_i and $b(A_i) = v_i$ are valid. We construct elements $\bar{u}$ $\bar{v}$, $\bar{x}_k$ $(k \in K)$ of $\prod\limits_{i \in I} A_i$ as follows:

(i) for $i \in I \setminus I(1)$ we put

$$\bar{u}(A_i) = \bar{v}(A_i) = \bar{x}_k(A_i) = a(A_i);$$

(ii) for $i \in I(1)$ we put

$$\bar{u}(A_i) = a(A_i), \quad \bar{v}(A_i) = b(A_i), \quad \bar{x}_k(A_i) = x_k^i.$$

Then $\{\bar{u}, \bar{v}, \bar{x}_k\}_{k \in K} = Z$ is a sublattice of B which is isomorphic to L_β.

First, suppose that $a(\Omega) \neq b(\Omega)$. Then (since the lattice Z has no proper congruence relations) the elements $\bar{u}(\Omega)$, $\bar{v}(\Omega)$, $\bar{x}_k(\Omega)$ $(k \in K)$ are pairwise distinct. Hence $Z' = \{\bar{u}(\Omega), \bar{v}(\Omega), \bar{x}_k(\Omega)\}_{k \in K}$ is a sublattice of B/Ω which is isomorphic to L_β; but this is impossible, since B/Ω is isomorphic to L_α.

Therefore $a(\Omega) = b(\Omega)$. Hence according to 1.2 the lattice B/Ω is distributive. We conclude that B/Ω cannot be isomorphic to L_α, completing the proof. □

Theorem 1.4. *There exists an injective mapping of the class of all cardinals α with $\alpha \geqslant 3$ into the class $\mathcal{C}_{pm}$.*

P r o o f. For each cardinal α with $\alpha \geqslant 3$ we put $f(\alpha) = HCP\{L_\alpha\}$. It is obvious that $f(\alpha) \in \mathcal{C}_{pm}$. According to 1.3, the mapping f is injective. □

Hence we have verified the validity of the conjecture expressed by E. Fried in [7].

2.

Now we shall establish some additional result on convexities of lattices; we shall also propose two open questions.

Though the collection $\mathcal{C}$ fails to be a set we can apply to it the usual notions concerning the partial order.

The least element of $\mathcal{C}$ is the class X_0 consisting of all one-element lattices; the greatest element of $\mathcal{C}$ is the class $\mathcal{L}$. If $\{X_i\}_{i \in I}$ is a nonempty subcollection of $\mathcal{C}$, then $\bigcap_{i \in I} X_i$ is the greatest element of $\mathcal{C}$ contained in all X_i; thus $\bigcap_{i \in I} X_i = \bigwedge_{i \in I} X_i$. In view of existence of the greatest element in $\mathcal{C}$ we conclude

Proposition 2.1. *$\mathcal{C}$ is a complete lattice.*

Next, 1.1 implies

Lemma 2.2. *Let $\{X_i\}_{i\in I}$ be a nonempty subcollection of $\mathcal{C}$. Then $\bigvee_{i\in I} X_i = HCP\bigcup_{i\in I} X_i$.*

In the previous formula, the meaning of $P\bigcup_{i\in I} X_i$ must be, in fact, considered to be the collection of all lattices L having the property that there is a set of indices $I(1)\subseteq I$ such that L is isomorphic to $\prod_{i\in I(1)} A_i$, where $A_i\in X_i$ for each $i\in I(1)$.

Let $L^{(2)}$ be a two-element lattice. We denote $X^{(2)} = HCP\{L^{(2)}\}$.

In [7] (loc. cit) Fried suggested to investigate the convexity which is generated by the two-element lattice.

Proposition 2.3. *$X^{(2)}$ is an atom in $\mathcal{C}$.*

Proof. Obviously $X_0 < X^{(2)}$. Let $X\in\mathcal{C}$, $X_0 < X \leqslant X^{(2)}$. Hence there exists $L\in X$ with $\operatorname{card} L\geqslant 2$. Thus there are a and b in L such that $a<b$.Because of $L\in X^{(2)}$ we infer that $[a,b]\in X^{(2)}$ and hence $[a,b]$ is relatively complemented and distributive. Therefore $[a,b]$ is a Boolean algebra. There exists a congruence relation Ω on $[a,b]$ such that $[a,b]/\Omega$ is a two-element lattice. Hence $L^{(2)}\in X$ and so $X = X^{(2)}$. □

It is clear that if X is a convexity and L is an element of X such that either (i) L is a Boolean algebra, or (ii) L is a chain with $\operatorname{card} L\geqslant 2$, then $X^{(2)}\leqslant X$.

Let us remark that an element of $X^{(2)}$ need not be a Boolean algebra. In fact, the following stronger result is valid.

Proposition 2.4. *No class of Boolean algebras is a convexity.*

Proof. Let X be a nonempty class of Boolean algebras. By way of contradiction, suppose that X is a convexity. We apply the same consideration as in the proof of 2.3 and so we conclude that the two-element Boolean algebra $\{0,1\}$ belongs to X. Let I be an infinite set and for each $i\in I$ let $A_i=\{0,1\}$; put $L=\prod_{i\in I} A_i$. Choose $I(1)\subset I$ such that both $I(1)$ and $I\setminus I(1)$ are infinite. Next, let us construct an element $x\in L$ such that $x(a_i)=1$ if $i\in I(1)$ and $x(A_i)=0$ otherwise. We denote by L_1 the set of all $y\in L$ having the property that the set $\{i\in I\colon y(A_i)\neq x(A_i)\}$ is infinite. Then L_1 is a convex sublattice of L, whence $L_1\in X$. But L_1 has neither the greatest element nor the least element and thus L_1 fails to be a Boolean algebra. □

Let us remark that if L_1 is the lattice as in the proof of 2.4, then $\operatorname{card} L_1=\aleph_0$. More generally, we have

Proposition 2.5. *Let X be a convexity such that $X^{(2)} \leqslant X$. Then there is $L_1 \in X$ such that* card $L_1 = \aleph_0$.

The proof follows the same idea as in 2.4; it will be omitted.

The following questions remain open:

1. Is $X^{(2)}$ the only atom of $\mathcal{C}$?

2. Let α be an infinite cardinal. We denote by X the class of all lattices L such that, whenever C is a convex chain in L, then $\operatorname{card} C < \alpha$. Is X a convexity?

References

[1] *M. K.Bennet*: Lattices of convex sets, Trans. Amer. Math. Soc. *345* (1977), 279–288.

[2] *M. K.Bennet*: Separation conditions on convexity lattices, Lecture Notes Math. no. 1149 (1985), 22–36.

[3] *M. K.Bennet, G. Birkoff*: Convexity lattices, Alg. Univ. *20* (1985), 1–26.

[4] *G. Birkhoff*: Lattice theory, Third edition, Providence, 1967.

[5] *G. Birkhof, M. K. Bennet*: The convexity lattice of a poset, Order *2* (1985), 223–242.

[6] *M. Kolibiar*: Intervals, convex sublattices and subdirect representations of lattices, Universal algebra and applications, vol. 9, Banach Center publ., Warsaw, 1982, pp. 335–339.

[7] *R. Mlitz* (editor): General algebra 1988, Proc. Internat. Conf. Krems 1988, North Holand, Amsterdam-New York-Tokyo-Oxford, 1990.

Author's address: Matematický ústav SAV, dislokované pracovisko, Grešákova 6, 040 01 Košice, Czechoslovakia.